Thomas N. Vougiouklis
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ON AFFINE KAC-MOODY LIE ALGEBRAS

Thomas N. VOUGIOUKLIS

Abstract: In this paper we deal with affine Kac-Moody Lie algebras of type $D_n^{(1)}$, $n \geq 4$. We give a method of computation of the eigenvalues needed for the realization of the basic representation that appeared in [3].

Key words: Affine Kac-Moody Lie algebras, graded Lie algebras.

Classification: 17B65, 17B70

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1. Introduction. In the paper [3] there is given a construction of the basic representation of Euclidean algebras. This is a generalization of the construction in [5]. In the main result of the paper [3] Theorem 4.1 one needs some constants λ_{ij}. The aim of this paper is to give a method to compute those constants for the affine Kac-Moody Lie algebras $D_n^{(1)}$, $n \geq 4$. Also we give an appropriate gradation of Lie algebras of type D_n.

2. Fix $n \geq 4$. Let $\{E_{ij}\}_{i,j=1,\ldots,2n}$ be the standard basis of the space of $2n\times 2n$ complex matrices, so that the matrix E_{ij} is 1 in the ij-entry and 0 in all the other entries.

We focus our attention on the basic representation of the affine Kac-Moody Lie algebra $\mathfrak{g}(A)$ of type $D_n^{(1)}$ $(n\geq 4)$, see KAC [2] and MOODY [6]. In this case we have

$$(1)\qquad \mathfrak{g} = o(2n,\mathbb{C}) \quad , \quad (x|y) = \operatorname{tr} xy \ .$$

So from the classical Lie algebras with symmetric Cartan matrices, see [2], we consider the Lie algebras of type D_n $(n\geq 4)$. We know that $\mathfrak{g}$ consists of all $2n\times 2n$ complex matrices X such that $X.J+J^t.X = 0$ where

$$J = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{pmatrix} \quad , \text{ see } [1] \ .$$

We take in $\mathfrak{g}$ the elements

$$(2)\begin{cases} e_o = E_{2n-1,1}-E_{2n,2}\ ,\ e_i = E_{i,i+1}-E_{2n-i,2n-i+1} \quad (i=1,\ldots,n-1), \\ e_n = E_{n-1,n+1}-E_{n,n+2} \\ f_o = E_{1,2n-1}-E_{2,2n}\ ,\ f_i = E_{i+1,i}-E_{2n-i+1,2n-i} \quad (i=1,\ldots,n-1), \\ f_n = E_{n+1,n-1}-E_{n+2,n} \\ h_o = E_{2n,2n}+E_{2n-1,2n-1}-E_{22}-E_{11} \ , \\ h_i = E_{2n-i,2n-i}+E_{ii}-E_{2n-i+1,2n-i+1}-E_{i+1,i+1} \quad (i=1,\ldots,n-1), \\ h_n = E_{nn}+E_{n-1,n-1}-E_{n+2,n+2}-E_{n+1,n+1} \end{cases}$$

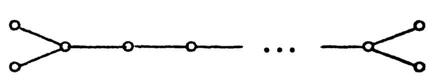
It is easy to see that the above elements satisfy the following relations

$$\left.\begin{array}{ll} [e_i\ ,\ f_j] = \delta_{ij}h_i & [h_i\ ,\ h_j] = 0 \\ [h_i\ ,\ e_j] = a_{ij}e_j & [h_i\ ,\ f_j] = -a_{ij}f_j \end{array}\right\} (i,j=0,\ldots,n)$$

where the $(n+1)\times(n+1)$ matrix

$$A = (a_{ij}) = \left[\begin{array}{cccccccc} 2 & 0 & -1 & 0 & & & & \\ 0 & 2 & -1 & 0 & & 0 & & \\ -1 & -1 & 2 & -1 & & & & \\ 0 & 0 & -1 & 2 & & & & \\ & & & & \ddots & & & \\ & & & & 2 & -1 & 0 & 0 \\ & 0 & & & -1 & 2 & -1 & -1 \\ & & & & 0 & -1 & 2 & 0 \\ & & & & 0 & -1 & 0 & 2 \end{array}\right]$$

be the generalized Cartan matrix corresponding to the Dynkin diagram



of $D_n^{(1)}$, see [2] .

