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MULTI-PHASE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF MOTION OF GENERAL FLUIDS

Atusi TANI

Abstract The nonstationary multi-phase free boundary problem for the equations of motion of general fluids is investigated. The proof is given by the well-known theory of parabolic system in Hölder spaces.

Key words. Multi-phase free boundary problem, General fluids.

AMS classification. 35Q99 , 76N10 , 76T05

1. Introduction. There are many famous and interesting problems in hydrodynamics, whose outstanding feature is somewhat paradoxical fact that the boundary of the flow is itself not given. While there is a great variety of problems with free boundaries, some of which were already investigated in Newton's day, it seems to the present author that they do study just a little from both a real physical and a strict mathematical point of view.

The one-phase free boundary problems for incompressible viscous fluids are discussed by Solonnikov [5] and Beale [1,2] and those for compressible ones, by Tani [7] and Secchi-Valli [3].

But concerning the multi-phase free boundary problems both for incompressible and compressible viscous fluids there is only one result [8,9], as far as the author knows until now.

In this paper, we confine ourselves to the multi-phase free boundary problem for the system of differential equations of motion of compressible viscous isotropic Newtonian fluids, say general fluids.

Notation. For a domain Ω in $\mathbb{R}^3$, any non-negative integer n and $\alpha \in (0,1)$, we define:

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$C^{n+\alpha}(\bar{\Omega})=\{f(x)$, defined on $\bar{\Omega}\mid \|f\|_{\Omega}^{(n+\alpha)}\equiv \sum_{|s|=0}^{n}|D_x^s f|_{\Omega}^{(0)}+\sum_{|s|=n}|D_x^s f|_{\Omega}^{(\alpha)}<+\infty\}$,

$C_{x,t}^{n+\alpha,(n+\alpha)/2}(\bar{Q}_T)=\{g(x,t)$, defined on $\bar{Q}_T\equiv\bar{\Omega}\times[0,T]\mid \|g\|_{Q_T}^{(n+\alpha)}\equiv$

$$\equiv \sum_{2r+|s|=0}^{n}|D_t^r D_x^s g|_{Q_T}^{(0)}+\sum_{2r+|s|=n}|D_t^r D_x^s g|_{x,Q_T}^{(\alpha)}+$$

$$+\sum_{2r+|s|=\max(n-1,0)}^{n}|D_t^r D_x^s g|_{Q_T}^{((n-2r-|s|+\alpha)/2)}<+\infty\},$$

$B^n(\bar{Q}_T)=\{h(x,t)$, defined on $\bar{Q}_T\mid \sum_{r+|s|=0}^{n}|D_t^r D_x^s h|_{Q_T}^{(0)}<+\infty\}$,

$B^{n+\alpha}(\bar{Q}_T)=\{h(x,t)$, defined on $\bar{Q}_T\mid \sum_{r+|s|=0}^{n}|D_t^r D_x^s h|_{Q_T}^{(0)}+\sum_{r+|s|=n}|D_t^r D_x^s h|_{Q_T}^{(\alpha)}<+\infty\}$,

(D_x^s, $D_t^r D_x^s$ and $|s|$ are defined in a conventional way)

$$|f|_{\Omega}^{(0)}=\sup_{\Omega}|f(x)|,\quad |f|_{\Omega}^{(\alpha)}=\sup_{x,x'\in\bar{\Omega},x\neq x'}|x-x'|^{-\alpha}|f(x)-f(x')|,$$

$$|g|_{Q_T}^{(0)}=\sup_{Q_T}|g(x,t)|,\quad |g|_{x,Q_T}^{(\alpha)}=\sup_{(x,t),(x',t)\in\bar{Q}_T,x\neq x'}|x-x'|^{-\alpha}|g(x,t)-g(x',t)|,$$

$$|g|_{t,Q_T}^{(\alpha)}=\sup_{(x,t),(x,t')\in\bar{Q}_T,t\neq t'}|t-t'|^{-\alpha}|g(x,t)-g(x,t')|,\quad |g|_{Q_T}^{(\alpha)}=|g|_{x,Q_T}^{(\alpha)}+|g|_{t,Q_T}^{(\alpha/2)}.$$

Using local coordinates, it is not difficult to define such spaces for functions defined on the boundary of Ω. The same notations will be used for the spaces of vector functions, whose norms are supposed to be equal to the sum of the norms of all its components. For the Hölder exponent $\alpha=1$, notations such as $|g|_{x,Q_T}^{(L)}$ are used. By $\mathcal{O}_{loc}^{n+L}((0,\infty)\times(0,\infty))$, we mean the set of all functions $q(\rho,\theta)$ which are defined on $(0,\infty)\times(0,\infty)$, n-times partially differentiable and their n-th order derivatives are locally Lipschitz continuous there.

