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TWO NON-HOMEOMORPHIC COUNTABLE SPACES HAVING
HOMEOMORPHIC SQUARES
M. M. MARJANOVIĆ and A. R. VUCEMILOVIĆ

Abstract: A pair of non-homeomorphic countable metrizable spaces having homeomorphic squares is exhibited. This answers a question of V. Trnková from [4].

Key words: Countable metrizable spaces, homeomorphism, squares of spaces.

Classification: 54B10

1. Introduction. A class \mathcal{K} of topological spaces is said to have the unique square root property if for any two objects A and B in \mathcal{K} , $A \times A \approx B \times B$ implies $A \approx B$.

Several naturally organized classes of topological spaces do not have this property (see [4]). In [4], V. Trnková asked the following question: Is the unique square root property valid in the class of all countable metrizable spaces?

In this paper, we exhibit a pair of non-homeomorphic countable metrizable spaces having homeomorphic squares.

2. A classification of points of a space. Now we consider a classification of points of a countable metric space, following the case of classification of points of a compact metric 0-dimensional space (see [2]).

When we say "a space", it will mean invariably "a countable

le metric space".

For a space X , let X_0 be the set of all isolated points of X and X_1 the set of those points of X which have a neighborhood without isolated points. Let $X_{(0)} = X \setminus (X_0 \cup X_1)$. Since $X_{(0)} \subseteq \bar{X}_0$ (\bar{A} denotes the closure of the set A), the set $X_{(0)}$ is split again into two parts $X_2 = X_{(0)} \setminus \bar{X}_1$ and $X_{(0)(1)} = X_{(0)} \cap \bar{X}_1$. In words, the set $X_{(0)}$ is split into the set X_2 of those points which are not accumulation points of X_1 and the set $X_{(0)(1)}$ of those points which are accumulation points of X_1 .

Now we have the following inductive definition: Suppose that the sets X_0, X_1, \dots, X_n and $X_{(0)}, X_{(0)(1)}, \dots, X_{(0)(1)\dots(n-1)}$ have been already defined. Put

$$X_{n+1} = X_{(0)(1)\dots(n-1)} \setminus \bar{X}_n, \quad X_{(0)(1)\dots(n)} = X_{(0)(1)\dots(n-1)} \cap \bar{X}_n.$$

In this way, we have defined a sequence of sets $X_0, X_1, \dots, X_n, \dots$ which are disjoint and for each n , the set $X_0 \cup X_1 \cup \dots \cup X_n$ is open and $X_{(0)(1)\dots(n-1)}$ closed.

Let

$$X_\omega = \bigcap \{ X_{(0)(1)\dots(n)} : n \in \mathbb{N} \}.$$

The following statement is immediately derived from the given definition.

Statement 1.

- (a) $\bar{X}_n = X_n \cup (\cup \{ X_k : k = n+2, \dots, \omega \})$
 (b) If $X_n = \emptyset$, then $X_k = \emptyset$ for $k = n+2, \dots, \omega$.

Call a point $x \in X$ n-point if $x \in X_n$ for some $n = 0, 1, \dots, \omega$. The number n is called the accumulation order of x and we write $\text{ord}(x) = n$.

To the space X , for which $X_{n-2} \neq \emptyset$, $X_{n-1} = \emptyset$ and $X_n \neq \emptyset$ (and according to 1 (b), $X_k = \emptyset$ for $k > n$) the sequence

$$s(X) = (0, 1, \dots, n-2, \emptyset, n),$$

and to the space X for which $X_{n-1} \neq \emptyset$, $X_n \neq \emptyset$ and $X_k = \emptyset$ for $k > n$, the sequence

$$s(X) = (0, 1, \dots, n-1, n)$$

is attached respectively. The sequence $s(X)$ is called the accumulation sequence of the space X (we avoid here the case $X_w \neq \emptyset$).

3. Q-full spaces. Denote by Q the space of rational numbers. Every countable metric space without isolated points is homeomorphic to Q (Sierpinski's theorem, [1], p. 290).

Call a space X Q-full if for each $n > 0$, $X_n \neq \emptyset$ implies X_n has no isolated point (or $X_n \approx Q$).

Now we construct a sequence of Q-full spaces.

Let $Q_{-1} = \emptyset$ be the empty set, Q_0 a one point space and $Q_1 = Q$, where Q is the set of rationals realized geometrically as the set of all end points of removed intervals of the Cantor discontinuum C (when C is constructed in the usual way of removing the middle third intervals).