We know [2], that a Lie algebra $\mathfrak{g}$ is said to be a graded modh Lie algebra if it can be written in the form

$$\mathfrak{g} = \bigoplus_{i=0}^{h-1} \mathfrak{g}_i \qquad \text{(direct sum)}$$

with the property

(3) $[\mathfrak{g}_i , \mathfrak{g}_j] \subset \mathfrak{g}_{(i+j)\mathrm{mod}h}$.

From now on for $i \in \mathbb{Z}$ we set $\mathfrak{g}_{i+h\mathbb{Z}} = \mathfrak{g}_i$ so we have

$$\mathfrak{g} = \bigoplus_{i=0}^{h-1} \mathfrak{g}_i = \bigoplus_{i \in \mathbb{Z}/h\mathbb{Z}} \mathfrak{g}_i$$

An element x of $\mathfrak{g}_i$ is said to be of degree i .

In our case we take the Coxeter number [3] for $\mathfrak{g}$ as h i.e. $h = 2(n-1)$.

Let's denote by $\underline{\varkappa}$ the number defined in the following way

$$\left.\begin{array}{ll} \underline{\varkappa} = \varkappa & \text{if } \varkappa \le n \\ \underline{\varkappa} = \varkappa - 1 & \text{if } \varkappa > n \end{array}\right\} \quad \varkappa \in \mathbb{Z}$$

PROPOSITION 1

A Lie algebra $\mathfrak{g}$ of type D_n is a graded modh where the 1-principal $\mathbb{Z}/h\mathbb{Z}$- gradation of $\mathfrak{g}$ is given by setting

(4) $\deg E_{ij} = (\underline{j}-\underline{i})\mathrm{mod}h$

PROOF

It is a simple calculation, observing that all the elements $e_o , e_1 , \ldots , e_n$ have degrees 1.

So the elements e_i , f_i , h_i , $i = 0,1,\ldots,n$, defined by (2), are the Chevalley generators for a complex Lie algebra $\mathfrak{g}^{\sim}(A)$, which we quotient by its largest $\mathbb{Z}^{n+1}$-graded ideal intersecting trivially the span of $h_o , \ldots , h_n$ and we take the Kac-Moody Lie algebra $\mathfrak{g}(A)$ of type $D_n^{(1)}$ $(n \ge 4)$. The images of e_i , f_i , h_i $(i=0,1,\ldots,n)$ in $\mathfrak{g}(A)$ will be denoted by the same letters.

3. We set

$$e = \sum_{i=0}^{n} e_i$$

This is a 1-cyclic element of $\mathfrak{g}$ studied by KOSTANT in [4]. Note that [3] a cyclic element is conjugate to a multiple of any other cyclic element by an automorphism of $\mathfrak{g}(A)$ defined by

$$e_i \longmapsto \beta_i e_i \quad , \quad f_i \longmapsto \beta_i^{-1} f_i \quad , \quad i = 0,1,\dots,n.$$

Although there is not known any natural normalization of the cyclic element in general, we give the following normalization in case $D_n^{(1)}$:

Note that

$$(\beta_i e_i)^{h+1} = (-1)^n 4\beta_0\beta_1\beta_{n-1}\beta_n \prod_{j=2}^{n-2} \beta_j^2 (\beta_i e_i)$$

so putting $\beta_j=1$, $j=2,\dots,n-2$ and $\beta_0=\beta_1=\beta_{n-1}=\beta_n=1/\sqrt{2}$ we obtain the simplest, for real β_i's, relation

$$(\beta_i e_i)^{h+1} = (-1)^n (\beta_i e_i)$$

From now on we take the normalized 1-cyclic element

(5) $E = \beta_i e_i$ where $\beta_0=\beta_1=\beta_{n-1}=\beta_n=1/\sqrt{2}$, $\beta_j=1, j=2,\dots,n-2$

