Barry J. Gardner

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SOME ASPECTS OF RADICAL THEORY FOR FULLY ORDERED ABELIAN GROUPS

B. J. GARDNER

Abstract: It is shown that hereditary classes (with respect to convex subgroups) of fully ordered abelian groups determine hereditary lower radical classes and homomorphically closed classes generate homomorphically closed semi-simple classes. Some radical and semi-simple classes determined by the chain of principal convex subgroups are presented, including a large collection of semi-simple radical classes.

Key words: Radical class, semi-simple class, fully ordered abelian group.

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Radical and semi-simple classes of fully ordered groups were first investigated by Chehata and Wiegandt [2]. Subsequently Jakubík [7], [8], [9] and Pringerová [11], [12] have studied radical theory for fully ordered groups and abelian groups. Our concern here is with the abelian case, of which we treat two aspects. We first show that not only does a hereditary class define a hereditary lower radical, but also (and this is a bit unusual) a homomorphically closed class generates a homomorphically closed semi-simple class. The rest of the paper is mostly devoted to radical and semi-simple classes which are determined in some way or other by real groups (subgroups of the reals with standard order): more specifically, by conditions on the skeletons of member groups. Examples of homomorphically closed semi-simple classes which are not radical, and of hereditary radical classes which are not semi-simple, are presented. (Analogous examples for fully ordered groups were given in [2]. In view of the symmetry alluded to above, the first type of example should not be more surprising than the second.) In addition, two large families of semi-simple radical classes are given.

We mention a few conventions and pieces of notation. All groups discussed are abelian; "f.o.group" accordingly means "fully ordered abelian group". A convex subgroup is indicated by the symbol $\leq_c$. Convex subgroups are normal subobjects, and accordingly we call a class of f.o. groups *hereditary* if it is closed under convex subgroups. If $a \in A$, a f.o. group, we denote by [a] the convex subgroup generated by a and call subgroups like this *principal*. We make heavy use of an invariant called the skeleton, due to Ribenboim (see [13]) though in the slightly different version used by Fuchs [3]. A jump in the chain of convex subgroups of a group will be indicated by the symbol $\prec$. Lexicographic products are denoted by $\Gamma_{\lambda \in \Lambda} A_\lambda$ or $X \,\Gamma\, \dot{Y}$. Following [6] we let $\|\Phi\|$ denote the order type of an ordered set Φ.

The lower (resp. upper) radical class defined by a class X will be denoted by $L(X)$(resp. $U(X)$) while $S(X)$ will denote the smallest semi-simple class containing X.

1. Characterizing Radical and Semi-simple Classes

Chehata and Wiegandt [2] give several characterizations of radical and semi-simple classes of (not necessarily abelian) f.o. groups which are equally valid in the abelian case. We shall find other characterizations convenient, and introduce them in this section.

THEOREM 1.1. *A non-empty class R of f.o. groups is a radical class if and only if*

(i) *R is homomorphically closed,*

(ii) *joins of chains of convex subgroups from R are in R and*

(iii) *R is closed under extensions, i.e. if $A \leq_c C$ and both A and $C/A \in R$, then $C \in R$.*

This theorem is well known (at least for some other structures) and goes back to Amitsur's early paper on radical theory [1]. A proof in an adequately general setting is given in [4].

THEOREM 1.2. *A non-empty class S of f.o. groups is a semi-simple class if and only if*

(i)* *S is hereditary,*

(ii)* *$A/\bigcap N_\lambda \in S$ whenever $A \in S$ and $\{N_\lambda | \lambda \in \Lambda\}$ is a set (chain) of convex subgroups of A with each $A/N_\lambda \in S$ and*

(iii) *S is closed under extensions.*

Proof. Suppose S is a semi-simple class, and let R be the corresponding radical class (i.e. $S = \{G | R(G) = 0\}$). Transitivity of convexity of subgroups gives (i)*. Let A and the N_λ be as in (ii)*, and let $B/\bigcap N_\lambda = R(A/\bigcap N_\lambda)$. If some $N_{\lambda_0} \subseteq B$, then

$$B/N_{\lambda_0} \leq_c A/N_{\lambda_0} \in S, \text{ so } B/N_{\lambda_0} \in S \text{ and}$$

$$B/N_{\lambda_0} \cong (B/\bigcap N_\lambda)/(N_{\lambda_0}/\bigcap N_\lambda) \in R,$$

so $B = N_{\lambda_0}$. It follows that $B \subseteq \bigcap N_\lambda$, so $A/\bigcap N_\lambda \in S$ and (ii)* is satisfied. Finally, if $C \leq_c D$ and $C, D/C \in S$, then either $R(D) \leq_c C \in S$ or $R(D)/C \leq_c D/C \in S$, so $R(D) \leq_c C$ anyway, and then $R(D) = 0$. Thus (iii) is also satisfied.