2. Statement of the problem. It is natural and plausible, to the present author, that the movement of one fluid acts upon those of others and the movement necessarily accompanies heat change and vice versa, so that we consider the multi-phase free boundary problem arising from the movement of a finite number, say n, of nonmiscible general fluids.

Let Ω_0 [resp. $\Omega_1,\Omega_2,\cdots,\Omega_n$]. be a bounded or unbounded domain in $\mathbb{R}^3$ [resp. Ω_0] with a boundary Γ_0 [resp. $\Gamma_1,\Gamma_2,\cdots,\Gamma_n$]; the distances between Γ_j and Γ_k $(j,k=0,1,\cdots,n;j\neq k)$ be supposed to be positive; the exterior boundary Γ_0 be assumed to be rigid. We set

$$\omega_0=\Omega_0-\bigcup_{i=1}^{n}\Omega_j,$$

$$\omega_j = \begin{cases} \Omega_j & \text{if there is no } k\in\{1,2,\cdots,n\}-\{j\} \text{ such that } \Omega_j\supset\Omega_k, \\ \Omega_j - \bigcup_{k:\Omega_k\subset\Omega_j} \Omega_k & \text{if there are } k\in\{1,2,\cdots,n\}-\{j\} \text{ such that } \Omega_j\supset\Omega_k. \end{cases}$$

Denoting by $\omega_j(t)$, the domain of the fluid at time t which initially occupies ω_j $(j=0,1,\cdots,n)$, then our *problem consists of finding the domain* $\omega_j(t)$ and the function $(\rho^{(j)},v^{(j)},\theta^{(j)})$ defined on $\omega_j(t)$ $(j=0,1,\cdots,n)$ satisfying the system of differential equations:

$$(1)\quad \begin{cases} [\frac{D}{Dt}]^{(j)}\rho^{(j)} = -\rho^{(j)}\nabla\cdot v^{(j)}, \\ \rho^{(j)}[\frac{D}{Dt}]^{(j)}v^{(j)} = \nabla\, P^{(j)} + \rho^{(j)}f^{(j)}, \\ \rho^{(j)}\theta^{(j)}[\frac{D}{Dt}]^{(j)}S^{(j)} = \nabla\cdot(\kappa^{(j)}\nabla\theta^{(j)})+\mu'^{(j)}(\nabla\cdot v^{(j)})^2+2\mu^{(j)}D^{(j)}:D^{(j)} \end{cases}$$

$$\text{in } \mathcal{D}_T^{(j)} \equiv \quad ,t)\in\mathbb{R}^4 \mid x\in\omega_j(t),\ t\in(0,T)\}\ (T>0),$$

the initial conditions

$$(2)\qquad (\rho^{(j)},v^{(j)},\theta^{(j)})\big|_{t=0} = (\rho_0^{(j)},v_0^{(j)},\theta_0^{(j)})(x) \quad (x\in\omega_j),$$

the boundary conditions

$$(3)\qquad \begin{cases} v^{(j)} = v^{(j')}, \quad P^{(j)}n^{(j)}(x,t) = P^{(j')}n^{(j')}(x,t), \\ \theta^{(j)} = \theta^{(j')}, \quad \kappa^{(j)}\nabla\theta^{(j)}\cdot n^{(j)}(x,t) = \kappa^{(j')}\nabla\theta^{(j')}\cdot n^{(j')}(x,t) \end{cases}$$

$$\text{on } \partial\omega_{j,T}\cap\partial\omega_{j',T}$$

for $\forall j,j'\in\{0,1,\cdots,n\}$ $(j\neq j')$ satisfying $\partial\omega_{j,T}\cap\partial\omega_{j',T}\neq\emptyset$,

$$(4)\qquad v^{(0)} = 0 \text{ (non-slip condition)}, \quad \theta^{(0)} = \theta_e \quad \text{on } \Gamma_{0,T}\equiv\Gamma_0\times[0,T],$$

and the equations

$$(5)\qquad [\frac{D}{Dt}]^{(j=j')}F^{(j=j')}(x,t) = 0 \quad \text{on } \partial\omega_{j,T}\cap\partial\omega_{j',T}\ (\neq\emptyset).$$

