

Rodolfo Salvi

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ON THE EXISTENCE OF WEAK SOLUTIONS OF A NON LINEAR  
MIXED PROBLEM FOR NON HOMOGENEOUS FLUIDS IN A TIME  
DEPENDENT DOMAIN  
Rodolfo SALVI

**Abstract:** We consider the flow of a viscous incompressible non-homogeneous fluid in a tube with time dependent boundary where we give physically expressive conditions. We prove the existence of a weak solution of the problem via the Rothe method and an elliptic approximation.

**Key words:** Non-homogeneous fluid, weak solution, time dependent domain, Rothe method, elliptic approximation, compact set, boundary strip.

Classification: 35Q10

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1. Introduction. In this paper we study a non linear mixed problem for non homogeneous fluids in a non cylindrical domain in  $R^3 \times (0, T)$ . We deal with the flow of a fluid in a tube with time dependent initial and final sections where we give physically expressive boundary conditions.

In [9] we considered this problem for the Navier-Stokes equations.

We shall prove the existence of a weak solution via the Rothe method and an elliptic approximation. It has to be pointed out that many authors considered problems of this type (see [1], [2], [4], [5], [8]) but our approach is new and works well to

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study more general non linear mixed problems.

The essential point of our proof is the estimate of a time difference quotient. The paper is composed of two sections. In § 2 we describe the problem introduce particular functional spaces and give the definition of weak solutions.

2. Statement of the problem and notations. Let  $\Omega(t)$  be an open set of  $R^3$  depending on  $t \in (0, T)$ ,  $T$  is a finite positive number. As  $t$  increases over  $(0, T)$ ,  $\Omega(t)$  generates an  $(x, t)$ -domain  $\hat{\Omega}$  and the boundary  $\Gamma(t)$  of  $\Omega(t)$  generates an  $(x, t)$ -hypersurface  $\hat{\Gamma}$ . We assume that  $\hat{\Gamma}$  is a  $C^3$ -hypersurface and  $\Gamma(t) = \Gamma_1(t) \cup \Gamma_2(t) \cup \Gamma_3$  ( $\Gamma_3$  is independent of  $t$ ) with  $\text{mes. } \Gamma_3 \neq 0$ . Then we can represent  $\hat{\Gamma}$  by  $X_1 = \psi(x_1, x_2, t)$  in terms of  $C^3$ -functions  $\psi$  (in each path of a finite covering of  $\hat{\Gamma}$ ).

The motion of an inhomogeneous incompressible fluid of viscosity 1 subject to the external force  $f = f(x, t) = \{f_1(x, t), f_2(x, t), f_3(x, t)\}$  is governed by the equations

$$\begin{aligned} \rho \frac{\partial u}{\partial t} - \Delta u + \rho u \cdot \nabla u + \nabla p &= \rho f \\ (2.1) \quad \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

where  $u = u(t) = u(x, t) = \{u_1(x, t), u_2(x, t), u_3(x, t)\}$  denotes the velocity,  $\rho = \rho(t) = \rho(x, t)$  the density and  $p = p(x, t)$  the pressure. In the first relation of (2.1)  $\rho u \cdot \nabla u = \sum \rho u_1 \partial u / \partial x_1$  and the second is the equation of continuity.

Denoting by  $\nu_t$  the outside normal to  $\Gamma(t)$ , we shall consider the boundary and initial conditions defined by the relations

$$(2.2) \quad \begin{aligned} \frac{1}{2} \rho |u|^2 \nu_t - \frac{1}{2} \rho u \frac{\partial v}{\partial t} + p \nu_t - \frac{\partial u}{\partial x_i} &= -\alpha \text{ on } ((\Gamma_1(t) \cup \Gamma_2(t)), t) \\ u &= 0 \quad \text{on } \Gamma_3 \times (0, T) \end{aligned}$$

$$\varphi(x, 0) = \varphi_0, \quad u(x, 0) = u_0 \text{ in } \Omega(0).$$

The first relation in (2.2) determines the value of the density of the energy flux of the fluid on  $\Gamma_1(t) \cup \Gamma_2(t)$ . The other conditions are standard.

We shall give the weak formulation of the problem (2.1), (2.2). Let us begin by giving some definitions and basic notations.

