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A constructive proof of the Tychonoff's theorem for locales

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Abstract: A choice- and replacement-free proof of the Tychonoff's theorem is given for compact locales.

Key words: Locales, compact locales, Tychonoff's theorem.

Classification: 54D30, 54H99

The Tychonoff's theorem ([12]) stating that a product of compact spaces is compact is well known to be equivalent to the axiom of choice (see [10]). A surprising result was obtained by P.T. Johnstone in [8]: if we consider compact locales (i.e., spaces represented as lattices of "open sets" - with points disregarded and, indeed, often not present in any form), the analogon of the Tychonoff's theorem can be proved without the axiom of choice. This is particularly interesting in connection with the fact that compact locales are always spatial, i.e. open-sets lattices of classical topological spaces ([12]; thus, the use of AC is localized in the formation of points, not in the preservation of the compactness property).

The proof in [8] contains a non-constructive element, namely the axiom of replacement. P.T. Johnstone formulated the problem whether one can get rid of this, too (for the special case of the locally compact locales he presented a positive answer himself). In this article, this problem is solved in...
the affirmative in full generality. The procedure is based on a new description of the product of locales, considerably more constructive as compared with the usually used ones ([5],[8]).

1. Locales. The basic theory of locales has been developed by Bénabou [1], Dowker and Strauss [3, 4, 5], Isbell [6] and Simmons [11]. There are considerable differences in the terminology; we follow that of Johnstone [8]. A frame is a complete lattice $A$ in which the infinite distribution law

$$a \land (\bigvee S) = \bigvee \{ a \land s \mid s \in S \}$$

holds for all $a \in A$, $S \subseteq A$. We shall denote the maximal resp. minimal element of $A$ by $1$ resp. $0$. A frame homomorphism $A \to B$ is a map preserving finite meets and arbitrary joins (i.e., in particular, the elements $0, 1$). Thus, we have a category Frm of frames. If $X$ is a topological space, the lattice $\Omega(X)$ of its open sets is a frame. If $f: X \to Y$ is a continuous map, then $f^{-1}: \Omega(Y) \to \Omega(X)$ is a frame homomorphism. Thus $\Omega$ is a contravariant functor from the category Top of topological spaces to Frm.

Following Isbell [6] and Johnstone [8] we shall write Loc for the opposite category Frm$^{\text{op}}$, and call its objects locales. This dual terminology enables us to make $\Omega: \text{Top} \to \text{Loc}$ a co-variant functor and, in consequence, to generalize familiar concepts from topology to Loc (see [7],[8]).

2. Products of locales. Products in the category Loc (sums in Frm) were defined by Dowker-Strauss [5] and Johnstone [8]. Their description is elegant, but rather non-constructive. It does not give any explicit formula for the join operation in the sum $\bigvee_{j} X_{j}$ of frames $X_{j}$. Johnstone [7] suggests to
construct the sum of $I_j$ ($j \in J$) as a free frame over the cartesian product $\prod_{j \in J} I_j$ of the sets $I_j$, factorized through a congruence generated by certain relations. (In the case of an infinite $J$, it is of an advantage to exclude from $\prod_{j \in J} I_j$ those $(a_j)_{j \in J}$ in which we have $a_j < 1$ for infinitely many $j$.) This shows an analogy between frames and commutative rings (see [9]). However, frames, being $\infty$-ary algebras, turn out to be in this respect much more complex. In fact, the congruence generated by the obvious relations is rather obscure.

In this section we give a quite explicit description of the congruence generated by the relations [7], which enables us to describe the structure of $\prod_{j \in J} I_j$, explicit formulas for finite meets and arbitrary joins included.