<u>Remark</u> The 1-cyclic element E satisfies the following obvious relations:

(6) $E^{h+1} = (-1)^n E$

(7) $E^{2\varkappa-1} = (-1)^n \, {}^tE^{2(n-\varkappa)-1}$ for $\varkappa \in N$; $\varkappa \le \frac{n}{2}$

Let S be the centralizer of E in $\mathfrak{g}$. Then, see Lemme 6.4B in [4], S is a Cartan subalgebra of $\mathfrak{g}$. It is clear that in our case a basis for S is the set

$$\{ E, E^3, \ldots, E^{2n-3}, E_o \}$$

where $E_o = E_{1n}-E_{n,n+1}+E_{n1}+E_{n,2n}-E_{n+1,1}-E_{n+1,2n}+E_{2n,n}-E_{2n,n+1}$

The element E_o has $\deg E_o = n-1$ and satisfies the relation

$$E_o^2 = 4I+(-1)^{n-1}4E^h$$

We need a basis T_i, $i=1,\ldots,n$ such that

$$(T_i | T_{n-j}) = \delta_{ij}$$

By virtue of (7) such a basis is the following one :

For $n = 2\varkappa$

$$\begin{cases} T_i = \dfrac{1}{\sqrt{h}} E^{2i-1} , & T_{\varkappa+i+1} = {}^tT_{\varkappa-i} , \quad i=1,\ldots,\varkappa-1 \\ T_\varkappa = \dfrac{1}{2\sqrt{2}} E_o , & T_{\varkappa+1} = \dfrac{1}{2\sqrt{2}} E_o + \dfrac{1}{\sqrt{h}} E^{2\varkappa-1} \end{cases}$$

For $n = 2\varkappa+1$

$$\begin{cases} T_i = \dfrac{1}{\sqrt{h}} E^{2i-1} , & T_{\varkappa+i+1} = -{}^tT_{\varkappa-i+1} , \quad i=1,\ldots,\varkappa \\ T_{\varkappa+1} = \dfrac{1}{2\sqrt{2}} E_o \end{cases}$$

According to [3] the subspace $\mathfrak{g}_o$ of the elements of $\mathfrak{g}$ of degree 0 is the linear span of the projections of all the root spaces of $\mathfrak{g}$ with respect to S. Our problem is to choose n root vectors $A_1, \ldots, A_n$, with respect to S, corresponding to the roots $\beta_1, \ldots, \beta_n$ such that their projections on $\mathfrak{g}_o$ form a basis of this space, then to compute the constants λ_{ij} defined by

$$\lambda_{ij} = \beta_i(T_j) \quad , \; i,j=1,\ldots,n$$

Note that if we decompose the vectors $A_1, \dots, A_n$ with respect to the 1-principal gradation :

$$A_i = \sum_j A_{ij} \quad , \quad i=1,\dots,n \ , \quad j \in \mathbb{Z}/h\,\mathbb{Z}$$

then the elements

$$A_{ij} \ , \ T_\varkappa \qquad \text{where} \quad i,\varkappa=1,\dots,n \ ; \quad j=0,\dots,h-1$$

form a basis of $\mathfrak{g}$.

Using the above notation we obtain the following proposition for $D_n^{(1)}$.

PROPOSITION 2

The constants λ_{ij} belong to the h-roots of the real numbers τ_ν , τ_μ such that

$$(8) \quad (\mathrm{ad}T_{h-\nu})(\mathrm{ad}T_\nu)A_{io} = \tau_\nu A_{io} \quad , \quad (\mathrm{ad}T_\mu)^2 A_{io} = \tau_\mu A_{io}$$

where $\nu=1,\dots,\varkappa-1$ and $\mu=\varkappa,\varkappa+1$ for $n = 2\varkappa$,

$\nu=1,\dots,\varkappa$ and $\mu=\varkappa+1$ for $n = 2\varkappa+1$.