Let  $\Omega$  be a bounded open set in  $R^3$  with boundary  $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ . We will need the following function spaces.

$$\begin{aligned} D(\Omega) &= \{ \varphi \mid \varphi \in (C^\infty(\bar{\Omega}))^3, \varphi = 0 \text{ on } \Gamma_3, \nabla \cdot \varphi = 0 \} \\ H(\Omega) &= \{ \text{the completion of } D(\Omega) \text{ under the } (L^2(\Omega))^3\text{-norm} \} \\ V(\Omega) &= \{ \text{the completion of } D(\Omega) \text{ under the } (H^1(\Omega))^3\text{-norm} \} \end{aligned}$$

We let

$$\begin{aligned} (u, v)_\Omega &= \sum \int_\Omega u_i v_i \, dx; \quad |u|_\Omega^2 = (u, u)_\Omega \\ ((u, v))_\Omega &= \sum \int_\Omega \nabla u_i \cdot \nabla v_i \, dx; \quad \|u\|_\Omega^2 = ((u, u))_\Omega \\ (u, v)_\Gamma &= \sum \int_\Gamma u_i v_i \, dx; \quad \|u\|_{\mathcal{L}} = \text{norm in } \mathcal{L}. \end{aligned}$$

Let  $\hat{\Omega}$  be the  $(x, t)$ -domain  $(\Omega(t), t)$ . For functions  $u$  defined in  $\hat{\Omega}$  we define  $\beta(u)$  by

$$\beta(u) = \int_0^T \|u\|_{\hat{\Omega}(t)}^2 \, dt$$

whenever the integral makes sense. Then we introduce

$$\begin{aligned} D(\hat{\Omega}) &= \{ \varphi \mid \varphi \in (C^\infty(\bar{\hat{\Omega}}))^3, \varphi = 0 \text{ on } \Gamma_3, \nabla \cdot \varphi = 0 \} \\ V(\hat{\Omega}) &= \{ \text{the completion of } D(\hat{\Omega}) \text{ under the norm } \beta(u) \} \end{aligned}$$

By  $U(\hat{\Omega})$  we mean the set of all  $u \in V(\hat{\Omega})$  such that

$\sup_t |u|_\Omega(t)$  over  $(0, T)$  is finite. We set the definition of weak solutions.

$(u, \rho)$  will be a weak solution of the problem (2.1), (2.2) if one has,

- i)  $u \in U(\hat{\Omega}), \rho \in L^\infty(\hat{\Omega})$
- ii)  $\forall \varphi \in D(\hat{\Omega})$  with  $\varphi(T) = 0$

$$(2.3) \int_0^T \left\{ (\rho u, \frac{\partial \varphi}{\partial t})_{\Omega(t)} + (\rho u, u \cdot \nabla \varphi)_{\Omega(t)} + \frac{1}{2} (\rho u, \varphi \frac{\partial \nu}{\partial t})_{\Gamma(t)} + \frac{1}{2} (\rho |u|^2 \nu_t, \varphi)_{\Gamma(t)} + (\alpha, \varphi)_{\Gamma(t)} + (\rho f, \varphi)_{\Omega(t)} - ((u, \varphi))_{\Omega(t)} - (\rho u \nu_t, u \cdot \varphi)_{\Gamma(t)} \right\} dt = (\rho_0 u_0, \varphi(0))_{\Omega(0)}$$

iii)

$$(2.4) \quad \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0 \quad (\text{in the distribution sense}).$$

(2.3) is obtained from the first part of (2.1) multiplying it by a test function  $\varphi$  integrating over  $\hat{\Omega}$  and bearing in mind the equation of continuity and the three conditions of (2.2).

The following theorem holds:

Theorem 1. We assume

$$u_0 \in H(\Omega(0)); f \in L^2(0, T; H(\Omega(t))); D < R_1 \leq \rho_0 \leq R_2$$

( $R_1, R_2$  are positive constants)

Then there exists a weak solution  $(u, \rho)$  of the problem (2.1), (2.2) such that  $0 < R_1 \leq \rho \leq R_2$ .

3. Proof of the theorem 1. Let us begin by considering the following approximating problem.

3.1 Auxiliary problem. We look for  $u^m, \rho^m$  such that

$$\forall \varphi \in (H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$$

$$\begin{aligned}
 (3.11) \quad & \int_0^T \left\{ \frac{1}{m} \left( \frac{\partial u^m}{\partial t}, \frac{\partial \varphi}{\partial t} \right)_{\Omega(t)} + ((u^m, \varphi))_{\Omega(t)} + \right. \\
 & + (\varphi^m u^m \cdot \nabla u^m, \varphi)_{\Omega(t)} + (u^m \cdot \nabla \varphi^m, u^m \varphi)_{\Omega(t)} - \\
 & - (\varphi^m, u^m, \frac{\partial \varphi}{\partial t})_{\Omega(t)} - \frac{1}{2m} (\Delta \varphi^m u^m, \varphi)_{\Omega(t)} - \\
 & - \frac{1}{2} \left( \varphi^m u^m, \varphi \frac{\partial \psi}{\partial t} \right)_{\Gamma(t)} - \frac{1}{2} (|u^m|^2 \cdot \nu_t, \varphi^m \varphi)_{\Gamma(t)} - \\
 & - (\alpha, \varphi)_{\Gamma(t)} - (\varphi^m f, \varphi)_{\Omega(t)} \Big\} dt = (\varphi_0^m u_0^m, \varphi(0))_{\Omega(0)} - \\
 & - (\varphi^m(T) u^m(T), \varphi(T))_{\Omega(T)},
 \end{aligned}$$