Let $J$ be a set. We call a $J$-connector a system $(M, R_j^+, R_j^-)$ ($j \in J$), $M_1, M_2 \subseteq M$, $R_j^+ \subseteq 2^M \times M$, $R_j^- \subseteq M \times 2^M$ for $j \in J$ such that the following condition holds:

Let $K \subseteq M$. Whenever

\begin{align*}
(M_1 \subseteq K) \land [(N \land R_j^+ x) \land (N \land K) \Rightarrow (x \land K) \land [(x \land R_j^- N) \land (x \land K) \Rightarrow N \land K] \\
\text{(C)} \quad \text{or} \\
(M_2 \subseteq K) \land [(N \land R_j^+ x) \land (x \land K) \Rightarrow N \land K] \land [(x \land R_j^- N) \land (N \land K) \Rightarrow x \land K]
\end{align*}

holds, it is $K = M$.

Now let $X_j$ ($j \in J$) be a system of frames. Denote by $B$ the cartesian product $\prod_{j \in J} X_j$. There is a natural ordering "$\preceq$" of $B$, making $\prod_{j \in J} X_j$ a Prn-product of $X_j$ (see [5]). Let $B' \subseteq B$ be the subset of all $x = \prod_{j \in J} a_j x_j \in B$ such that we have $a_j < 1$ for at most finitely many $j \in J$. It is easy to see that $B'$ is a sublattice of $B$, preserving finite meets and non-empty joins, but it is not a locale: There is no minimal element in $B'$. Denote by $Z$
the lattice of all subsets of $\mathcal{B}$ ordered by inclusion.

We call $m^*G_1^GZ$ strongly equivalent $(m, j \sim \eta)$, if there exists an $m \in Z$ and a $J$-connector $(m, R_j^1, R_j^2, m^1, m^2)$ (in the sequel called simply the connector) such that it holds

$$(x R_j^1 m^1 \lor m^2 R_j^2 x) \Rightarrow (a_j^1 = \bigvee_{m^1} a_j^2 \land (a_j^k = a_j^\eta \text{ for } k \neq \eta, y \in m \land (m^1 \neq \emptyset)).$$

(3)

We will call a kernel of $m \in Z$ the set

$$s(m) = \{x \in m \mid (\forall j \in J) a_j^1 > 0\}.$$

We set $u \sim v := df s(u) \sim s(v)$. The element $u$ is called standard, if $u = s(u)$.

2.1. Observation: "\(\sim\)" is an equivalence relation, containing "\(\sim_{\#}\)".

Proof: It suffices to show that $u \sim s v \Rightarrow s(u) \sim s(v)$.

Let $(m, R_j^1, R_j^2, u, v)$ be a connector. Denoting by $\overline{R}_j^1, \overline{R}_j^2$ the restrictions of $R_j^1 (x R_j^1 m^1 \lor m^2 R_j^2 x)$ to $s(m) \times Z(m), 2s(m) \times s(m)$, respectively, we obtain a connector $(s(m), \overline{R}_j^1, \overline{R}_j^2, s(u), s(v))$.

Denote by $[m]$ the class of $m \in Z$ in $(Z/\sim)$.

2.2. Further observations: 1. Assume $x, y, z \in Z, x \subseteq y, x \sim z$.

Then there exists a $t \in Z$ such that $z \subseteq t$, $y \sim t$. Thus, we can define a canonical ordering on $(Z/\sim)$ by the formula $[x] \leq [y] := df (\exists z \in Z) (z \sim y \land x \subseteq z)$.

Proof: Let $(m, R_j^1, R_j^2, s(z), s(z))$ be a connector. Putting $t = (y \setminus x) \cup z$, we obtain an obvious connector $(m \cup s(t), R_j^1, R_j^2, s(y), s(t))$. \(\square\)

2. If $u \subseteq v \subseteq w$ and $u \sim w$. Hence, "\(\leq\)" is a partial ordering.

Proof: It suffices to show that $v \sim w$. But if $(m, R_j^1, R_j^2)$, - 622 -
s(u), s(w)) is a connector, then (m, R^+_j, R^-_j, s(v), s(w)) is a connector, as well. □

