Commentationes Mathematicae Universitatis Carolinae

Marian Dawidowski

On the existence of bounded solutions of differential equations in Banach spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 3, 611--617

Persistent URL: http://dml.cz/dmlcz/106397

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
26,3 (1885)

ON THE EXISTENCE OF BOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

Marian DAWIDOWSKI

Abstract: In this note we shall give sufficient conditions for the existence of bounded solutions of the differential equation $y' = f(t,y)$, $y(0) = x_0$, on the half-line $t \geq 0$. Here f is a function with values in a Banach space satisfying some conditions expressed in terms of an axiomatic measure of noncompactness μ. The proof of our theorem is suggested by the paper of Stokes [7] concerning finite dimensional vector differential equations.

Key words: Ordinary differential equations in Banach spaces, fixed point, measure of noncompactness.

Classification: 34G20, 47H09

Introduction: Let $(E, \|\cdot\|)$ be a Banach space. The closure of a subset A of E, its convex hull and its closed convex hull will be denoted, respectively, by $\bar{A}$, conv A and $\overline{\text{conv}}$ A. If A and B are subsets of E and t, s are real numbers, the $t\cdot A + s\cdot B$ is the set of all $t\cdot x + s\cdot y$, where $x \in A$ and $y \in B$. Further let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ - the family of all relatively compact and nonempty subsets of E.

A function $\mu: \mathfrak{M}_E \longrightarrow [0,+\infty)$ is said to be a measure of noncompactness if it satisfies the following conditions:

1° the family $\mathcal{P} = \{A \in \mathfrak{M}_E: \mu(A) = 0\}$ is nonempty and $\mathcal{P} \subset \mathfrak{N}_E$,

2° $\mu(\{x\}) = 0$ for all $x \in E$,

3° $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$,

4° $\mu(\overline{A}) = \mu(A)$,

5° $\mu(\text{conv } A) = \mu(A)$,

6° $\mu(t\cdot A) = |t| \cdot \mu(A)$ for every $t \in \mathbb{R}$,

7° $\mu(A + B) \le \mu(A) + \mu(B)$,

8° $\mu(A \cup B) \le \max(\mu(A), \mu(B))$.

We put

$\|A\| = \sup \{\|x\|: x \in A\}$, $K(0,1) = \{x \in E: \|x\| \le 1\}$.

The following property of the function μ is true:

Lemma 1. If $A \in \mathfrak{M}_E$ then $\mu(A) \le \|A\| \cdot \mu(K(0,1))$.

Now let $J = [0,+\infty)$ and denote by $C(J)$ the set of all continuous functions from J to E. The set $C(J)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of J.

Let us put $X(t) = \{x(t): x \in X\}$, $X_t = \cup \{X(s): 0 \le s \le t\}$ for $t \in J$ and $X \subset C(J)$. We have

Lemma 2. If $X \subset C(J)$ is bounded and almost equicontinuous then $\mu(X_t) = \sup \{\mu(X(s)): 0 \le s \le t\}$ for $t \in J$.

For properties of μ see [1],[2],[3],[4].

The Ascoli theorem we state as follows: $X \subset C(J)$ is conditionally compact if and only if X is almost equicontinuous and $X(t)$ is compact for each $t \in J$.

We shall use the following fixed-point theorem of Sadovskii type (see [3],[5],[6]):

Let $\mathfrak{X}$ be a nonempty closed convex subset of $C(J)$. Let $\Phi: 2^{\mathfrak{X}} \to [0,+\infty)$ be a function with the following properties:

(1) $\Phi(X) = 0 \Rightarrow \overline{X}$ is compact,

(2) $\Phi(\overline{\text{conv}}\, X) = \Phi(X)$,

(3) $\Phi(X \cup \{x\}) = \Phi(X)$

for every subset X of $\mathfrak{X}$ and for each $x \in \mathfrak{X}$.

Suppose that T is a continuous mapping of $\mathfrak{X}$ into itself and $\Phi(T[X]) < \Phi(X)$ for $\Phi(X) > 0$. Then T has a fixed point in $\mathfrak{X}$.

Main result.