Suppose the sequence Q_0, Q_1, \dots, Q_n has already been defined (and all the spaces Q_i , $i = 0, \dots, n$ are the subspaces of $[0, 1]$).

Define Q_{n+1} to be the space Q plus a copy of the disjoint topological sum $Q_{n-2} + Q_{n-1}$ being interpolated in each of the removed intervals. Now by induction, the sequence of spaces

$$Q_0, Q_1, \dots, Q_n, \dots$$

is defined and it is easy to see that all these spaces are Q-full as well as the sums $Q_{n-1} + Q_n$, ($n \in \mathbb{N}$).

As for the accumulation sequences, we have

$$s(Q_0) = (0), \quad s(Q_1) = (\emptyset, 1), \quad s(Q_0 + Q_1) = (0, 1)$$

and for $n > 1$,

$s(Q_n) = (0, \dots, n-2, \emptyset, n)$, $s(Q_{n-1} + Q_n) = (0, \dots, n-1, n)$.

In particular, $s(Q_2 + Q_3) = (0, 1, 2, 2)$, $s(Q_5) = (0, 1, 2, 3, \emptyset, 5)$, what shows that $Q_2 + Q_3 \neq Q_5$.

We quote [5] for the following two easily proved statements.

Statement 2.

(a) A compact space cannot be Q-full.

(b) If every infinite sequence in X_0 has an accumulation point then \bar{X}_0 is compact.

Call two Q-full spaces X and Y equivalent if their accumulation sequences are finite and equal, and if $\text{card}(X_0) = \text{card}(Y_0)$.

According to the statement 2.6 in [5], which can be considered as a variation on the already mentioned Sierpinski's theorem, two equivalent spaces are homeomorphic (Sierpinski's theorem being the case $s(X) = s(Y) = (\emptyset, 1)$).

We give here a sketch of a (new) proof.

In order to simplify the proofs which follow, notice that according to the statement 1, a space X such that $s(X) = (0, \dots, n-1, n)$ has both parts X_{n-1} and X_n closed in X. Then, it easily follows that X can be decomposed into two closed and open parts X' and X'' such that $s(X') = (0, \dots, \emptyset, n)$, $s(X'') = (0, \dots, \emptyset, n-1)$ (see also 2.3 in [5]).

Notice also that a closed and open subset of a Q-full space is Q-full again.

A pointed Q-full space is a pair (X, x_0) where X is Q-full space and $x_0 \in X$ a point of highest accumulation order.

Statement 3. Let (X, x_0) and (Y, y_0) be two pointed Q-full spaces such that X and Y are equivalent and $s(X) = s(Y) = (0, \dots, \emptyset, n)$. If $X = X' \cup X''$ is a decomposition into two closed

and open subsets such that $x_0 \in X'$, then there exists a decomposition of Y into two closed and open subsets, $Y = Y' \cup Y''$ such that $y_0 \in Y'$ and X' is equivalent to Y' and X'' to Y'' .

Proof. The statement is easily seen to be true in the cases $s(X) = 0$, $s(X) = (\emptyset, 1)$. Suppose $n \geq 2$. We have two cases

a) $s(X') = (0, \dots, \emptyset, n)$, $s(X'') = (0, \dots, \emptyset, m)$

b) $s(X') = (0, \dots, \emptyset, n)$, $s(X'') = (0, \dots, m-1, m)$.

a) If $m = 0$ and $\text{card } X'' < \aleph_0$, we take $Y'' \subset Y_0$ such that $\text{card } Y'' = \text{card } X''$ and $Y' = Y \setminus Y''$.

If $m = 0$ and $\text{card } X'' = \aleph_0$, then by 2, there exists a closed and open subset $Y'' \subset Y_0$ such that $\text{card } Y'' = \aleph_0$ and $Y' = Y \setminus Y''$ has the required properties.

If $1 \leq m < n$, then since $Y_0 \cup Y_1 \cup \dots \cup Y_{m-2} \cup Y_m$ is open, take a small enough closed and open neighborhood Y'' of a point $y \in Y_m$ such that $Y'' \subset Y_1$ if $m = 1$ and $Y'' \subset Y_0 \cup Y_1 \cup \dots \cup Y_{m-2} \cup Y_m$ if $m > 1$. Let $Y' = Y \setminus Y''$.