$$(3.12) \quad \frac{\partial \varphi^m}{\partial t} + u^m \cdot \nabla \varphi^m = \frac{1}{m} \Delta \varphi^m \quad \text{in } \hat{\Omega}$$

$$(3.13) \quad u^m \in (H^1(\hat{\Omega}))^3 \cap H(\Omega); \quad \varphi^m \in L^\infty(\hat{\Omega}) \cap H^1(\hat{\Omega}) \cap L^2(0, T);$$

$$H^2(\Omega(t))$$

with

$$\begin{aligned}
 & \varphi_0^m \in C^1(\Omega(0)); \quad \varphi_0^m \rightarrow \varphi_0 \quad \text{in } L^2(\Omega(0)); \quad \frac{\partial \varphi_0^m}{\partial \nu} = 0 \quad \text{on } (\Gamma(t), t) \\
 & 0 < R_1 \leq \varphi_0^m \leq R_2; \quad |\nabla \varphi_0^m|^2_{\Omega(0)} \leq m; \quad u_0^m \rightarrow u_0 \quad \text{in } L^2(\hat{\Omega}).
 \end{aligned}$$

Assuming  $u^m$  to be known and applying the Rothe method one has a solution  $\varphi^m$  of (3.12) in  $\hat{\Omega}$  (as to details see [3]). The following uniform estimates for  $\varphi^m$  hold.

By the maximum principle we have

$$(3.14) \quad 0 < R_1 \leq \varphi^m \leq R_2$$

Then multiplying (3.12) by  $\varphi^m$  and integrating over  $\Omega(t)$  one obtains

$$\frac{1}{2} \int_{\Omega(t)} \frac{\partial}{\partial t} (\varphi^m)^2 \, dx + \frac{1}{m} \|\varphi^m\|_{\Omega(t)}^2 = \frac{1}{2} \int_{\Omega(t)} u^m \cdot \nabla (\varphi^m)^2 \, dx$$

or

$$\begin{aligned}
 \frac{1}{2} |\varphi^m|_{\Omega(t)}^2 + \frac{1}{m} \int_0^t \|\varphi^m\|_{\Omega(t)}^2 \, dt &= \frac{1}{2} |\varphi_0^m|^2 + \\
 + \int_0^t \left\{ (u^m \cdot \nu_t + \frac{\partial \psi}{\partial t}, (\varphi^m)^2)_{\Gamma(t)} \right\} dt &
 \end{aligned}$$

hence

$$\frac{1}{m} \int_0^T \|\varphi^m\|_{\Omega(t)}^2 dt < c.$$

Now multiplying (3.12) by  $\Delta \varphi^m$  and integrating over  $\Omega(t)$ , and after the integration by parts one gets

$$\left( \frac{\partial \nabla \varphi^m}{\partial t}, \nabla \varphi^m \right)_{\Omega(t)} + \frac{1}{m} |\Delta \varphi^m|_{\Omega(t)}^2 = - (\nabla \varphi^m \cdot \nabla u^m, \nabla \varphi^m)_{\Omega(t)}$$

Using the estimates (3.14), (3.15) and the interpolation inequality (see [7])

$$\|\nabla \varphi^m\|_{L^4(\Omega(t))}^2 \leq c |\Delta \varphi^m|_{\Omega(t)} \|\varphi^m\|_{L^\infty(\Omega(t))}$$

one gets

$$\begin{aligned} \frac{1}{2} |\nabla \varphi^m|_{\Omega(t)}^2 + \frac{1}{m} \int_0^T |\Delta \varphi^m|_{\Omega(t)}^2 dt \leq \frac{1}{2} |\nabla \varphi_0^m|_{\Omega(0)} - \\ - \int_0^T \left( \nabla \varphi^m, \nabla \varphi^m \frac{\partial \psi}{\partial t} \right)_{\Gamma(t)} dt + c \end{aligned}$$