3. Let u, v ∈ Z, u ≤ v. Then (∀ y ∈ v)(∃ x ∈ u)(x ≤ y) ⇒ u ∼ v.

Proof: For u ∈ Z put d(u) = {x ∈ B' | (∃ y ∈ u) x ≤ y}. Since evidently (∀ y ∈ v)(∃ x ∈ u)(x ≤ y) & u ≤ v ⇒ d(u) = d(v), it suffices to show that u ∼ d(u) for u ∈ Z. Let R^+_j, R^-_j be maximal relations on 2^{d(u)} × 2^{d(u)}, d(u) × 2^{d(u)}, satisfying (3). (The condition (3) is obviously preserved by the union of relations.) From the fact that for x ∈ B' there are only finitely many j with a_j ≤ 1, we easily obtain that (d(u), R^+_j, R^-_j, u, d(u)) is a connector. □

4. For any u ∈ Z we have [\bigvee_{i \in I} u_i] = \bigvee_{u \in I} [u_i].

Proof: The union t(α) of all elements of a given class α ∈ (Z/∼) belongs to α, since a union of connectors (in the obvious meaning) is a connector. Moreover, the mapping t: (Z/∼) → Z preserves ordering and for arbitrary z ∈ Z, α ∈ (Z/∼) it holds z ≤ t(α) = [z] ∼ α. Thus, " [ ] " is a left adjoint to t so that it preserves joins. □

5. Denote by ∧_{B'} the meet operation in B'. For u, v ∈ Z let u ∧ v = {x ∧ y | x ∈ u; y ∈ v}. Then [u ∧ v] depends only on [u], [v].

Proof: Assume that (m^{(1)}, R^+_j^{(1)}, R^-_j^{(1)}, u^{(1)}, v^{(1)}) are connectors, i = 1, 2. Put R^+_j = {((x ∧ y, m ∧ (y)) ∈ B' × B' | (x R^+_j^{(1)} m ∧ y ∈ m^{(2)}) or (x R^+_j^{(2)} m ∧ y ∈ m^{(1)})} . R^-_j = {((m ∧ (y), x ∧ y) ∈ B' × B' | (m R^-_j^{(1)} m ∧ y ∈ m^{(2)}) or (m R^-_j^{(2)} m ∧ y ∈ m^{(1)})}. It is easy to see that (m^{(1)} ∧ m^{(2)}, R^+_j, R^-_j, u^{(1)} ∧ u^{(2)}, v^{(1)} ∧ v^{(2)}) is a connector. □

6. The operation " ∧ " in (Z/∼) defined by [u] ∧ [v] =
= [u \wedge v] is the ordinary meet (= infimum in \leq) in (Z/\sim).

**Proof:** By 2, 4, \((Z/\sim)\) is a complete lattice. Denote by "^w (Z/\sim)" the true meet in \((Z/\sim)\). By 3, we have

\[(\forall x \in u)(\exists y \in v)(x \leq y) \Rightarrow [u] \leq [v],\]

and hence trivially \([u] \wedge (Z/\sim) [v] \leq [u] \wedge [v]\). Moreover, \([u] \wedge (Z/\sim) [v] \leq [u], [v]\), by definition. Thus, by 7, there exist \(s \sim u, t \sim v\) such that for some representative \(uv\) of the class \([u] \wedge (Z/\sim) [v]\) it holds \(uv \leq s, uv \leq t\). By 5, (+), we have now \([u] \wedge (Z/\sim) [v] \leq [s \wedge t] = [u] \wedge [v]\).

7. Given a system \(f_j : X_j \rightarrow C\) of join-preserving mappings, there exists a unique join-preserving mapping \(f : (Z/\sim) \rightarrow C\) such that it holds that

\[(4) \quad f([x]_1) = \bigwedge_j f_j (a^j_x) \text{ for any } x \in B'.\]

**Proof:** By 4, the mapping \(f\) is uniquely determined by the formula \(f([m]) = \bigvee_{x \in m} f([\{x\}])\), and it obviously preserves joins. Our only task is to show that \(f\) is correctly defined. Let \((m, R^+_j, R^-_j, u, v)\) be a connector. We will show that, by our definition, \(f([u]) = f([v])\). (This will be enough, since the definition obviously gives \(f([u]) = f([s(u)])\).) In fact, since the set \(K = \{ x \in M \mid f([x]) \notin f([m]) \}\) trivially satisfies the condition (1), it is \(K = m\). Thus, \(f([v]) \leq f([m]) \leq f([u])\). Analogously, \(f([u]) \leq f([v])\). □