Here $\rho^{(j)}=\rho^{(j)}(x,t)$ is the density, $v^{(j)}=v^{(j)}(x,t)=(v_1^{(j)},v_2^{(j)},v_3^{(j)})$ is the velocity of the fluid at time t at the point $x=(x_1,x_2,x_3)$, $f^{(j)}=f^{(j)}(x,t)$ is a vector of external forces and $\theta^{(j)}=\theta^{(j)}(x,t)$ is the absolute temperature. The pressure $p^{(j)}$, the entropy $S^{(j)}$, the coefficients of viscosity $\mu^{(j)}$ and $\mu'^{(j)}$, and the coefficient of heat conduction $\kappa^{(j)}$ are given functions of the variables $\rho^{(j)}$ and $\theta^{(j)}$ satisfying the conditions $\mu^{(j)}$, $\kappa^{(j)}$, $S^{(j)}_{\theta^{(j)}}(=\partial S^{(j)}/\partial\theta^{(j)}) > 0$, $2\mu^{(j)} + 3\mu'^{(j)} \geqq 0$. $\nabla = (\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)$; $[\frac{D}{Dt}]^{(j)} = \partial/\partial t + (v^{(j)}\cdot\nabla)$; $P^{(j)} = [-p^{(j)}+\mu'^{(j)}(\nabla\cdot v^{(j)})]I + 2\mu^{(j)}D^{(j)}$; I is the identity matrix of order 3; $D^{(j)}=D^{(j)}(v^{(j)})$ is a matrix with elements $D^{(j)}_{ik} = \frac{1}{2}(\partial v_i^{(j)}/\partial x_k + \partial v_k^{(j)}/\partial x_i)$, $i,k=1,2,3$; $D^{(j)}:D^{(j)}=\sum_{i,k=1}^3 D_{ik}^{(j)}D_{ik}^{(j)}$; $\partial\omega_j(t)$ is the boundary of $\omega_j(t)$; $\partial\omega_{j,T}=\{(x,t)\,|\,x\in\partial\omega_j(t),\ t\in[0,T]\}$; $F^{(j=j')}(x,t)$ is such as $\partial\omega_j(t)\cap\partial\omega_{j'}(t) = \{x\in\mathbb{R}^3 \mid F^{(j=j')}(x,t) = 0\}$; $n^{(j)}=n^{(j)}(x,t)$ is a unit normal vector at $x \in \partial\omega_j(t) \cap \partial\omega_{j'}(t)$ pointing into the interior of $\omega_j(t)$ ($n^{(j')} = -n^{(j)}$).

Throughout this paper we assume that the compatibility conditions are valid even if they are not written down explicitly.

Our main result is the following:

<u>Theorem</u>. *Suppose (i)* $\Gamma_0, \Gamma_1, \cdots, \Gamma_n \in C^{2+\alpha}$ $(\alpha \in (0,1))$, $dis(\Gamma_j, \Gamma_k) > 0$ $(j,k=0,1,\cdots,n; j \neq k)$ *(ii)* $(\rho_0^{(j)}, v_0^{(j)}, \theta_0^{(j)}) \in C^{1+\alpha}(\bar{\omega}_j) \times C^{2+\alpha}(\bar{\omega}_j) \times C^{2+\alpha}(\bar{\omega}_j)$ $(0 < \underline{\rho}_0^{(j)} \leq \rho_0^{(j)}(x) \leq \bar{\rho}_0^{(j)}$, $0 < \underline{\theta}_0^{(j)} \leq \theta_0^{(j)}(x) \leq \bar{\theta}_0^{(j)}$; $\underline{\rho}_0^{(j)}$, $\bar{\rho}_0^{(j)}$, $\underline{\theta}_0^{(j)}$, $\bar{\theta}_0^{(j)}$ *are constants)* $(j=0,1,\cdots,n)$ *(iii)* $f^{(j)} \in B^1(\bar{\mathbb{R}}_T^3 \equiv \mathbb{R}^3 \times [0,T])$, $\sum_{r+|s|=1} \overline{|D_t^r D_x^s f^{(j)}|}^{(L)}_{x,\bar{\mathbb{R}}_T^3} < +\infty$ $(j=0,1,\cdots,n)$ *(iv)* $\mu^{(j)}$, $\mu'^{(j)}$, $\kappa^{(j)}$, $p^{(j)}$, $S^{(j)} \in \mathcal{O}_{loc}^{2+L}((0,\infty) \times (0,\infty))$, $2\mu^{(j)} + 3\mu'^{(j)} \geq 0$, $\mu^{(j)}$, $\kappa^{(j)}$, $S_{\theta^{(j)}}^{(j)} > 0$ $(j=0,1,\cdots,n)$ *(v)* $\theta_e \in C_{x,t}^{2+\alpha,1+\alpha/2}(\Gamma_{0,T})$.