From (3.15) one has

$$\frac{1}{m} \int_0^T |\Delta \varphi^m|_{\Omega(t)}^2 dt \leq c |\nabla \varphi_0^m|^2 + c$$

hence

$$(3.16) \quad \int_0^T |\Delta \varphi^m|_{\Omega(t)}^2 dt \leq c m^2.$$

Next we consider the existence of the solution of (3.11). We set

$$\begin{aligned} a(\varphi^m u^m, u^m, \varphi) = \int_0^T \left\{ \frac{1}{m} \left( \frac{\partial u^m}{\partial t}, \frac{\partial \varphi}{\partial t} \right)_{\Omega(t)} + \right. \\ + (\varphi^m u^m \cdot \nabla u^m, \varphi)_{\Omega(t)} + (u^m \cdot \nabla \varphi^m, u^m \varphi)_{\Omega(t)} - \frac{1}{2} (\varphi^m |u^m|^2 \cdot \nu_t, \varphi) - \\ - \frac{1}{2} \left( \varphi^m u^m, \varphi \frac{\partial \psi}{\partial t} \right)_{\Gamma(t)} - (\varphi^m u^m, \frac{\partial \varphi}{\partial t})_{\Omega(t)} - \frac{1}{2m} (\Delta \varphi^m u^m, \varphi)_{\Omega(t)} + \\ \left. + ((u^m, \varphi))_{\Omega(t)} \right\} dt + (\varphi^m(T) u^m(T), \varphi(T))_{\Omega(T)} \\ \langle L, \varphi \rangle = \int_0^T \left\{ (\varphi^m f, \varphi)_{\Omega(t)} + (\alpha, \varphi)_{\Gamma(t)} \right\} dt + (\varphi_0^m u_0^m, \varphi(0))_{\Omega(0)}. \end{aligned}$$

By the following well known theorem one obtains the existence

of a solution in  $(H^1(\hat{\Omega}))^3$  of the equation

$$(3.17) \quad a(\varphi^m u^m, u^m, \varphi) = \langle L, \varphi \rangle$$

(for convenience we denote different constants by the same symbol  $c$ ).

Theorem 2. If

1) there exists a constant  $c > 0$  such that

$$a(\varphi^m u^m, u^m, u^m) \geq c \|u^m\|_{(H^1(\hat{\Omega}))^3}^2$$

ii) the form  $u^m \rightarrow a(\varphi^m u^m, u^m, \varphi)$  is weakly continuous in  $H^1(\hat{\Omega})$  i. e.

$$u_n^m \rightarrow u^m \text{ weakly in } (H^1(\hat{\Omega}))^3 \text{ implies}$$

$$\lim_{m \rightarrow \infty} a(\varphi^m u^m, u_n^m, \varphi) = a(\varphi^m u^m, u^m, \varphi).$$

Then (3.17) has a solution in  $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$ .

The condition ii) is obvious. The condition i) can be easily proved; in fact

$$\begin{aligned} a(\varphi^m u^m, u^m, u^m) &= \int_0^T \left\{ \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)}^2 + \|u^m\|_{\Omega(t)}^2 \right\} dt + \\ &+ \frac{1}{2} \left| \sqrt{\varphi^m} u^m(T) \right|_{\Omega(T)}^2 + \frac{1}{2} \left| \sqrt{\varphi_0^m} u^m(0) \right|_{\Omega(0)}^2 \geq c \|u^m\|_{H^1(\hat{\Omega})}^2 \end{aligned}$$

Then there exists a solution in  $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$  of (3.17).

To passing to the limit in (3.11) we will need a priori estimates of the approximations  $u^m$ .

3.2. Standard a priori estimates. We can replace in (3.11)

$\varphi$  by  $u^m$ , it comes

$$\begin{aligned} &\int_0^T \left\{ \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)}^2 + \|u^m\|_{\Omega(t)}^2 - (\varphi^m u^m, \frac{\partial u^m}{\partial t})_{\Omega(t)} \right\} + \\ &+ (\varphi^m u^m \cdot \nabla u^m, u^m)_{\Omega(t)} + (u^m \cdot \nabla \varphi^m, u^m)_{\Omega(t)} - \\ &- \frac{1}{2} (\varphi^m |u^m|^2 \cdot \nu_t, u^m)_{\Gamma(t)} - \frac{1}{2} (u^m \varphi^m, u^m \frac{\partial \varphi}{\partial t})_{\Gamma(t)} - (\alpha, u^m)_{\Gamma(t)} - \end{aligned}$$



$$\begin{aligned}
 & - \frac{1}{2m} (\Delta \varphi^m u^m, u^m)_{\Omega(t)} - (\varphi^m f, \varphi)_{\Omega(t)} \} dt = |\sqrt{\varphi^m} u^m|_{\Omega(0)}^2 - \\
 & - |\sqrt{\varphi^m} u^m|_{\Omega(T)}^2
 \end{aligned}$$

Bearing in mind (3.12), after some calculations, one has

$$\begin{aligned}
 & \int_0^T \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)}^2 dt + \int_0^T \|u^m\|_{\Omega(t)}^2 dt + |\sqrt{\varphi^m} u^m|_{\Omega(T)}^2 + \\
 & + |\sqrt{\varphi^m} u^m|_{\Omega(0)}^2 < c,
 \end{aligned}$$

hence

$$\begin{aligned}
 (3.21) \quad & \frac{1}{m} \int_0^T \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)}^2 dt < c; \quad \int_0^T \|u^m\|_{\Omega(t)}^2 dt < c \\
 & |u^m(T)|_{\Omega(T)}^2 < c; \quad |u^m(0)|_{\Omega(0)}^2 < c.
 \end{aligned}$$

By virtue of (3.14), (3.21) one gets

$$\lim_{m \rightarrow \infty} u^m \rightharpoonup u \text{ in the weak topology}$$

$$\lim_{m \rightarrow \infty} \varphi^m \rightharpoonup \varphi \text{ in the weak* topology}$$

To passing to the limit in the non linear terms of (3.11), we need the convergence of  $u^m$  in a suitable strong topology for example in  $L^2(\hat{\Omega})$ . To do this we will prove appropriate estimates.