2.3. **Theorem:** The set \((Z/\sim)\) ordered by ";=;" is a frame with joins and meets given by the formulas

\[(5) \quad i \wedge [u] = [i \wedge u]\]
\[
[u] \wedge [v] = \{ x \wedge y \mid x \in u, y \in v \}.
\]

If we define \(\tau_j : X_j \rightarrow (Z/\sim)\) by \(\tau_j (a) = \{ \tau_j (a) \}\), where
\( \zeta_j(a) \in B \) and \( a^j_j(a) = a, a^k_j(a) = 1 \) for \( k \neq j \), then \( \zeta_j \) are frame homomorphisms and \((\mathcal{Z}/\sim)\) is the sum of \( X_j \) with injections \( \zeta_j \).

**Proof:** By 2.2.2, 2.2.4, 2.2.6, \((\mathcal{Z}/\sim)\) is a complete lattice with joins and meets given by (5). However, (5) trivially implies the distributive law so that \((\mathcal{Z}/\sim)\) is a frame. The mappings \( \zeta_j \) are frame homomorphisms by (5). (Note that namely the behaviour of the zero element forces us to set \( u \sim v \equiv s(u) \sim s(v) \).)

Given homomorphisms \( f_j : X_j \rightarrow C \), there exists (by 2.2.7) a unique join-preserving mapping \( f : (\mathcal{Z}/\sim) \rightarrow C \) satisfying (4). This mapping obviously preserves finite meets. □

2.4. Observation: For arbitrary standard \( x, y \in B \) we have \([\{x\}] \leq [\{y\}] \equiv x \sim y\).

**Proof:** Consider the mapping \( \zeta_j : X_j \rightarrow B' \) defined by Theorem 2.3. Obviously \( \zeta_j \) preserve joins, and thus, by 2.2.7, there exists a unique join-preserving \( \zeta : \bigvee_j X_j \rightarrow B' \) satisfying (4).

Since \( B \) is the product of \( X_j \) and \( B' \) is a sublattice of \( B \), we have a canonical join- and finite meet-preserving map \( \iota : B' \rightarrow \bigvee_j X_j \) induced by \( \zeta_j : X_j \rightarrow \bigvee_j X_j \). By (4), the diagram

\[
\begin{array}{ccc}
B' & \xrightarrow{\text{Id}} & B' \\
\downarrow{\iota} & & \downarrow{\iota} \\
\bigvee_j X_j & \xrightarrow{\zeta} & \bigvee_j X_j
\end{array}
\]

commutes. Thus, \( \iota \) is injective and hence \( \{x\} \sim \{y\} \equiv x \sim y \) (for standard \( x, y \)). Now \([\{x\}] \leq [\{y\}] \equiv [\{x\}] \wedge [\{y\}] = [\{x\}] \equiv [\{x \wedge y\}] = [\{x\}] \equiv x \wedge y = x \sim x \sim y. □

2.5. Remark: This result is proved in [5] and it can be reformulated to say that \( \zeta_j \) preserve arbitrary (even infinite) meets. This property could be called the openness of \( \zeta_j \). This
is motivated by the following

**Fact:** Let \( X, Y \) be topological \( T_1 \)-spaces. Then a continuous \( f: X \to Y \) is open iff \( f^{-1}: \Omega(Y) \to \Omega(X) \) preserves arbitrary (even infinite) meets.