Then there exists a unique solution $(\rho^{(j)}, v^{(j)}, \theta^{(j)})$ $(j=0,1,\cdots,n)$ *of* (1)~(5), *which belongs to* $B^{1+\alpha}(\bar{\mathcal{O}}_{T'}^{(j)}) \times C_{x,t}^{2+\alpha,1+\alpha/2}(\bar{\mathcal{O}}_{T'}^{(j)}) \times C_{x,t}^{2+\alpha,1+\alpha/2}(\bar{\mathcal{O}}_{T'}^{(j)})$ $(0 < \rho^{(j)} \leq \bar{\rho}^{(j)} = constant$, $0 < \theta^{(j)} \leq \bar{\theta}^{(j)} = constant)$ *for some* $T' \in (0,T)$ $(j=0,1,\cdots,n)$.

3. <u>Sketch of the proof of Theorem</u>. Since we have already proved in [8] the analogous theorem for two-phase free boundary problem of general fluids in detail and the same arguements are applicable in the present case, we give here only the sketch of the proof of the above theorem.

1°. First of all, we transform the equations (1) by the characteristic transformation $\Pi_{x_0,t_0}^{x,t} : (x,t) \mapsto (x_0,t_0)$ which is defined by the relation

$$x = x_0 + \int_0^{t_0} \hat{v}^{(j)}(x_0,\tau)d\tau \equiv x(x_0,t_0;\hat{v}^{(j)}) \quad (\hat{v}^{(j)}(x_0,t_0) = \Pi_{x_0,t_0}^{x,t} v^{(j)}(x,t))$$

into the form

$$(6)\quad \begin{cases} \dfrac{\partial}{\partial t_0}\hat{\rho}^{(j)} = -\hat{\rho}^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)}, \\ \hat{\rho}^{(j)}\dfrac{\partial}{\partial t_0}\hat{v}^{(j)} = \nabla_{\hat{v}^{(j)}}(\mu'^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)}) + 2\nabla_{\hat{v}^{(j)}}\cdot(\mu D_{\hat{v}^{(j)}}(\hat{v}^{(j)})) - \\ \qquad - \nabla_{\hat{v}^{(j)}}p^{(j)} + \hat{\rho}^{(j)}\hat{f}^{(j)}, \\ \hat{\rho}^{(j)}\hat{\theta}^{(j)}S_{\hat{\theta}^{(j)}}^{(j)}\dfrac{\partial}{\partial t_0}\hat{\theta}^{(j)} = \nabla_{\hat{v}^{(j)}}\cdot(\kappa^{(j)}\nabla_{\hat{v}^{(j)}}\hat{\theta}^{(j)}) + \mu'^{(j)}(\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)})^2 + \\ \qquad + 2\mu^{(j)}D_{\hat{v}^{(j)}}(\hat{v}^{(j)}):D_{\hat{v}^{(j)}}(\hat{v}^{(j)}) + \hat{\rho}^{(j)2}\hat{\theta}^{(j)}S_{\hat{\rho}^{(j)}}^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)} \end{cases}$$

$(j=0,1,\cdots,n)$.