3.3. Time difference quotients. We denote by  $\bar{u}^m(x, t)$  the extension to  $R^3$  of  $u^m$  for every  $t \in (0, T)$ ; moreover, we put  $\bar{u} = 0$  for  $t < 0, t > T$ . We let

$$u_h^m = \frac{1}{h} \int_{t-h}^t \bar{u}^m(x, s) ds \quad (h > 0)$$

We can replace in (3.11)  $\varphi$  by  $u_h^m$  and get

$$\begin{aligned}
 & \int_0^T \frac{1}{m} \left( \frac{\partial u^m}{\partial t}, \frac{\bar{u}^m(t) - \bar{u}^m(t-h)}{h} \right)_{\Omega(t)} dt - \frac{1}{h} \int_0^T (\varphi^m(t) u^m(t), \bar{u}^m(t) - \\
 & - \bar{u}^m(t-h))_{\Omega(t)} dt + \int_0^T \{ ((u^m, u_h^m))_{\Omega(t)} - (\varphi^m u^m, u^m \cdot \nabla u_h^m)_{\Omega(t)} + \\
 & + (\varphi^m u^m \cdot \nu_t, u^m \cdot u_h^m)_{\Gamma(t)} - \frac{1}{2} (\varphi^m |u^m|^2 \cdot \nu_t, u_h^m)_{\Gamma(t)} -
 \end{aligned}$$

$$\begin{aligned}
 & - (\alpha, u_h^m)_{\Gamma(t)} - \frac{1}{2} (\varphi^m u^m, u_h^m \frac{\partial \psi}{\partial t})_{\Gamma(t)} - \frac{1}{2m} (\Delta \varphi^m u^m, u_h^m)_{\Omega(t)} - \\
 & - (\varphi^m f, u_h^m)_{\Omega(t)} \} dt + (\varphi^m(T) u^m(T), u_h^m(T))_{\Omega(T)} = 0
 \end{aligned}$$

By virtue of (3.14), (3.21), Jensen inequality and the smoothness of  $\hat{r}$  one has

$$\begin{aligned}
 & \left| \frac{1}{m} \int_0^T \left( \frac{\partial u^m}{\partial t}, \frac{\bar{u}^m(t) - u^m(t-h)}{h} \right)_{\Omega(t)} dt \right| \leq \\
 & \leq \frac{1}{m} \int_0^T \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)} \left| \frac{\bar{u}^m(t) - \bar{u}^m(t-h)}{h} \right|_{\Omega(t)} dt \leq \frac{c}{m} \int_0^T \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)}^2 dt \leq c \\
 & \left| \int_0^T \left( u^m, \frac{1}{h} \int_{t-h}^t \bar{u}^m(x, s) ds \right)_{\Omega(t)} dt \right| \leq \\
 (3.31) \quad & \leq c \int_0^T \| u^m \|_{\Omega(t)} \left\| \frac{1}{h} \int_{t-h}^t \bar{u}^m(x, s) ds \right\|_{\Omega(t)} dt \leq \\
 & \leq c \int_0^T \| u^m \|_{\Omega(t)} \cdot \frac{1}{\sqrt{h}} \left( \int_{t-h}^t \| \bar{u}^m(s) \|_{R^3}^2 ds \right)^{\frac{1}{2}} dt \leq \frac{c}{\sqrt{h}}
 \end{aligned}$$

$$\begin{aligned}
 & \left| \int_0^T (\varphi^m u^m, u^m \cdot \nabla \frac{1}{h} \int_{t-h}^t \bar{u}^m(x, s) ds)_{\Omega(t)} dt \right| \leq \\
 & \leq c \int_0^T \| u^m \|_{\Omega(t)} \| u^m \|_{\Omega(t)} \cdot \frac{1}{\sqrt{h}} \left( \int_{t-h}^t \| \bar{u}^m \|_{R^3}^2 ds \right)^{1/2} dt \leq \frac{c}{\sqrt{h}}
 \end{aligned}$$