Conversely, suppose S satisfies (i)*, (ii)* and (iii). Clearly (i)* implies (S1) of [2], p.145, so we pass to (S2) of [2], p.145. Let X be a f.o. group of which every non-zero convex subgroup has a non-zero homomorphic image in S. Let

$$I = \cap\{Y \mid Y \leq_c X \text{ and } X/Y \in S\},$$
$$J = \cap\{W \mid W \leq_c I \text{ and } I/W \in S\}.$$

Then by (ii)*, X/I, I/J $\in S$, so by (iii) X/J $\in S$, whence I = J. Thus I has no non-zero homomorphic image in S, so by the hypothesis of (S2), I = 0. Thus X $\cong$ X/I $\in S$, (S2) is satisfied and S is a semi-simple class. //

We note in passing that the two theorems of this section remain true for non-abelian groups. In Theorem 1.2 some extra work is needed to show that J is normal in X, but the essential ideas are contained in [2].

2. Hereditary Radical Classes; Homomorphically Closed Semi-simple Classes.

In most contexts where radical theory is studied, hereditary classes determine hereditary lower radical classes. Somewhat less commonly, radical classes are hereditary if and only if they have semi-simple classes which are closed under essential extensions (in an appropriate sense). More rarely, there are analogous results involving homomorphically closed semi-simple classes and small (normal) subobjects. (For some fairly inconclusive remarks on all this, see [5].) For f.o. groups, all proper convex subgroups are both essential and small, but all the results just mentioned are valid.

THEOREM 2.1. *Let R be a radical class of f.o. groups, S the* *corresponding semi-simple class. Then* R *is hereditary if and only if* S *satisfies the condition*

$$0 \neq A \leqslant_c B \ \& \ A \in S \Rightarrow B \in S. \qquad (*)$$

Proof. Let R be hereditary, $0 \neq A \in S$, $A \leqslant_c B$. Since R is hereditary, we cannot have $A \subseteq R(B)$, so $R(B) \subseteq A$. But then $R(B) \leqslant_c A$, so $R(B) \subseteq R(A) = 0$, i.e. $B \in S$. Conversely, if S satisfies (*) and $X \leqslant_c Y \in R$, then $X/R(X) \leqslant_c Y/R(X)$, whence $Y/R(X) \in S$. But $Y/R(X) \in R$, so $X/R(X) \in S \cap R = \{0\}$ and thus $X \in R$. //

COROLLARY 2.2. *Let* M *be hereditary and homomorphically closed. Then* $L(M)$ *is hereditary.*

Proof. Let S be the semi-simple class corresponding to $L(M)$. Then S consists of all those A which have no non-zero convex subgroups in M (cf. [2], p.149). If $0 \neq A \in S$ and $A \leqslant_c B$, let $M \leqslant_c B$ with $M \in M$. If $M \subseteq A$, then $M = 0$, as $A \in S$. If $A \subseteq M$, then $A \leqslant_c M$, so $M \in S$ and thus $M = 0$. Thus in any case $M = 0$, so $B \in S$. Accordingly, S satisfies (*), so $L(M)$ is hereditary. //

THEOREM 2.3. *Let* R *be a radical class of f.o. groups,* S *the corresponding semi-simple class. Then* S *is homomorphically closed if and only if* R *satisfies the condition*

$$0 \neq A/B \in R \Rightarrow A \in R. \qquad (\dagger)$$

Proof. Let S be homomorphically closed, $0 \neq A/B \in R$. Suppose $R(A) \subseteq B$. Then

$$A/B \cong [A/R(A)]/[B/R(A)] \in S,$$

so $A/B = 0$ - a contradiction. Hence we have $B \subseteq R(A)$. But then

$$A/R(A) \cong [A/B]/[R(A)/B] \in R \cap S,$$

so $A = R(A) \in R$. Conversely, if R satisfies (†), then for $X \leq_c Y \in S$, there exists $W \leq_c Y$ such that $W/X = R(Y/X) \in R$, and thus $W = X$ or $W \in R$. The second possibility requires that $W = 0$ as $W \leq_c Y \in S$, so in any case, $W/X = 0$, i.e. $Y/X \in S$. ⫽