**Proof:** If \( f: X \to Y \) is open, then the image mapping \( f_1: \Omega(X) \to \Omega(Y) \) is evidently left adjoint to \( f^{-1} \). Thus, \( f^{-1} \) preserves meets. On the other hand, if \( f^{-1} \) preserves meets, it has a left adjoint \( f_\ast \). For \( U \in \Omega(X), V \in \Omega(Y) \) we have
\[
f_\ast(U) = f^{-1}(\bigwedge_{U \subseteq V} V) = f_1(U) \quad \text{(for, since } Y \text{ is } T_1, \text{ we have } x \in f^{-1}(V) \in V = f_1(U)).
\]
On the other hand, \( f^{-1}f_\ast(U) = f^{-1}(\bigwedge_{U \subseteq V} V) = f_1(V) \in U \), and hence \( f_\ast(U) \subseteq f_1(U) \). Thus, \( f_\ast = f_1 \). \( \Box \)

3. **The Tychonoff’s theorem.** A frame (locale) is said to be **compact**, if for any \( S \subseteq A \) with \( \bigvee S = 1 \) there exists a finite \( F \subseteq S \) with \( \bigvee F = 1 \). In this section we give a choice- and replacement-free proof of the theorem that the product of compact locales is compact.

Let \( A \) be a frame. A set \( S \subseteq A \) is called a covering of \( A \), if it holds \( \bigvee S = 1 \). For coverings \( s, t \) of a frame \( A \) we set \( s \leq t \), if it holds \( (\forall x \in s)(\exists y \in t) x \leq y \). (This is the ordinary concept of refinement.) Let now \( s \) be a covering of \( A \) and let \( t \subseteq A \) such that \( \bigvee t \geq a \). We will use the notation \( s \land_a t = \{ x \in s \mid x \leq a \} \cup \{ x \land y \mid x \leq s \land y \leq t \} \). Obviously, \( s \land_a t \) is a covering of \( A \) and \( s \land_a t \leq s \). Analogously, \( (\forall x \in s \land_a t)((x \leq a) \Rightarrow (\exists y \leq t) (x \leq y)) \).

Now let \( I_j \) (\( j \in J \)) be a system of frames. Consider a system \( s_j \) of coverings such that \( s_j = 1 \) except for, at most, finitely many \( j \). Then the system
is a covering of $\bigvee_j X_j$.

In the last section we remarked that $\bigvee_j X_j \rightarrow \bigvee_j X_j$ preserve arbitrary meets. Thus, they have left adjoints $p_j: \bigvee_j X_j \rightarrow X_j$ (which, of course, are not frame homomorphisms).

We can easily check that

$$p_j(\{u\}) = \bigvee \{x \in u\}.$$  

3.1. Lemma: Let $\{u, v\}$ be a covering of $\bigvee_{j \in \{0, 1\}} A_j$. Then

$$p_0(u) = 1 \text{ or } p_1(v) = 1.$$  

Proof: There should exist a connector $(m, x_1, x_2, j \in \{0, 1\}, u \cup v, \emptyset)$ for some standard representatives $u, v$ of the classes $u, v$. Consider a system $x_i \in B, i \in I$ such that $x_i$ differ at most at one coordinate. Then the statement $(\forall i \in I)[(a_1^0 \leq p_0(u)) \text{ or } (a_1^1 \leq p_1(v))]$ implies the statement $(a_1^0 \leq p_0(u)) \text{ or } (a_1^1 \leq p_1(v))$. Thus, by (3), the set $K = \{x \in m | (a_1^0 \leq p_0(u)) \text{ or } (a_1^1 \leq p_1(v))\}$ satisfies (1), and hence $K = m$. In particular, $1 \leq p_0(u)$ or $1 \leq p_1(v)$. \qed

3.2. Observation: Any element of a finite lattice is a join of join-irreducible elements.

Proof: An obvious induction. \qed

3.3. Lemma: Consider a finite covering $t = \{\{x_i\} \mid i \in n, x_i \in B\}$ of the frame $\bigvee_j X_j$. Then there exist finite coverings $a_j$ of the frames $X_j$ such that $\bigvee_{j \in t} a_j \leq t$.

Proof: will be done for $J = \{0, 1\}$. This, by induction, obviously implies the case of $J$ finite; the case of $J$ infinite
is executed by the finiteness of \( t \). Let, hence, \( J = \{0,1\} \).