If $m = n$, let Y' be a small enough closed and open neighborhood of y_0 such that $Y_n \setminus Y' \neq \emptyset$. Take $Y'' = Y \setminus Y'$.

b) If $m = 0$, we do the same as under a) (and it is the same case). If $s(X'') = (0, 1)$, take a closed and open neighborhood U of a point in Y_1 such that $U \subset Y_1$ and let $V \subset Y_0$, closed in Y , be equivalent to $X'' \cap X_0$. Take $Y'' = U \cup V$, $Y' = Y \setminus Y''$.

Now we have left the case $1 < m \leq n - 2$. Take U and V to be closed and open neighborhoods of a point $y_1 \in Y_m$ and $y_2 \in Y_{m-1}$ respectively such that $U \subset Y_0 \cup Y_1 \cup \dots \cup Y_m$ and $V \subset Y_0 \cup Y_1 \cup \dots \cup Y_{m-1}$. Take $Y'' = U \cup V$ and $Y' = Y \setminus Y''$. This concludes the proof.

Statement 4. If X and Y are equivalent spaces, then they are homeomorphic.

Proof. We can suppose that X and Y are subspaces of the interval $[0, 1]$. Since X and Y are countable, we have the enumera-

tions of each of them $X = \{x_1, \dots, x_1, \dots\}$, $Y = \{y_1, \dots, y_1, \dots\}$. Let x_{i_1} and y_{j_1} be the first elements of highest order (i.e. of order n) in the enumerations of X and Y respectively. Consider the pointed spaces (X, x_{i_1}) , (Y, y_{j_1}) .

Now let the term "to point a closed and open part A " of X or Y mean to form the pair (A, a) , where $a \in A$ is the point of highest order in A which stands first in the given enumeration and has not been already used in the process of pointing.

If $s(X) = s(Y) = (0, \dots, n-1, n)$, then both of these spaces can be decomposed into two parts each, so that the accumulation sequences of the parts are $(0, \dots, \emptyset, n)$ and $(0, \dots, \emptyset, n-1)$, and the pointed parts having the sequence $(0, \dots, \emptyset, n)$. Point the non-pointed parts, if any. Then, each of these parts of X , or X itself, if $s(X) = (0, \dots, \emptyset, n)$, can be decomposed into two closed and open parts which are of diameter less than $2/3$ of the diameter of X . Applying 3, we also have equivalent parts of the parts of Y . Now the decompositions of X and Y have at most 4 elements and let us point non-pointed parts. The parts of Y , having the sequence $(0, \dots, m-1, m)$ decompose into two parts having each the sequences $(0, \dots, \emptyset, m)$ or $(0, \dots, \emptyset, m-1)$, point them and correspond to each the equivalent parts of the corresponding parts of X . Point also parts of X . Now, we have at most 8 parts in each of the spaces. Finally decompose the parts of Y so that the diameters of the parts are less than $2/3$ of diameter of Y . Point non-pointed parts and do the same with the equivalent non-pointed parts of X .

In this way X and Y are decomposed into at most 16 pointed parts. If x_{i_k} points a part of X , denote such a part by $X^1(x_{i_k})$ and the corresponding part of Y with $Y^1(y_{j_k})$. The parts $X^1(x_{i_k})$ and $Y^1(y_{j_k})$ are all of diameter less than $2/3$ and they are equi-

valent pointed Q -full spaces.

Now starting with the pairs $X^1(x_{1_k}), Y^1(y_{j_k})$. We decompose them into at most 16 parts $X^2(x_{1_k}), Y^2(y_{j_k})$ having the diameters less than $(2/3)^2$.

Proceeding inductively, in the m -th step, we have the parts $X^m(x_{1_k}), Y^m(y_{j_k})$ of diameter less than $(2/3)^m$.

Define the mapping $f: X \rightarrow Y$ by $f(x_{1_k}) = y_{j_k}$. If $x_{1_s} \in X_t$, then by 1, the set $X_0 \cup \dots \cup X_t$ is open, and for a large enough m , there will exist a part X^m of X contained in $X_0 \cup \dots \cup X_t$ and disjoint from the set of those points of order t which precede x_{1_s} . So x_{1_s} , if not already used in pointing, will be used in the m -th step. The same is valid for the points of Y , so that f is a mapping defined from the whole X onto Y . It is easily seen that f is 1-1 and on both sides continuous. Hence, X and Y are homeomorphic.

Thus the term "equivalent Q -full spaces" means topologically equivalent and it was only a working term.