COROLLARY 2.4. *Let M be hereditary and homomorphically closed. Then the smallest semi-simple class* S(M) *containing M is homomorphically closed.*

Proof. The radical class corresponding to S(M) is U(M). Let $0 \neq A/B \in U(M)$, and let A/D be in M. If $B \subseteq D$, then $A/D \cong [A/B]/[D/B]$ is a homomorphic image of $A/B \in U(M)$, so $A/D = 0$. If $D \subseteq B$, then $A/B \cong [A/D]/[B/D] \in M \cap U(M)$, so $A/B = 0$ - a contradiction. Thus $A/D = 0$ and $A \in U(M)$. Since U(M) therefore satisfies (†) of Theorem 2.3, we have the result. ⫽

3. <u>The Lower Radical Class and Snallest Semi-simple Class Defined By a Set of Real Groups.</u>

In this section we first look at the effects of certain f.o. group constructions on the skeleton, and then use the information gained to obtain description of L(M) and S(M) for a set M of real groups.

LEMMA 3.1. *Let* $A \leq_c B$, *and let* $[\Pi_A, B_\pi(\pi \in \Pi_A)]$. $[\Pi_B, B_\pi(\pi \in \Pi_B)]$ *and* $[\Pi_{B/A}, B_\pi(\pi \in \Pi_{B/A})]$ *denote the skeletons of* A, B, B/A *respectively. Then*

(i) $\|\Pi_B\| = \|\Pi_A\| + \|\Pi_{B/A}\|$ *and (up to isomorphism)*

(ii) $\{B_\pi | \pi \in \Pi_B\} = \{B_\pi | \pi \in \Pi_A\} \cup \{B_\pi | \pi \in \Pi_{B/A}\}$.

Proof. Let a be in A. Then a generates the same principal convex subgroup [a] in A and B. If $X \leqslant_c B$ and $a \notin X$, then $[a] \nsubseteq X$, so $X \subseteq [a]$ and therefore $X \leqslant_c A$. From this it follows that a determines the same jump $D_a < [a]$ in A and B.

Let b be in $B \backslash A$. Then $[b] \nsubseteq A$, so $A \subsetneq [b]$. Also $[b]/A \leqslant_c B/A$. But if $b + A \in Y/A \leqslant_c B/A$, then $b \in Y \leqslant_c B$, so $[b] \subseteq Y$, whence it follows that $[b]/A = [b+A]$. If $W/A \leqslant_c B/A$ and $b + A \notin W/A$, then $b \notin W$ so (as $W \leqslant_c B$) $W \subsetneq [b]$. Conversely, if $A \subseteq V \leqslant_c B$ and $b \notin V$, then $V \subsetneq [b]$. Let $G < [b]$, $H/A < [b+A]$ be the jumps corresponding to b in B, b + A in B/A respectively. Then

$$\begin{aligned} H/A &= \bigcup\{K/A \leqslant_c B/A \mid b + A \notin K/A\} \\ &= \bigcup\{K/A \mid A \subseteq K \leqslant_c B \text{ and } b \notin K\} \\ &= \bigcup\{K \mid A \subseteq K \leqslant_c B \text{ and } b \notin K\}/A \\ &= \bigcup\{L \mid L \leqslant_c B \text{ and } b \notin L\}/A = G/A. \end{aligned}$$

It follows that the chain of principal convex subgroups of B/A is order-isomorphic to that of the principal convex subgroups of B which contain A, and that at corresponding jumps $G < [b]$, $H/A < [b+A]$ we have

$$\begin{aligned} [b+A]/(H/A) = [b+A]/(G/A) &\cong ([b]/A)/(G/A) \\ &\cong [b]/G. \; /\!\!/ \end{aligned}$$