Analogously one obtains

$$\begin{aligned}
 & \left| \int_0^T (\varphi^m u^m, \frac{\partial \psi}{\partial t})_{\Gamma(t)} dt \right| \leq c/\sqrt{h}; \quad \left| \int_0^T (\alpha, \bar{u}_h^m)_{\Gamma(t)} dt \right| \leq c/\sqrt{h} \\
 (3.32) \quad & \left| \int_0^T \frac{1}{2m} (\Delta \varphi^m u^m, u_h^m)_{\Omega(t)} dt \right| \leq c/\sqrt{h}; \quad \left| \int_0^T (\varphi^m f, \bar{u}_h^m)_{\Omega(t)} dt \right| \leq \\
 & \leq c/\sqrt{h} \\
 & \left| \int_0^T (\varphi^m |u^m|^2 \cdot \nu_t, u_h^m)_{\Gamma(t)} dt \right| \leq c/\sqrt{h}; \quad \left| \int_0^T (\varphi^m u^m \cdot \nu_t, u^m \cdot \right. \\
 & \left. \cdot u_h)_{\Gamma(t)} dt \right| \leq c/\sqrt{h}.
 \end{aligned}$$

Finally we will estimate

$$- \frac{1}{h} \int_0^T (\varphi^m(t) u^m(t), \bar{u}^m(t) - \bar{u}^m(t-h))_{\Omega(t)} dt.$$

First we integrate (3.12) from  $t-h$  to  $t$  and obtain

$$(3.33) \quad \varphi(t) - \varphi(t-h) = \frac{1}{m} \int_{t-h}^t \Delta \varphi^m ds - \int_{t-h}^t \nabla \cdot (\varphi^m u^m) ds$$

obviously  $x \in C(t) = \bigcap_{s \in (t-h, t)} \Omega(s)$  with  $s \in (t-h, t)$ . Multiplying (3.33) by  $\bar{u}^m \cdot \bar{u}^m / 2$  and integrating, one obtains

$$(3.34) \quad \begin{aligned} & \frac{1}{2} \int_{t-h}^t ((\varphi^m(t) - \varphi^m(t-h))u^m, u^m)_{C(t)} dt = \\ & = \frac{1}{2m} \int_{t-h}^t ((\int_{t-h}^t \Delta \varphi^m ds)u^m, u^m)_{C(t)} dt + \\ & + \frac{1}{2} \int_{t-h}^t ((\int_{t-h}^t \nabla(u^m \varphi^m) ds)u^m, u^m)_{C(t)} dt \approx c \sqrt{h} \end{aligned}$$

Next setting  $\Delta \Omega(t) = \Omega(t) \setminus \Omega(t-h)$ ;  $\Delta \Omega(t+h) = \Omega(t+h) \setminus \Omega(t)$

and bearing in mind the smoothness of  $\hat{\Gamma}$  one has

$$(3.35) \quad \begin{aligned} & - \frac{1}{h} \int_0^T (\varphi^m(t)u^m(t), \bar{u}^m(t) - \bar{u}^m(t-h))_{\Omega(t)} dt = \\ & - \frac{1}{h} \int_0^T |\sqrt{\varphi^m(t)u^m(t)}|_{\Omega(t)}^2 dt + \frac{1}{2h} \int_0^T |\sqrt{\varphi^m(t)u^m(t)}|_{\Omega(t)}^2 dt + \\ & \frac{1}{2h} \int_0^T |\sqrt{\varphi^m(t)\bar{u}^m(t-h)}|_{\Omega(t)}^2 dt - \\ & \frac{1}{2h} \int_0^T |\sqrt{\varphi^m(t)(u^m(t) - \bar{u}^m(t-h))}|_{\Omega(t)}^2 dt \leq \\ & - \frac{1}{2h} \int_0^T |\sqrt{\varphi^m(t)u^m(t)}|_{\Delta \Omega(t+h)}^2 dt - \\ & \frac{1}{2h} \int_0^T |\sqrt{\varphi^m(t)u^m(t)}|_{\Omega(t) \cap \Omega(t+h)}^2 dt + \\ & \frac{1}{2h} \int_{t-h}^t |\sqrt{\varphi^m(t-h)\bar{u}^m(t-h)}|_{\Delta \Omega(t)}^2 dt + \\ & \frac{1}{2h} \int_{t-h}^t |\sqrt{\varphi^m(t-h)u^m(t-h)}|_{\Omega(t) \cap \Omega(t-h)}^2 dt \\ & \frac{1}{2h} \int_{t-h}^t ((\varphi^m(t) - \varphi^m(t-h))u^m(t-h), u^m(t-h))_{C(t)} dt + \\ & \frac{1}{2h} \int_{t-h}^t ((\varphi^m(t) - \varphi^m(t-h))\bar{u}^m(t-h), \bar{u}^m(t-h))_{\Omega(t) \setminus C(t)} dt - \\ & \frac{c}{h} \int_{t-h}^t |u^m(t) - u^m(t-h)|_{\Omega(t)}^2 dt \leq \\ & - \frac{1}{2h} \int_0^{T-h} |\sqrt{\varphi^m(t)u^m(t)}|_{\Omega(t+h) \cap \Omega(t)}^2 dt + \\ & \frac{1}{2h} \int_0^{T-h} |\sqrt{\varphi^m(t)u^m(t)}|_{\Omega(t+h) \cap \Omega(t)}^2 dt + \frac{c}{\sqrt{h}} - \\ & \frac{c}{\sqrt{h}} \int_{t-h}^t |u^m(t) - u^m(t-h)|_{\Omega(t)}^2 dt = \frac{c}{\sqrt{h}} - \\ & \frac{c}{h} \int_{t-h}^t |u^m(t) - u^m(t-h)|_{\Omega(t)}^2 dt. \end{aligned}$$