Let \( A_j (j = 0,1) \) be sets of all possible elements of \( X_j \) obtained from \( a_j^y \) (\( i \leq n \)) by join and meet-operations in \( X_j \). Obviously \( A_j \) are finite lattices. Write \( s_j \) for the set of all join-irreducible elements in \( A_j \). By 3.2, \( s_j \) is a covering of \( X_j \).

We will show that \( s_0 \times s_1 \leq t \). Suppose the contrary. Then there exists a \( y \in B' \) such that \( a_j^y \in s_j \) for \( j = 0,1 \) and \( x_1 \neq y \) for any \( i \leq n \). From the join-irreducibility of \( a_j^y \) it follows that

\[
p_j([x_1 | a_j^y + a_j^y]) \geq s_j^y \text{ for } j = 0,1.
\]

By 2.4 and by the properties of \( y \), however,

\[
\bigvee_{j \in \{0,1\}} [x_1 | a_j^y + a_j^y] = 1,
\]

contradicting 3.1. \( \square \)

3.4. Lemma: Consider compact frames \( X_j (j \in J) \). Let 

\((m, R_j^+, R_j^-) (j \in J), k, \{1\}\) be a connector. Then for any finite \( m' \subseteq m \) such that \( m' \sim s 1 \) and for any \( x \in m' \) there exists a finite \( m'' \subseteq (m' \{x\}) \cup k \) such that \( m'' \sim s 1 \).

**Proof:** Let \( \bar{k} \) be the set of all \( x \in m \), satisfying the statement of Lemma 3.4. We will show that \( k \) satisfies the condition (1), and hence \( \bar{k} = m \).

The inclusion \( k \subseteq \bar{k} \) is obvious.

\( \alpha \) Let \( y R_j^+ u \land x \in u \). Then, of course, \( x \sim y \) so that if \( 1 \sim s m' \exists x \), it is \( 1 \sim s (m' \{x\}) \cup y \). Thus, \( x \in \bar{k} \).

\( \beta \) Let \( u R_j^\uparrow x \land u \in \bar{k} \). Assume \( 1 \sim s m' \exists x \). Put \( M' = \{ \{y\} \mid y \in \in m' \} \). Then \( M' \) is a covering of \( \bigvee_j X_j \). By 3.3, there exists a covering \( \bigvee_j s_j \preceq M' \). We take the covering \( s_{m'} \bigotimes \bigvee_j a_j^y \mid y \in u \}

of the frame \( X_{m'} \). By compactness, it possesses a finite subcovering \( \bar{s}_{m'} \). Putting \( \bar{s}_j = s_j \) for \( j \neq \infty \), we obviously obtain

\( \bigvee_j \bar{s}_j \preceq (M' \{x\}) \cup \{x \} \} \mid y \in u \}. 

The left hand set is

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finite. Thus, there exists a finite subset \( t \subseteq u \) with \((m' \setminus x) \cup u \setminus 1\). From \( t \subseteq \bar{k} \) we easily obtain \( x \in \bar{k} \) (by induction on \( \text{card } t \)). □

3.5. Theorem: In the Zermelo set theory (without the axioms of choice and replacement) Tychonoff’s theorem holds for locales; i.e., the product of compact locales is compact.

Proof: Let \( I_j \) \((j \in J)\) be a system of compact frames and let \( S \) be a covering of \( \bigvee_{j \in J} I_j \). Put \( k = s(\bigcup_{x \in S} t(x)) \), where \( s \) is the kernel and \( t \) is defined in 2.2.4. It will be \( k \sim 1 \).

By Lemma 3.4 (with \( m' = \{1, x = 1\} \)), there exists a finite subset \( k' \subseteq k \) with \( k' \sim 1 \). Since \( k' \) is finite, however, there exists a finite \( F \subseteq S \) such that \((\forall x \in k')(\exists \alpha \in F)(x \in s(t(\alpha)))\).

Thus, of course, \( \bigvee F = 1 \). □

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References

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