An *ascending convex series* for a f.o. group A is an ordinally labelled series

$$0 = A_0 \leqslant_c A_1 \leqslant_c \cdots \leqslant_c A_\alpha \leqslant_c A_{\alpha+1} \leqslant_c \cdots \leqslant_c A_\mu = A \qquad (*)$$

in which $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ whenever β is a limit ordinal. The f.o.

groups $A_{\alpha+1}/A_\alpha$ are called the *factors* of (*). A *descending convex series* for A is an ordinally-labelled series

$$A = A^0 \geqslant_c A^1 \geqslant_c \dots \geqslant_c A^\alpha \geqslant_c A^{\alpha+1} \geqslant_c \dots \geqslant_c A^\lambda = 0 \qquad (**)$$

in which $A_\beta = \bigcap_{\alpha<\beta} A_\alpha$ whenever β is a limit ordinal. The $A^\alpha/A^{\alpha+1}$ are called the *factors* of (**).

Series of these two kinds will be used in the description of radical and semi-simple classes respectively. We now describe the skeletons of f.o. groups having series of either kind. For infinite sums of order types, we refer to Sierpiński [14]. pp.246-247.

THEOREM 3.2. *Let* A *be a f.o. group with an ascending convex series* (*). *Let* Φ *be the set of ordinals* $\{\alpha | 0 \leqslant \alpha < \mu\}$, *equipped with reverse order. For each* $\alpha \in \Phi$ *let* $A_{\alpha+1}/A_\alpha$ *have skeleton* $[\Pi_\alpha, B_\pi(\pi \in \Pi_\alpha)]$, *and let* A *have skeleton* $[\Pi, B_\pi(\pi \in \Pi)]$. *Then* $\|\Pi\| = \sum_{\alpha\in\Phi} \|\Pi_\alpha\|$ *and* $\{B_\pi | \pi \in \Pi\} = \bigcup_{\alpha\in\Phi} \{B_\pi | \pi \in \Pi_\alpha\}$.

Proof. If $a \in A$, there exists an ordinal γ such that $a \in A_\gamma$ and γ is minimal for this. It is clear that γ is not a limit, and we therefore have $A_{\gamma-1} \subsetneqq [a] \subseteq A_\gamma$. By Lemma 3.1, that part of the skeleton of A determined by jumps occurring after $A_{\gamma-1}$ and not after A_γ is equivalent to the skeleton of $A_\gamma/A_{\gamma-1}$. It follows that $\|\Pi\| = \sum_{\alpha\in\Phi} \|\Pi_\alpha\|$. The second assertion also follows from Lemma 3.1. //

THEOREM 3.3. *Let* A *be a f.o. group with a descending convex series* (**). *Let* Ψ *be the set of ordinals* $\{\alpha | 0 \leqslant \alpha < \lambda\}$, *equipped with standard order. For each* $\alpha \in \Psi$ *let* $A^\alpha/A^{\alpha+1}$ *have skeleton* $[\Pi_\alpha, B_\pi(\pi \in \Pi_\alpha)]$, *and let* A *have skeleton* $[\Pi, B_\pi(\pi \in \Pi)]$. *Then* $\|\Pi\| = \sum_{\alpha\in\Psi} \|\Pi_\alpha\| + 1$ *and* $\{B_\pi | \pi \in \Pi\} = \underset{\alpha\in\Psi}{} \{B_\pi | \pi \in \Pi_\alpha\} \cup \{0\}$.

Proof. If $0 \neq a \in A$, there is an ordinal γ such that $a \notin A^{\gamma}$ and γ is minimal for this. Since $a \in A^{\alpha}$ for every $\alpha < \gamma$, we see that γ is not a limit. We therefore have $A^{\gamma-1} \supsetneq [a] \supseteq A^{\gamma}$. It now follows as in the previous proof that the order type of the "non-zero part" of the skeleton of A is $\sum_{\alpha \in \Psi} \|\Pi_{\alpha}\|$. Hence (adding in the zero jump) we get $\|\Pi\| = \sum_{\alpha \in \Psi} \|\Pi_{\alpha}\| + 1$ as desired. The other assertion follows similarly. //

THEOREM 3.4. *Let M be a set of real groups. The following conditions are equivalent for a f.o. group* A.

(i) $A \in L(M)$.