From (3.32), (3.32), (3.35) we conclude

$$(3.36) \quad \int_{h^v}^T |u^m(t) - u^m(t-h)|_{\Omega(t)}^2 dt \leq c \sqrt{h}.$$

By the classical characterization of M. Riesz and A. Kolmogorov of compact sets in  $L^2(\hat{\Omega})$  (see [6]) we can prove that the set  $\{u^m\}$  of  $u^m$  satisfying (3.21), (3.36) is relatively compact in  $L^2(\hat{\Omega})$ .

From (3.21) and the relative compactness of  $\{u^m\}$  in  $L^2(\hat{\Omega})$  we can choose a subsequence again denoted by  $u^m$  such that

$$\lim_{m \rightarrow \infty} \int_0^T (\varphi^m u^m, u^m \cdot \nabla \varphi)_{\Omega(t)} dt = \int_0^T (\varphi u, u \cdot \nabla \varphi)_{\Omega(t)} dt$$

$\forall \varphi \in D(\hat{\Omega})$ .

Now it remains to prove

$$(3.37) \quad \lim_{m \rightarrow 0} \int_0^T (\varphi^m u^m \cdot \nu_t, u^m \varphi)_{\Gamma(t)} dt = \int_0^T (\varphi u \cdot \nu_t, u \varphi)_{\Gamma(t)} dt$$

$$\lim_{m \rightarrow 0} \int_0^T (\varphi^m |u^m|^2 \cdot \nu_t, \varphi)_{\Gamma(t)} dt = \int_0^T (\varphi |u|^2 \cdot \nu_t, \varphi)_{\Gamma(t)} dt$$

First we will need the following compactness theorem (see [10]).

**Theorem 3.** Let  $B_0 \subset B_1 \subset B_2$  be three reflexive Banach spaces and the injection of  $B_0$  into  $B_1$  is compact. For given  $h > 0$  we define the space

$$W = \{ \varphi \mid \varphi(t) \in L^2(0, \cdot; B_0); \sup_{h^v} \frac{1}{\sqrt{h}} \int_0^{T-h^v} \|\varphi(t+h) - \varphi(t)\|_{B_2}^2 dt < \infty \}.$$

Then the injection of  $W$  into  $L^2(0, T; B_1)$  is compact.

Now geometric notions are needed below.

$\omega_1(t, \sigma)$  means the interior boundary strip of  $\Omega(t)$  with width  $\sigma$  that is

$$\omega_1(t, \sigma) = \{x \mid x \in \Omega(t), \text{dist.}(x, \Gamma(t)) < \sigma\}.$$

Then let  $\{t_j\}$  be a countable dense subset of  $(0, T)$ . For positive integers  $j, k, \ell$  we put  $G_{j,k,\ell} = (t_j, t_k) \times \Omega^\ell(t_j)$  where

$$\Omega^{\ell}(t_j) = \Omega(t_j) \setminus \omega_1(t_j, 1/\ell) = \{x | x \in \Omega(t_j), \text{dist.}(x, \Gamma(t)) > 1/\ell\}.$$

We denote by  $S$  the totality of  $G_{j,k,\ell}$  such that  $G_{j,k,\ell}$  is non-void open set of  $\hat{\Omega}_1$ . An element  $G$  of  $S$  is called a slab of type  $S$ . For  $G \in S$  the following lemma is an immediate consequence of (3.21), (3.37) and Theorem 3.