Here $(\hat{\rho}^{(j)}, \hat{\theta}^{(j)})(x_0,t_0) = \Pi_{x_0,t_0}^{x,t}(\rho^{(j)}, \theta^{(j)})(x,t)$, $\mathcal{J}^{(j)} = (\partial x(x_0,t_0,v^{(j)})/\partial x_0)^{-1}$,

$\nabla_{\hat{v}^{(j)}} = (\nabla_{\hat{v}^{(j)},1}, \nabla_{\hat{v}^{(j)},2}, \nabla_{\hat{v}^{(j)},3}) = \mathcal{G}^{(j)}\mathring{\nabla}$, $\mathring{\nabla} = (\partial/\partial x_{0,1}, \partial/\partial x_{0,2}, \partial/\partial x_{0,3})$, $D_{\hat{v}^{(j)}}(\hat{v}^{(j)})$ is a matrix with elements $\frac{1}{2}(\nabla_{\hat{v}^{(j)},k}\hat{v}_i^{(j)} + \nabla_{\hat{v}^{(j)},i}\hat{v}_k^{(j)})$ $(i,k=1,2,3)$.

Integrating the equations $(6)_1$, we can reduce our problem to the initial-boundary value problem for the parabolic system $(6)_{2,3}$ with

$$\hat{\rho}^{(j)}(x_0,t_0) = \rho_0^{(j)}(x_0)\ \exp[-\int_0^{t_0}\nabla_{\hat{v}^{(j)}}\ \hat{v}^{(j)}(x_0,\tau)d\tau]$$

and with the initial-boundary conditions

(7) $$(\hat{v}^{(j)},\hat{\theta}^{(j)})(x_0,0) = (v_0^{(j)},\theta_0^{(j)})(x_0) \quad \text{on} \quad \omega_j \ (j=1,2,\cdots,n),$$

(8) $$\begin{cases} \hat{v}^{(j)} = \hat{v}^{(j')}, \quad \dfrac{\hat{P}^{(j)}\cdot\mathcal{G}^{(j)}n^{(j)}(x_0)}{|\mathcal{G}^{(j)}\mathring{\nabla}F_0^{(j)}|} = \dfrac{\hat{P}^{(j')}\cdot\mathcal{G}^{(j')}n^{(j')}(x_0)}{|\mathcal{G}^{(j')}\mathring{\nabla}F_0^{(j')}|}, \quad \hat{\theta}^{(j)} = \hat{\theta}^{(j')}, \\ \dfrac{\kappa^{(j)}\mathcal{G}^{(j)}n^{(j)}(x_0)}{|\mathcal{G}^{(j)}\mathring{\nabla}F_0^{(j)}|}\cdot(\nabla_{\hat{v}^{(j)}}\hat{\theta}^{(j)}) = \dfrac{\kappa^{(j')}\mathcal{G}^{(j')}n^{(j')}(x_0)}{|\mathcal{G}^{(j')}\mathring{\nabla}F_0^{(j')}|}\cdot(\nabla_{\hat{v}^{(j')}}\hat{\theta}^{(j')}) \end{cases}$$

on $[\partial\omega_j \cap \partial\omega_{j'}]\times[0,T]$ for $\forall j,j' \in \{0,1,\cdots,n\}$ $(j\neq j')$ satisfying $\partial\omega_j \cap \partial\omega_{j'} \neq \emptyset$,

(9) $$\hat{v}^{(0)} = 0, \quad \hat{\theta}^{(0)} = \hat{\theta}_e \quad \text{on} \quad \Gamma_{0,T},$$

where $\hat{P}^{(j)} = [-p^{(j)} + \mu'^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)}]I + 2\mu^{(j)}D_{\hat{v}^{(j)}}(\hat{v}^{(j)})$, $F_0^{(j)}(x_0) = F^{(j)}(x_0,0)$, $n^{(j)}(x_0) = n^{(j)}(x_0,0)$.