(ii) *There is an ascending convex series*

$$0 = A_0 \leqslant_c A_1 \leqslant_c \cdots \leqslant_c A_{\alpha} \leqslant_c A_{\alpha+1} \leqslant_c \cdots \leqslant_c A_{\mu} = A$$

for which every factor is in M.

(iii) *The skeleton of* A *has the form* $[\Pi, B_{\pi} (\pi \in \Pi)]$ *where* Π *is inversely well-ordered and each* $B_{\pi} \in M$.

Proof. As normality of subobjects (i.e. convexity of subgroups) is transitive, the equivalence of (i) and (ii) follows as in [10], pp.276-277.

(ii) $\Rightarrow$ (iii): We first show that every convex subgroup of A is an A_{α}. If $0 \neq X \leqslant_c A$, let $\gamma = \min\{\alpha \mid X \subsetneq A_{\alpha}\}$. If $\gamma - 1$ exists, then $A_{\gamma-1} \subseteq X \subseteq A_{\gamma}$, so $X/A_{\gamma-1} \leqslant_c A_{\gamma}/A_{\gamma-1}$. But $A_{\gamma}/A_{\gamma-1}$ is a real group, so $X/A_{\gamma-1} = 0$, i.e. $X = A_{\gamma-1}$. If γ is a limit, on the other hand, then $A_{\delta} \subseteq X \subseteq A_{\gamma}$ for every $\delta < \gamma$, so $A_{\gamma} = \bigcup_{\delta<\gamma} A_{\delta} \subseteq X \subseteq A_{\gamma}$ - a contradiction.

We conclude that the set of convex subgroups of A is well-ordered. So, therefore, is the set of principal convex subgroups, whence Π is inversely well-ordered. Let $D_\pi < C_\pi$ be a jump in the chain of convex subgroups. Then there exists an ordinal λ such that

$$D_\pi = A_\lambda \text{ and } C_\pi = A_{\lambda+1},$$

so $B_\pi = C_\pi/D_\pi = A_{\lambda+1}/A_\lambda \in M$.

(iii) $\Rightarrow$ (ii): If Π is inversely well-ordered and if $B_\pi = C_\pi/D_\pi$ for each $\pi \in \Pi$, then the set $\{C_\pi | \pi \in \Pi\}$ of principal converse subgroups is well-ordered by inclusion. We augment this well-ordered list as follows: for every jump $D_\pi < C_\pi$ for which D_π is non-principal, insert D_π before C_π. Then $D_\pi = \bigcup\{C_\rho | C_\rho \subsetneqq C_\pi\} = \bigcup\{C_\rho | C_\rho \subsetneqq C_\pi\} \cup \bigcup\{D_\rho | C_\rho \subsetneqq C_\pi\}$. There results a well-ordered list

$$0 = A_0, A_1, \ldots, A_\alpha, A_{\alpha+1}, \ldots, A_\mu = A$$

of the convex subgroups of A, labelled by ordinals, with the order matching that of inclusion. For each α, $A_\alpha < A_{\alpha+1}$ is a jump, so $A_{\alpha+1}/A_\alpha \in M$. If β is a limit, then A_β is not a successor and is therefore non-principal, so as above $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$. Thus we have the kind of series required in (ii). //

We have an analogous result for semi-simple classes.

THEOREM 3.5. *Let M be a set of real groups and let* S(M) *denote the smallest semi-simple class containing M. The following conditions are equivalent for a f.o. group* A.

(i) $A \in S(M)$.

(ii) *There is a descending convex series*

$$A = A^0 \geqslant_c A^1 \geqslant_c \cdots \geqslant_c A^\alpha \geqslant_c A^{\alpha+1} \geqslant_c \cdots \geqslant_c A^\mu = 0$$

for which each factor is in M.

(iii) *The skeleton of* A *has the form* $[\Pi, B_\pi(\pi \in \Pi)]$ *where* Π *is well-ordered and each* $B_\pi \in M$.

Proof. The equivalence of (i) and (ii) follows as in [10], p.283, Theorem 7.

