Charles W. Swartz
The Farkas lemma of Glover

Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 4, 651--654

Persistent URL: http://dml.cz/dmlcz/106403

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1985

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
THE FARKAS LEMMA OF GLOVER
Charles SWARTZ

Abstract: We use standard functional analysis techniques to establish a result of Glover which he employs to obtain a nonlinear version of the classical Farkas Lemma.

Key words: Farkas Lemma, convex functions, subgradients, Krein-Smulian Theorem.

Classification: 90C25

In this brief note we present a proof of a theorem which has been used in optimization to establish a nonlinear version of the Farkas Lemma ([1], Lemma 3) and to establish Kuhn-Tucker Theorems ([3], 2.1, [4], 2.3, 2.4). The proof given by Glover in [1] uses machinery from set-valued mappings; we present a proof below which only employs standard topics from functional analysis, namely, the Krein-Smulian Theorem.

Let $X$ and $Y$ be locally convex spaces with $S$ a closed convex cone in $Y$. Let $g: X \to Y$ be positive homogeneous, $S$-convex and such that $s'g = s'g$ is lower semi-continuous for each $s \in S^*$, where $S^* = \{ s' \in Y^* : (s',s) \geq 0 \forall s \in S \}$ is the dual cone of $S$. As usual we write $\partial f(0)$ for the subgradient of a convex function $f: X \to \mathbb{R}$ at 0 ([7]). Glover shows that if $A = \bigcup_{s' \in S^*} \partial (s'g)(0)$, then $- (g^{-1}(-S))^* = \overline{A}$, where the closure is in the weak* topology of $Y^*$ ([1] Lemma 1), and then uses this result to establish a
general nonlinear Farkas Lemma ([1] Theorem 1). Glover's Farkas Lemma contains the linear Farkas Lemmas of Zalinescu ([6]) and Schirotzek ([5]). In order to obtain a sharper form of the Farkas Lemma, Glover gives sufficient conditions which guarantee that the set $A$ above is weak* closed ([1] Lemma 3). We state and prove a version of this result which uses only standard functional analysis techniques whereas Glover's proof uses results of Robinson on set-valued mappings. Our proof also covers the case when $X$ is only metrizable and not necessarily a normed space, but we must assume that the range space is barrelled although not necessarily normed.

**Theorem 1.** Let $X$ be complete, metrizable and let $Y$ be barrelled and suppose that $g(X) + S = Y$. Then $- (g^{-1}(-S))^* = A$ so in particular $A$ is weak* closed.

**Proof:** By Lemma 1 of [1] it suffices to show $A$ is weak* closed and by the Krein-Smulian Theorem ([2], 3.10.2) it suffices to show that $A \cap U^0$ is weak* closed for each closed, absolutely convex neighborhood of $0$, $U$, in $X$. Let $(x'_n)$ be a net in $A \cap U^0$ such that $(x'_n)$ is weak* convergent to $x'$. It suffices to show that $x' \in A \cap U^0$. Let $p$ be the Minkowski functional of $U$. Choose $s'_n \in S^*$ such that $x'_n \in \mathcal{A}(s'_n, g)(0)$ and let $y \in Y$. By hypothesis, $y = g(x) + s$ for some $x \in X$, $s \in S$. Then

1. $\langle s'_n, y \rangle = \langle s'_n, g(x) \rangle + \langle s'_n, s \rangle \geq \langle x'_n, x \rangle \geq -p(x)$.

Also $-y = g(\bar{x}) + \bar{s}$ for some $\bar{x} \in X$, $\bar{s} \in S$ so $\langle s'_n, -y \rangle = \langle s'_n, g(\bar{x}) \rangle + \langle s'_n, \bar{s} \rangle \geq \langle x'_n, \bar{x} \rangle \geq -p(\bar{x})$ and

2. $\langle s'_n, y \rangle \leq p(\bar{x})$.

Thus, if $r = \max \{ p(x), p(\bar{x}) \}$, (1) and (2) imply that $|\langle s'_n, y \rangle| \leq r$. Hence, $\{ s'_n \}$ is weak* bounded and, therefore, relatively weak* compact by the barrelledness ([2], 3.6.2).
Thus, \( i_s \) has a subnet, which we continue to denote by \( \{ s'_y \} \), which is weak* convergent to some \( y' \in Y \). Since \( \langle s'_y , s \rangle \geq 0 \) for \( s \in S \), \( \langle y' , s \rangle \geq 0 \) so that \( y' \in S^* \). For \( x \in X \), we have
\[
\langle y' , g(x) \rangle = \lim \langle s'_y , g(x) \rangle \geq \lim \langle x'_y , x \rangle = \langle x', x \rangle \text{ so } x' \in \partial (y'g)(0) \text{ and } x' \in A \cap U^0 \text{ since } U^0 \text{ is weak* closed.}
\]

In Glover's version he assumes that \( X \) is a Banach space and \( Y \) is a normed space, but there is no completeness assumption on \( Y \).

If \( f : X \to \mathbb{R} \) is lower semicontinuous and sublinear and \( x' \in X \), then under the assumptions of Theorem 1 Glover's Farkas Lemma ([1], Theorem 1) is: \( x' \in \partial f(0) \iff -g(x) \in S \) implies \( f(x) \geq \langle x' , x \rangle \). For the case when \( f \) and \( g \) are linear, this yields the Farkas Lemmas of Zalinescu ([6]) and Schirotzek ([5]).

References


- 653 -

Department of Mathematical Sciences, New Mexico State University
Las Cruces, NM 88003, U.S.A.

(Oblatum 15.4. 1985)