(6)∿(9) can be written in a shorter form

(10) $$\begin{cases} \dfrac{\partial}{\partial t_0}w^{(j)} = \mathcal{A}^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})w^{(j)} + \mathcal{B}^{(j)}(x_0,t_0,w^{(j)}) \quad \text{in} \quad Q_T^{(j)}, \\ w^{(j)}|_{t_0=0} = 0, \\ w^{(0)} = (0,\hat{\theta}_e - \theta_0^{(0)}) \quad \text{on} \quad \Gamma_{0,T}, \\ \begin{pmatrix} I \\ \dfrac{B^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})}{|\mathcal{G}^{(j)}n^{(j)}(x_0)|}\end{pmatrix} w^{(j)} - \begin{pmatrix} I \\ \dfrac{B^{(j')}(x_0,t_0,w^{(j')};\mathring{\nabla})}{|\mathcal{G}^{(j')}n^{(j')}(x_0)|}\end{pmatrix} w^{(j')} = \\ = \phi(x_0,t_0,w^{(j)},w^{(j')}) \quad \text{on} \quad [\partial\omega_j \cap \partial\omega_{j'}]\times[0,T], \end{cases}$$

where $w^{(j)} = (\hat{v}^{(j)}-v_0^{(j)}, \hat{\theta}^{(j)}-\theta_0^{(j)})$, $Q_T^{(j)} = \omega_j\times(0,T)$, $\mathcal{A}^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})$ and $B^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})$ are matrices with elements 2nd and 1st order differential operators respectively.

2°. We consider a linearlized initial-boundary value problem of (10):

$$
(11)\quad \begin{cases}
\dfrac{\partial}{\partial t_0} w^{(j)} = \mathcal{A}^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})w^{(j)} + \mathcal{B}^{(j)}(x_0,t_0,w^{(j)}) \quad \text{in } Q_T^{(j)}, \\
w^{(j)}|_{t_0=0} = 0, \\
w^{(0)} = (0,\hat{\vartheta}_e - \vartheta_0^{(0)}) \quad \text{on } \Gamma_{0,T}, \\
\left(\dfrac{I}{\dfrac{B^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})}{|\mathcal{G}^{(j)}(w^{(j)})n^{(j)}(x_0)|}}\right)w^{(j)} - \left(\dfrac{I}{\dfrac{B^{(j')}(x_0,t_0,w^{(j')};\mathring{\nabla})}{\mathcal{G}^{(j')}(w^{(j')})n^{(j')}(x_0)}}\right)w^{(j')} = \\
\quad = \phi(x_0,t_0,w^{(j)},w^{(j')}) \quad \text{on } [\partial\omega_j \cap \partial\omega_{j'}] \times [0,T].
\end{cases}
$$

Here $w^{(j)}(j=0,1,\cdots,n)$ are assumed to belong to the set

$$
\mathfrak{S}_T = \{(w^{(0)},\cdots,w^{(n)}) \in C^{2+\alpha,1+\alpha/2}_{x_0,t_0}(\bar{Q}_T^{(0)}) \times \cdots \times C^{2+\alpha,1+\alpha/2}_{x_0,t_0}(\bar{Q}_T^{(n)}) \mid w^{(j)}|_{t_0=0} = 0,
$$

$$
\|w^{(j)}\|^{(2)}_{\bar{Q}_T^{(j)}} < M_1^{(j)}, \quad |\mathring{\nabla}\mathring{\nabla}w^{(j)}|^{(\alpha)}_{x_0,\bar{Q}_T^{(j)}} < M_2^{(j)} \quad (j=0,1,\cdots,n)\}
$$

$\left(\|w^{(j)}\|^{(2)}_{\bar{Q}_T^{(j)}} = \sum_{2r+|s|=0}^{2} |D^r_{t_0} D^s_{x_0} w^{(j)}|^{(0)}_{\bar{Q}_T^{(j)}}\right)$ for any positive number $M_1^{(j)}$ and a positive number $M_2^{(j)}$ determined later.

We note the two facts:

(a) The system of differential equations (11) is uniformly parabolic in the sense of Petrowsky (modulo of parabolicity δ) for a suitably chosen T.

(b) When we consider the same problem as (11) in $\mathbb{R}^3_+ \equiv \{x_0=(x_{0,1},x_{0,2},x_{0,3}) \mid x_{0,3} > 0\}$, the complementing condition holds (see [6,8]).