(ii) $\Rightarrow$ (iii): Let X be a convex subgroup of A with $0 \neq X \subsetneqq A$. Then if $\gamma = \min\{\alpha | A^\alpha \subsetneq X\}$, we have $A^\gamma \subsetneq X \subseteq A^\delta$ for every $\delta < \gamma$. Now γ is not a limit, as otherwise $X \subseteq \bigcap_{\delta<\gamma} A^\delta = A^\gamma$, so $A^\gamma \subsetneqq X \subseteq A^{\gamma-1}$ and so $0 \neq X/A^\gamma \leqslant_c A^{\gamma-1}/A^\gamma \in M$, whence $X = A^{\gamma-1}$. It follows that the set of all convex subgroups of A is inversely well-ordered by inclusion. In particular, every convex subgroup is a successor, and therefore principal (cf. [13], p.15). Finally this means that Π is well-ordered. Let $D_\pi < C_\pi$ be a jump. Then there is an ordinal λ such that $D_\pi = A^\lambda$ and $C_\pi = A^{\lambda-1}$, whence $B_\pi = C_\pi/D_\pi = A^{\lambda-1}/A^\lambda \in M$.

(iii) $\Rightarrow$ (ii): If Π is well-ordered, then the set of principal convex subgroups is inversely well-ordered by inclusion. This means in particular that there is a largest principal convex subgroup, i.e. that A (as the union of all principal convex subgroups) is principal. Suppose there exists a non-principal convex subgroup Y. Then there is a jump $Y < C$, where C is principal and Y is the join of all the principal convex subgroups properly contained in C. But there is a largest such subgroup, so Y is principal. From this contradiction we deduce that all convex subgroups of A are principal. We can label them by ordinals to give a well-ordered list

$$A = A^0, A^1, A^2, \ldots, A^\alpha, \ldots, A^\mu = 0.$$

For each α, $A^{\alpha+1} < A^\alpha$ is a jump, so $A^\alpha/A^{\alpha+1} \in M$. Let β be a limit ordinal. Then A^β has no successor, but for every $\alpha_0 < \beta$ we have

$$A^{\alpha_0} \supseteq \bigcap_{\alpha < \beta} A^\alpha \supseteq A^\beta,$$

whence $\bigcap_{\alpha < \beta} A^\alpha = A^\beta$. Thus

$$A = A^0 \geqslant_c A^1 \geqslant_c \cdots \geqslant_c A^\alpha \geqslant_c A^{\alpha+1} \geqslant_c \cdots \geqslant_c A^\mu = 0$$

is a series of the kind required in (ii). //

4. Examples

We now consider some examples of radical and semi-simple classes.

EXAMPLE 1. Let M be a set of real groups. By Theorem 3.4 $L(M)$ consists of those f.o. groups whose skeletons are inversely well-ordered lists of groups from M. By Corollary 2.2, $L(M)$ is hereditary. It is not a semi-simple class, as the following example shows. Let $\Lambda = \{0,1,2,\ldots;\omega\}$ with its natural order, and for each $\lambda \in \Lambda$ let A_λ be a group in M. Consider the lexicographic product $\Gamma_{\lambda \in \Lambda} A_\lambda$.

Let $G_n = \Gamma\{A_\lambda \mid \lambda \geqslant n\}$ for each n. Then $\Gamma A_\lambda / G_n \cong \Gamma\{A_0,\ldots,A_{n-1}\} \in L(M)$ for each n (cf. Fuchs [3], p.55) while $\Gamma A_\lambda / \bigcap G_n = \Gamma A_\lambda / A_\omega$

$$\cong \Gamma\{A_1, A_2, \ldots\} \notin L(M),$$

as its skeleton is not inversely well-ordered. By Theorem 1.2, $L(M)$ is not a semi-simple class.

EXAMPLE 2. Let M be a set of real groups. Then $U(M)$ is not hereditary unless M contains all real groups, for if $X \in M$ and Y is a real group, then $X <_c Y \,\Gamma\, X$, so if $U(M)$ is hereditary

$Y \Gamma X$ is in $S(M)$ (by Theorem 2.1) whence (as by Corollary 2.4 $S(M)$ is homomorphically closed) $Y \in S(M)$ and hence $Y \in M$.