Theorem. Assume that $f\colon J \times E \to E$ is a function satisfying the following conditions:

1° for each fixed $x \in E$ the mapping $t \mapsto f(t,x)$ is measurable;

2° for each fixed $t \in J$ the mapping $x \mapsto f(f,x)$ is continuous;

3° $\| f(t,x) \| \leq G(t, \|x\|)$ for $(t,x) \in J \times E$, where the function G is nondecreasing in the second variable such that $t \mapsto G(t,u)$ is locally bounded for any fixed $u \in J$ and $t \mapsto G(t,y(t))$ is measurable for every continuous bounded function $y\colon J \to J$;

4° the scalar inequality

$$g(t) \geq \| x_0 \| + \int_0^t G(s, g(s))\,ds$$

has a bounded solution g existing on J;

(let us put $r_0 = \sup\{g(t)\colon t \in J\}$ and $Z_0 = \{x \in E\colon \|x\| \leq r_0\}$)

5° there exist functions m, p of J into itself such that

(i) m is measurable and integrable on compact subsets of J with

$$M = \sup \left\{ \int_0^t m(s)\,ds\colon t \in J \right\} < \infty,$$

(ii) p is nondecreasing such that $M \cdot p(t) < t$ for $t > 0$,

(iii) for any $t > 0$, $\varepsilon > 0$, $X \subset Z_0$ there exists a closed subset $Q \subset [0,t]$ such that $\operatorname{mes}([0,t] \setminus Q) < \varepsilon$ and $\mu(f[I \times X]) \leq \sup\{m(s)\colon s \in I\} \cdot p(\mu(X))$ for each closed subset I of Q.

Then the differential equation

$$y' = f(t,y)$$

with the initial condition $y(0) = x_0$ has at least one solution y defined on J and $\|y(t)\| \leq g(t)$ for $t \in J$.

Proof: Denote by $\mathfrak{X}$ the set of all $x \in C(J)$ such that $\|x(t)\| \leq g(t)$ on J and

$$\| x(t_1) - x(t_2) \| \leq \left| \int_{t_1}^{t_2} G(s, r_0) ds \right| \text{ for } t_1, t_2 \in J.$$

The set $\mathfrak{X}$ is nonempty closed convex bounded and almost equicontinuous subset of $C(J)$.

Let us put

$$\Phi(X) = \sup \{ \mu(X(t)) : t \in J \} \text{ for a subset } X \subset \mathfrak{X}.$$

Obviously $\Phi(X) < \infty$, $\Phi(X_1) \leq \Phi(X_2)$ for $X_1 \subset X_2$ and $\Phi(X \cup \{x\}) = \Phi(X)$ for $x \in \mathfrak{X}$.

Since

$$(\overline{\mathrm{conv}}\, X)(t) = (\overline{\mathrm{conv}}\, X)(t) \subset \overline{(\mathrm{conv}\, X)(t)} \subset \overline{\mathrm{conv}\,(X(t))}$$

so $\mu((\overline{\mathrm{conv}}\, X)(t)) \leq \mu(\overline{\mathrm{conv}(X(t))}) = \mu(X(t))$.

The inverse inequality immediately follows from the inclusion $X(t) \subset (\overline{\mathrm{conv}}\, X)(t)$. Hence $\Phi(\overline{\mathrm{conv}}\, X) = \Phi(X)$. If $\Phi(X) = 0$ then $\overline{X(t)}$ is compact for every $t \in J$; therefore Ascoli's theorem proves that $\overline{X}$ is compact in $C(J)$.

To apply our fixed-point theorem we define the mapping T as follows:

for $y \in \mathfrak{X}$, $(T(y))(t) = x_0 + \int_0^t f(s, y(s)) ds$.

It is easy to see that T is continuous and $T[\mathfrak{X}] \subset \mathfrak{X}$.

Let X be a subset of $\mathfrak{X}$ such that $\Phi(X) > 0$. To prove the theorem it remains to be shown that $\Phi(T[X]) < \Phi(X)$. To this end, fix t in J. Let $\varepsilon \in (0,1)$ and $\delta = \delta(\varepsilon) > 0$ be a number such that $\int_A G(s, r_0) ds < \infty$ for each measurable $A \subset [0,t]$ with mes $(A) < \delta$. By the Luzin theorem there exists a closed subset B_1 of $[0,t]$ with mes $([0,t] \setminus B_1) < \delta/2$ such that the function m is continuous on

B_1. Furthermore, by assumption 5°(iii) there exists a closed subset B_2 of $[0,t]$ such that mes $([0,t]\setminus B_2) < \delta/2$ and $\mu(f[I\times X_t]) \le \sup\{m(s)\colon s\in I\}\cdot p(\mu(X_t))$ for each closed subset I of B_2.