The statement 4 shows that the only Q -full spaces are the spaces

$$Q_n, Q_{n-1} + Q_n$$

adding to them at most countable discrete spaces and the topological sums of such a space and the space Q_1 .

4. The space $Q_2 + Q_3$ and Q_5 have homeomorphic squares.

Consider two spaces X and Y having no point of the accumulation order 4. (Such two spaces are $Q_2 + Q_3$ and Q_5 .) If $\text{ord}(x) = m, (x \in X)$ and $\text{ord}(y) = n, (y \in Y)$, we will denote the order of $(x, y) \in X \times Y$ by $m \times n$. The number $m \times n$ does not depend of the choi-

ce of spaces X and Y as it will become evident from the proofs which follow. The evident homeomorphism of the spaces $X \times Y$ and $Y \times X$ sends the point (x, y) onto (y, x) and so $m \times n = n \times m$.

Now we show that $n \times m$ dependently of n and m is given by the following table

\times	0	1	2	3	5
0	0	1	2	3	5
1	1	1	1	1	1
2	2	1	2	5	5
3	3	1	5	3	5
5	5	1	5	5	5

(a) $0 \times n = n$: Suppose $\text{ord}(x) = 0$, $\text{ord}(y) = n$. The set $\{x\} \times Y$ is mapped onto Y by a homeomorphism sending (x, y) onto y . Thus, $\text{ord}(x, y) = \text{ord}(y)$.

(b) $1 \times n = 1$: The point x has a neighborhood without isolated points and so the point (x, y) has also such a neighborhood.

(c) $2 \times 3 \geq 5$: In an arbitrary neighborhood of the point (x, y) , there exist two points (x', y') , (x'', y'') such that $\text{ord}(x') = 0$, $\text{ord}(y') = 3$, $\text{ord}(x'') = 2$, $\text{ord}(y'') = 0$. Thus, $\text{ord}(x', y') = 3$, $\text{ord}(x'', y'') = 2$ and the point (x, y) is an accumulation point of $(X \times Y)_2$ and $(X \times Y)_3$. By the statement 1 (a), it follows that $\text{ord}(x, y) \geq 5$.

(d) $2 \times 5 \geq 5$: The proof is the same as under (c).

(e) $3 \times 3 = 3$: By 1 (a), $\overline{X}_2 = X_2 \cup X_4 \cup X_5$ and the set $X_0 \cup X_1 \cup X_3$ is open. Take closed and open neighborhoods U and V of x and y respectively so that $U \subseteq X_0 \cup X_1 \cup X_3$, $V \subseteq Y_0 \cup Y_1 \cup Y_3$. Let (x', y') be in $U \times V$. If one of the numbers $\text{ord}(x')$, $\text{ord}(y')$ is less than 3, then $\text{ord}(x', y') = 0, 1$ or 3 . If $\text{ord}(x') = \text{ord}(y') = 3$, then $\text{ord}(x', y') \geq 3$, since $(x', y') \in \overline{(X \times Y)}_0$ and $(x', y') \in \overline{(X \times Y)}_1$. Thus, no point in $U \times V$ has the order 2. Thus, $\text{ord}(x', y') = 3$.

(f) $3 \times 5 \geq 5$: The proof is the same as under (c).

(g) $5 \times 5 \geq 5$: The proof as under (c).

(h) $2 \times 2 = 2$: The proof easier than (e).

Hence, the space $X \times Y$ has no point of order 4. By 1 (b), $X \times Y$ has no point of order greater than 5 and we have $2 \times 3 = 5$, $2 \times 5 = 5$, $3 \times 5 = 5$, $5 \times 5 = 5$.

It is immediately seen that the product of two Q-full spaces X and Y is a Q-full space.

Take $X = Q_2 + Q_3$, $Y = Q_5$. Then, $s(X) = (0, 1, 2, 3)$ and $s(Y) = (0, 1, 2, 3, \emptyset, 5)$ and X and Y are not homeomorphic. The spaces $X \times X$ and $Y \times Y$ are Q-full and $s(X \times X) = s(Y \times Y) = (0, 1, 2, 3, \emptyset, 5)$. By the statement 4, the spaces $X \times X$ and $Y \times Y$ are homeomorphic.

In a full analogy with the case of compact spaces (see [3]), it can be shown that there exists an infinite sequence of pairs of non-homeomorphic separable metric spaces having homeomorphic squares.

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