Lemma 1.  $\{u^m\}$  is relatively compact in  $L^2(t_j, t_k; H^{1-\epsilon}(\Omega^{\ell}(t_j))) \quad \forall \epsilon > 0$ . Hence  $\{u^m\}$  is relatively compact in  $L^2(t_j, t_k; \partial\Omega^{\ell}(t_j))$ . Moreover, one has (see [1])

Lemma 2. Let  $G = (\alpha, \beta) \times \Omega$  be a slab of type  $S$  and let  $\sigma'$  be a small positive number. Suppose that the lateral boundary  $(\alpha, \beta) \times \partial\Omega$  of  $G$  lies in the interior boundary strip  $\omega_1(x, \sigma')$ . Then for any  $u \in V(\Omega)$  we have

$$\int_{\alpha}^{\beta} |u|_{\Gamma}^2 dt \leq c \int_{\alpha}^{\beta} |u|_{\partial\Omega}^2 dt + \sigma' \int_{\alpha}^{\beta} \|u\|_{\Omega}^2 dt.$$

Now we are in condition to prove the strong convergence of  $u^m$  in  $L^2(\hat{\Gamma})$ . Suppose  $\epsilon > 0$  is given. If  $\sigma'_1 > 0$  is sufficiently small, then for each  $\sigma'$  in  $0 < \sigma' < \sigma'_1$  we can choose a finite number of slabs  $G_j^{(\sigma')}$  ( $j = 1, 2, \dots, N_{\sigma'}$ ) of type  $S$  with the following properties:

i) by  $K$  we denote the union of  $G_j^{(\sigma')}$ , then  $\hat{\Omega} \setminus K \subset \omega_1(x, \sigma')$ ;

ii) overlapping of  $G_j^{(\sigma')}$ 's is such that any point of  $K$  is contained in at most two of  $G_j^{(\sigma')}$ 's.

The smoothness assumptions of  $\hat{\Gamma}$  make this choice possible. We suppose the  $G_j^{(\sigma')}$  is expressed as  $(\alpha_j, \beta_j) \times \Omega_j$  (understanding the dependence on  $\sigma'$ ). We put  $w = u^m - u^n$  and attempt to show

$$\int_0^T |w|_{\Gamma}^2 dt \rightarrow 0 \quad \text{as } m, n \rightarrow \infty$$

Now

$$\int_0^T |w|_{\Gamma}^2 dt \leq \sum_1^{N_{\sigma}} \int_{\alpha_j}^{\beta_j} |w|_{\partial\Omega_j}^2 dt + \sigma \sum_1^{N_{\sigma}} \int_{\alpha_j}^{\beta_j} \|w\|_{\Omega_j}^2 dt.$$

Re-choosing  $\sigma$  if necessary, we get

$$\int_0^T |w|_{\Gamma}^2 dt \leq \sum_1^{N_{\sigma}} \int_{\alpha_j}^{\beta_j} |w|_{\partial\Omega_j}^2 dt + \varepsilon.$$

From Lemma 1 one has

$$\int_{\alpha_j}^{\beta_j} |w|_{\partial\Omega_j}^2 dt \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

consequently

$$\int_0^T |w|_{\Gamma}^2 dt < 2\varepsilon$$

provided that  $m, n$  are large enough.

From (3.21) one obtains (3.37).

Now passing to the limit  $m \rightarrow \infty$  in (3.11) all terms converge to the respective terms in (2.3).

Finally we prove

$$\sup_{0 < t < T} |u|_{\Omega}(t) < c$$

We let

$$u_{\bar{t}}^m(t) = \begin{cases} u^m(t) & 0 \leq t \leq \bar{t} \\ 0 & t > \bar{t}. \end{cases}$$

We replace in (3.11)  $\varphi$  by  $u_{\bar{t}}^m(t)$  and after some calculations obtain

$$\begin{aligned} |\sqrt{\varphi^m(\bar{t})} u^m(\bar{t})|_{\Omega(\bar{t})}^2 &\leq \frac{1}{m} \int_0^{\bar{t}} \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)}^2 dt + \\ &+ \frac{1}{m} \left( \frac{\partial u^m(\bar{t})}{\partial t}, u^m(\bar{t}) \right)_{\Omega(\bar{t})} + \frac{1}{2} |\sqrt{\varphi^m} u^m(0)|_{\Omega(0)}^2 + \\ &+ c \int_t^{\bar{t}} \|u^m\|_{\Omega(t)}^2 dt + c. \end{aligned}$$

By virtue of (3.21) and the compactness of  $\{u^m\}$  in  $L^2(\hat{\Omega})$  one obtains

$$\lim_{m \rightarrow \infty} \left| \frac{1}{m} \left( \frac{\partial u^m(\bar{t})}{\partial t}, u^m(\bar{t}) \right)_{\Omega(\bar{t})} \right| \leq$$

$$\leq \lim_{m \rightarrow \infty} \frac{1}{\sqrt{m}} \left| \frac{\partial u^m(\bar{t})}{\partial t} \right|_{\Omega(\bar{t})} \sqrt{m} |u^m(\bar{t})|_{\Omega(\bar{t})} = 0$$

hence

$$|u(\bar{t})|_{\Omega(\bar{t})} < c.$$

Finally in a standard way one has  $\varphi$  satisfies (2.4) in the distribution sense. The proof is completed.

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Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italia

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