EXAMPLE 3. Let M be a set of real groups. By Theorem 3.5, $S(M)$ consists of those f.o.groups whose skeletons are well-ordered lists of groups from M. Since M is homomorphically closed, so is $S(M)$,(by Corollary 2.4). However, $S(M)$ is not a radical class. To see this, consider the lexicographic product $\Gamma_{\lambda \in \Lambda} A_\lambda$, where $\Lambda = \{\ldots,-2,-1,0\}$ with the integers in their natural order and each $A_\lambda \in M$. For each n, let $H_n = \Gamma\{A_\lambda \mid \lambda \geqslant n\}$. Then as $\{n,n+1,\ldots,-2,-1,0\}$ is well-ordered, H_n is in $S(M)$ for each n. However, $\Gamma_{\lambda \in \Lambda} A_\lambda = \bigcup_n H_n \notin S(M)$.

EXAMPLE 4. Let X,Y be distinct real groups. Then $L(\{X \Gamma Y\})$ is not hereditary as the only homomorphic images of $X \Gamma Y$ are 0, X and $X \Gamma Y$, so $Y \notin L(\{X \Gamma Y\})$ although $Y \leqslant_c X \Gamma Y$.

We have seen that homomorphically closed semi-simple classes need not be radical (Example 3) and that hereditary radical classes need not be semi-simple (Example 1). We conclude with some discussion of semi-simple radical classes, which we shall call, for brevity, SSR classes.

Since a SSR class is homomorphically closed and hereditary, it must contain, together with any group A, every C/D, where $C \prec D$ is a jump in the chain of convex subgroups of A. Thus SSR classes can be expected, in some sense, to be determined by the real groups they contain. We shall consider two distinct ways of generating SSR classes from real groups.

EXAMPLE 5. Let M be a set of real groups and let $T(M)$ denote the class of f.o. groups for whose skeletons $[\Pi, B_\pi (\pi \in \Pi)]$ we have $B_\pi \in M$ for all π. By Lemma 3.1, $T(M)$ is hereditary, homomorphically closed and closed under extensions. If A has a chain $\{A_\lambda | \lambda \in \Lambda\}$ of convex subgroups in $T(M)$, let $\bar{A} = \bigcup A_\lambda$. If $0 \neq a \in \bar{A}$ let a be in A_{λ_0}. Then as in the proof of Lemma 3.1, the jump $D < [a]$ and the factor $[a]/D$ are the same whether considered in $\bar{A}$ or A_λ. Hence $[a]/D \in M$ and $\bar{A} \in T(M)$, so by Theorem 1.1, $T(M)$ is a radical class. Finally let a f.o. group G have a chain $\{G_\rho | \rho \in P\}$ of convex subgroups such that each $G/G_\rho \in T(M)$. If $g \notin \bigcap G_\rho$ then $g \notin$ some G_{ρ_0}. Since

$$G/G_{\rho_0} \cong [G/\bigcap G_\rho]/[G_{\rho_0}/\bigcap G_\rho]$$

it follows as in the proof of Lemma 3.1 that $[g + \bigcap G_\rho]$ determines the same real group as $[g + G_{\rho_0}]$ in the appropriate skeletons. We conclude that $G/\bigcap G_\rho \in T(M)$, whence by Theorem 1.2, $T(M)$ is a semi-simple class also.

For a given set M of real groups, $T(M)$ is not the smallest SSR class containing M. We now present a transfinite induction method of constructing the smallest such SSR class. For each ordinal α we construct a class M_α as follows:

$$M_1 = M; \ M_{\alpha+1} = SL(M_\alpha) \text{ for every } \alpha;$$
$$M_\beta = \bigcup_{\alpha < \beta} M_\alpha \text{ if } \beta \text{ is a limit.}$$

THEOREM 4.1. *For a set M of real groups, $\bigcup_\alpha M_\alpha$ is the smallest* SSR *class containing M.*

Proof. It is clear that there is a smallest SSR class containing M; call it $SSR(M)$. If $M_\alpha \subseteq SSR(M)$, then, since $SSR(M)$ is a radical class, $L(M_\alpha) \subseteq SSR(M)$, and then, since $SSR(M)$ is a semi-simple class, $M_{\alpha+1} = SL(M_\alpha) \subseteq SSR(M)$. Since everything is clear at limits, we conclude that $\bigcup_\alpha M_\alpha \subseteq SSR(M)$. Thus we need only show that $\bigcup_\alpha M_\alpha$ is a SSR class.