PROOF

From relation (7) we obtain that $\mathrm{ad}T_\nu$ and $\mathrm{ad}T_{h-\nu}$ have opposite eigenvalues with eigenvectors which are transpose to each other. That means that those eigenvectors have the same projections on $\mathfrak{g}_o$.

On the other hand from the relation

$$(\mathrm{ad}T_\nu)A_{ij} = \beta_i(T_\nu)A_{i,j+\nu}$$

the transformation $\mathrm{ad}T_\nu$ shifts the gradation by ν. Therefore we have for the projection A_{io} on $\mathfrak{g}_o$ the relation

$$(\mathrm{ad}T_{h-\nu})(\mathrm{ad}T_\nu)A_{io} = \beta_i(T_\nu)\beta_i(T_{h-\nu})A_{io}$$

According to KAÇ [2] an automorphism σ of order h of the Lie algebra $\mathfrak{g}$ is given by $\sigma(x) = \varepsilon^{i}x$, $x \in \mathfrak{g}_i$, where ε is a primitive h-root of unity. So we have

$$(\beta_i(T_\nu))^{h-\nu} = (\beta_i(T_{h-\nu}))^{\nu}$$

Therefore if ε is a primitive h-root of $\tau_\nu = \beta_i(T_\nu)\beta_i(T_{h-\nu})$ then we can take

$$\varepsilon^{\nu} = \beta_i(T_\nu) \;,\quad \varepsilon^{h-\nu} = \beta_i(T_{h-\nu}) \;. \qquad \text{Q.E.D.}$$

4. In order to compute the λ_{ij}'s we can take an integer $\nu<n$ such that $(\nu,h) = 1$ and then we can try to find n different τ_ν's . The eigenvector A_i corresponding to such an τ_ν will be defined by

$$A_i = \sum_{j=0}^{h-1} (\mathrm{ad}T_\nu)^{j} A_{io} \tag{9}$$

In the special case of $\nu = 1$ we have the following :

a) For $A_{10} = E_{11}-E_{2n,2n}$ and $A_{no} = E_{nn}-E_{n+1,n+1}$ we have $\tau = -1$. So

$$A_i = \sum_{j=0}^{h-1} (\mathrm{ad}T_\nu)^{j} A_{io} \;,\quad \lambda_{ij} = \varepsilon^{j}$$

where $i=1$ or n and ε be an h-primitive root of -1 .

b) For $i = 2,\dots,n-1$ we set

$$A_{io} = \mathrm{diag}(0,x_{i2},\dots,x_{i,n-1},0,0,-x_{i,n-1},\dots,-x_{i2},0) \tag{10}$$

Then from relation (8) we obtain that τ's are the eigenvalues of the matrix

$$D = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

and for some eigenvector $x = (x_{i2},\ldots,x_{i,n-1})$ we get the corresponding A_{io} from (10).

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REFERENCES

[1] D.Ž. DJOKOVIĆ, Classification of Z-graded real semisimpe Lie algebras, Journal of Algebra 76 (1982), 367-382.

[2] V.G. KAČ, Simple irreducible graded Lie algebras of finite growth, Isv.Akad. Nauk. SSSR Ser. Mat. Tom 32 (1968) No6, 1271-1311.

[3] V.G. KAČ, D.A. KAZHDAN, J. LEPOWSKY and R.L.WILSON, Realization of the basic representations of the Euclidean Lie algebras, Advances in Mathematics, 42 (1981), 83-112.

[4] B. KOSTANT, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, American Journal of Mathematics 81 (1959), 973-1032.

[5] J. LEPOWSKY and R.L. WILSON, Construction of the affine Lie algebra $A_1^{(1)}$, Comm. Math. Phys. 62 (1978), 43-53.

[6] R.V. MOODY, A new class of Lie algebras, Journal of algebra 10 (1968), 211-230.

Democritus University of Thrace, Math. Dept., Xanthi, Greece

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