Let us put $B = B_1\cap B_2$, $A = [0,t]\setminus B$. Hence mes $(A) < \delta$. Since m is uniformly continuous on B, for any given $\varepsilon' > 0$ there exists $\eta > 0$ such that $t', t'' \in B$ and $|t' - t''| < \eta$ implies $|m(t') - m(t'')| < \varepsilon'$. Let $t_0 = 0 < t_1 < \ldots < t_n = t$ be the partition of the interval $[0,t]$ with $\max\{|t_{j-1} - t_j|\colon 1\le j\le n\} < \eta$. Moreover, let $I_j = [t_{j-1}, t_j]\cap B$ and s_j be a point in I_j such that $m(s_j) = \sup\{m(s)\colon s\in I_j\}$.

Putting

$$\int_I f(s,X(s))ds = \{\int_I f(s,x(s))ds\colon x\in X\}$$

we get

$$\|\int_A f(s,X(s))ds\| \le \int_A G(s,r_0)ds < \varepsilon < 1.$$

By the mean-value theorem, for $x\in X$ we have

$$\int_B f(s,c(s))ds = \sum_{j=1}^{m} \int_{I_j} f(s,x(s))ds \in$$
$$\in \sum_{j=1}^{m} \text{mes } (I_j)\ \overline{\text{conv}}\ (\{f(s,x(s))\colon s\in I_j\}) \subset$$
$$\subset \sum_{j=1}^{m} \text{mes } (I_j)\ \overline{\text{conv}}\ (f[I_j\times X_t]),$$

hence $\int_B f(s,X(s))ds \subset \sum_{j=1}^{m} \text{mes } (I_j)\ \overline{\text{conv}}\ (f[I_j\times X_t])$. Thus

$$\mu(T[X](t)) \le \mu(\{x_0\} + \int_A f(s,X(s))ds + \int_B f(s,X(s))ds) \le$$
$$\le \mu(\{x_0\}) + \|\int_A f(s,X(s))ds\| \cdot \mu(K(0,1)) +$$
$$+ \sum_{j=1}^{m} \text{mes } (1_j) \cdot \mu(f[I_j\times X_t]) \le \varepsilon\cdot\mu(K(0,1)) +$$
$$+ \sum_{j=1}^{m} \text{mes } (I_j) m(s_j) p(\mu(X_t)) \le \varepsilon\cdot\mu(K(0,1)) +$$
$$+ p(\mu(X_t))\cdot(\sum_{j=1}^{m} \int_{I_j} |m(s_j) - m(s)|ds + \sum_{j=1}^{m} \int_{I_j} m(s)ds) \le$$
$$\le \varepsilon\cdot\mu(K(0,1)) + p(\mu(X_t))\cdot(\varepsilon'\cdot t + \int_0^t m(s)ds)$$

and therefore

$$\mu(T[X](t)) \leq \varepsilon \cdot \mu(K(0,1)) + M \cdot p(\mu(X_t)).$$

Since with respect to Lemma 2

$$\mu(X_t) = \sup\{\mu(X(s)):\ 0 \leq s \leq t\} \leq \Phi(X)$$

we obtain

$$\mu(T[X](t)) \leq \varepsilon \cdot \mu(K(0,1)) + M \cdot p(\Phi(X));$$

as $\varepsilon > 0$ is arbitrary, this implies

$$\mu(T[X](t)) \leq M \cdot p(\Phi(X)).$$

Hence $\Phi(T[X]) \leq M \cdot p(\Phi(X)) < \Phi(X)$, and consequently T has a fixed point in $\mathfrak{X}$. The proof is complete.

References

[1] J. BANAŚ: On measures of noncompactness in Banach spaces, Comment.Math. Univ. Carolinae 21(1980), 131-143.

[2] J. BANAŚ, K. GOEBEL: Measures of noncompactness in Banach spaces, Lect. Notes Pure Applied Mathematics, Marcel Dekker, vol. 60, New York and Basel, 1980.

[3] J. DANEŠ: On densifying and related mappings and their application in nonlinear functional analysis, Theory of nonlinear operators, Akademie-Verlag, Berlin 1974, pp. 15-56.

[4] I. KUBIACZYK: On the existence of solutions of differential equation in Banach space (to appear).

[5] B. RZEPECKI: Remarks on Schauder's fixed point principle and its applications, Bull. Acad. Polon. Sci. Sér. Math. 27(1979), 473-480.

[6] B.N. SADOVSKIĬ: Predel'no kompaktnye i uplotnjajuščije operatory, Uspehi Mat. Nauk XVII 1(163)(1972), 81-146 (in Russian).

[7] A. STOKES: The applications of a fixed-point theorem to a variety of nonlinear stability problems, Proc. Nat. Acad. Sci. USA 45(1959), 231-235.

Institute of Mathematics
A. Mickiewicz University
Ul. Matejki 48/49
60-769 Poznań
Poland

(Oblatum 18.12. 1984)