By Corollary 2.4, SL(C) is homomorphically closed for every class C. Hence $\bigcup_\alpha M_\alpha$ is homomorphically closed. Similarly (more obviously) $\bigcup_\alpha M_\alpha$ is hereditary. Let A be such that every non-zero homomorphic image has a non-zero convex subgroup in $\bigcup_\alpha M_\alpha$. Then for every $N \leqslant_c A$, A/N will have a non-zero convex subgroup in some $M_{\alpha(N)}$. If γ is chosen greater than every $\alpha(N)$, then every non-zero homomorphic image of A will have a non-zero convex subgroup in M_γ, so that $A \in L(M_\gamma) \subseteq SL(M_\gamma) = M_{\gamma+1} \subseteq \bigcup_\alpha M_\alpha$. Analogously, if every non-zero convex subgroup of A has a non-zero homomorphic image in $\bigcup_\alpha M_\alpha$, then for some δ we have

$$A \in S(M_\delta) \subseteq S(L(M_\delta)) = M_{\delta+1} \subseteq \bigcup_\alpha M_\alpha .$$

Thus (cf. (R2), (S2), p.145 of [2]) $\bigcup_\alpha M_\alpha$ is a SSR class. //

To show that SSR(M) $\neq$ T(M) we obtain a condition which the skeletons of groups in SSR(M) must satisfy. For this purpose we make use of some classes of chains which we now define. For a class $\mathcal{D}$ of chains, let

$$\mathcal{D}^* = \{\Lambda \mid \|\Lambda\| = \sum_{\phi \in \Phi} \|\Lambda_\phi\|, \text{ where } \Phi \text{ is inversely well-ordered and each } \Lambda_\phi \in \mathcal{D}\},$$

and let

$$\hat{\mathcal{D}} = \{\Lambda \mid \|\Lambda\| = \sum_{\psi \in \Psi} \|\Lambda_\psi\|, \text{ where } \Psi \text{ is well-ordered with greatest element and each } \Lambda_\psi \in \mathcal{D}^*\}.$$

Now let

C_1 = the class of singletons,

$C_{\alpha+1} = \hat{C}_\alpha$ for each ordinal α and

$C_\beta = \bigcup_{\alpha < \beta} C_\alpha$ for every limit ordinal β.

PROPOSITION 4.2. *Let M be a set of real groups. If $A \in SSR(M)$ then the order type of the skeleton of A is in some C_α.*

Proof. Obviously if $A \in M_1$ then the order type of the skeleton of A is in C_1. Suppose the analogous result is true for M_γ and C_γ for every $\gamma < \alpha$. Then clearly it persists if α is a limit, so we look at the case where $\alpha-1$ exists. Let G be in $M_\alpha = SL(M_{\alpha-1})$. By Theorem 3.2, [10], pp.276-277 and the inductive hypothesis, the order types of the skeletons of groups in $L(M_{\alpha-1})$ are in $C^*_{\alpha-1}$. Then by Theorem 3.3 and [10], p.383, the groups in $M_\alpha = SL(M_{\alpha-1})$ have skeletons with order types in $\hat{C}_{\alpha-1} = C_\alpha$. //

Thus to show that $SSR(M) \neq T(M)$ we simply have to exhibit a chain which is not in any C_α. Consider the interval $(0,1]$ of real numbers with its usual order. Clearly $(0,1] \notin C_1$. Suppose $(0,1] \in C_\alpha$, with α minimal. Then α is not a limit, so $(0,1]$ is a disjoint union $\dot{\bigcup}\{I_\alpha | 0 \leq \gamma \leq \beta\}$ of intervals I_α (labelled by the ordinals from 0 to some β) each of which is in $C^*_{\alpha-1}$. In turn each I_γ is an inversely well-ordered disjoint union of intervals from $C_{\alpha-1}$. Thus $(0,1]$ is a disjoint union of intervals (with greatest elements) from $C_{\alpha-1}$. Since $(0,1]$ is equivalent to any non-trivial such unit, each of them must be a singleton (by the minimality of α). But then each I_γ is inversely well-ordered, and an interval, hence a singleton, so $(0,1] = \dot{\bigcup} I_\gamma$ is well-ordered - a contradiction.

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Mathematics Department, University of Tasmania, G.P.O.Box 252C Hobart, Tasmania 7001, Australia

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