(b) garantees the possibility for the construction of the regularizer of (11) in the half space $\mathbb{R}^3_+$, from which, together with the partition of unity, follows the solvability of auxiliary linearized problem (11):

There exists a unique solution $w^{(j)} \in C^{2+\alpha,1+\alpha/2}_{x_0,t_0}(\bar{Q}_T^{(j)})$ of (11) satisfying the estimates

$$
(12)\quad \begin{cases}
\|w^{(j)}\|^{(2)}_{\bar{Q}_T^{(j)}} \le [C_1^{(j)}(T,M_1^{(j)}) + C_2^{(j)}(T,M_1^{(j)})M_2^{(j)}](T^{\alpha/2} + T^{1+\alpha/2}), \\
|\nabla\nabla w^{(j)}|^{(\alpha)}_{x_0,\bar{Q}_T^{(j)}} \le C_1^{(j)}(T,M_1^{(j)}) + C_2^{(j)}(T,M_1^{(j)})M_2^{(j)},
\end{cases}
$$

where $C_1^{(j)}$ and $C_2^{(j)}$ increase monotonically in T and $M_1^{(j)}$ and $C_2^{(j)} \to 0$ as $T \to 0$ $(j=0,1,\cdots,n)$. If we choose the constant $M_2^{(j)}$ and T_0 in such a way that $M_2^{(j)} > C_1^{(j)}(T,M_1^{(j)}) + M$ for any positive number M and for such $M_2^{(j)}$, $[C_1^{(j)}(T_0,M_1^{(j)}) + M]\,(T_0^{\alpha/2} + T_0^{1+\alpha/2}) \le M_1^{(j)}$ and $C_2^{(j)}(T_0,M_1^{(j)})M_2^{(j)} \le M$, then $W = (W^{(0)}, \cdots, W^{(n)}) \in \mathfrak{S}_{T_0}$. We denote T_0 by T for simplicity.

3°. Next we construct the sequence $\{w_m(x_0,t_0)\}$ of successive approximate solutions as follows:

$$\begin{cases} w_0(x_0,t_0) = 0 \\ w_m(x_0,t_0) \text{ is defined as a solution } W^{T'} \text{ of (11) assuming } w \equiv (w^{(0)}, \cdots, \\ w^{(n)}) = w_{m-1} \in \mathfrak{S}_T. \end{cases}$$

Then the result in 2°implies that w_m $(m=0,1,2,\cdots)$ are well defined and belong to $\mathfrak{S}_T$. Applying the estimates (12) to the equation concerning $w_m - w_{m-1}$, we obtain

$$|||w_m - w_{m-1}||| \leq C_3(T,M_1,M_2)\, |||w_{m-1} - w_{m-2}||| \tag{13}$$

$\left(|||w||| \equiv \sum_{j=0}^{n} \|w^{(j)}\|^{(2+\alpha)}_{\overline{Q}_T^{(j)}},\ M_1 = \sum_{j=0}^{n} M_1^{(j)},\ M_2 = \sum_{j=0}^{n} M_2^{(j)}\right)$ where $C_3 \to 0$ as $T \to 0$. Therefore the sequence $\{w_m(x_0,t_0)\}$ converges to $w(x_0,t_0)$ uniformly if we choose $T' \in (0,T]$ so as to satisfy $C_3(T',M_1,M_2) < 1$. Then $\hat{v} \equiv (v^{(0)}, \cdots, v^{(n)}) = w' + v_0$ $(w = (w', w_4),\ v_0 = (v_0^{(0)}, \cdots, v_0^{(n)}))$, $\hat{\theta} = w_4 + \theta_0$ $(\theta_0 = (\theta_0^{(0)}, \cdots, \theta_0^{(n)}))$, $\hat{\rho}(x_0,t_0) = \rho_0(x_0) \exp[-\int_0^{t_0} \hat{v}\, d\tau]$ is our desired solution of (6)∿(9). The uniqueness of the solution follows from the uniqueness of the solution of (10), which is proved by the fact that two solutions supposed to exist satisfy the inequality analogous to (13).

4°. The unique solution of the original free boundary problem (1)∿(5) can be obtained by the formulae

$$(\rho^{(j)}(x,t),\ v^{(j)}(x,t),\ \theta^{(j)}(x,t),\ \omega_j(t)) = \\ = \Pi_{x,t}^{x_0,t_0}(\hat{\rho}^{(j)}(x_0,t_0),\ \hat{v}^{(j)}(x_0,t_0),\ \hat{\theta}^{(j)}(x_0,t_0),\ \omega_j) \quad (j=0,1,\cdots,n).$$

The positivity and boundedness of ρ and θ are obvious from our construction method.

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