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ULTRAFILTERS AND ENDOMORPHIC UNIVERSES

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Abstract. This paper is in some respects a continuation of [T]. We transfer from the standard literature some further results concerning the Rudin-Keisler ordering and its minimal elements of the ultrafilters on Sd_V. Ramsey ultrafilters are established and we point out that the class of ultrafilters containing the supersets of a countable class is isomorphic to the class of ultrafilters on FN. At the same time we relate properties of ultrafilters with properties of endomorphic universes and show the existence of endomorphic universes satisfying certain particular conditions.

Key words: Alternative set theory, ultrafilter, minimal Ramsey, ω-complete, Rudin-Keisler ordering, endomorphic universe.

Classification: 02K10, 02K99

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§ 1. Preliminaries and some standard facts. All ultrafilters considered in the sequel are on the ring Sd_V of set-definable classes, unless otherwise stated; they are non-trivial and contain sets. Hence for every ultrafilter $\mathfrak{M}$, $\mathfrak{M} \cap V$ (V is the universe) is a base of $\mathfrak{M}$, and sometimes we identify $\mathfrak{M}$ to $\mathfrak{M} \cap V$. All functions considered here are set-definable. If F is a function and $\mathfrak{M}$ is an ultrafilter such that $dom(F) \in \mathfrak{M}$, let $F''\mathfrak{M} = \{F''u; u \subseteq dom(F) \wedge u \in \mathfrak{M}\}$; then $F''\mathfrak{M}$ is an ultrafilter.

We say that the ultrafilters $\mathfrak{M}$, $\mathfrak{N}$ are isomorphic (in symbols $\mathfrak{M} \cong \mathfrak{N}$) if there is a permutation $F: V \twoheadrightarrow V$ such that $F''\mathfrak{M} = \mathfrak{N}$. For each $\mathfrak{M}$ the class $[\mathfrak{M}] = \{\mathfrak{N}; \mathfrak{M} \cong \mathfrak{N}\}$ is the isomorphism class of $\mathfrak{M}$. Every isomorphism class is codable. As usual we shall not **make clear** distinction between $\mathfrak{M}$ and $[\mathfrak{M}]$.

We recall that the Rudin-Keisler ordering $\leq$ of (isomorphism classes of) ultrafilters is defined as follows:

$\mathfrak{M} \leq \mathfrak{N}$ if $(\exists f)(\mathrm{dom}(f) \in \mathfrak{N} \wedge f''\mathfrak{N} = \mathfrak{M})$.

The following well-known ZF-facts (which justify the term "ordering") hold in AST as well.

Lemma 1.1. (i) $f''\mathfrak{M} = \mathfrak{M}$ if $(\exists u \in \mathfrak{M})(f \restriction u = \mathrm{id})$.

(ii) $F''\mathfrak{M} \cong \mathfrak{M}$ iff $(\exists u \in \mathfrak{M})$ $(f \restriction u$ is one-to-one).

Proof. The proof of (i) is a trivial modification of the proof of Lemma 2.3 of [T].

(ii) Let $f''\mathfrak{M} \cong \mathfrak{M}$. There is a permutation F of the universe such that $F''f''\mathfrak{M} = (F \circ f)''\mathfrak{M} = \mathfrak{M}$.
By (i) there is $u \in \mathfrak{M}$, such that $F \circ f \restriction u = \mathrm{id}$. Then $f \restriction u$ is one-to-one.

Conversely, let $f \restriction u$ be one-to-one for $u \in \mathfrak{M}$. Put $w = u \cup f''u$. Divide u into two infinite disjoint sets u_1, u_2 and suppose $u_1 \in \mathfrak{M}$. Since $u_1 \mathrel{\hat{\approx}} f''u_1$, it follows $w - u_1 \mathrel{\hat{\approx}} w - f''u_1$. Hence there is a bijection $g: w - u_1 \to w - f''u_1$ and the function

$$\pi(x) = \begin{cases} f(x) & \text{if } x \in u_1 \\ g(x) & \text{if } x \in f''u_1, \end{cases}$$

is a permutation of w. Extend π to a permutation F of V by putting $F(x) = x$ for $x \notin w$. Then $F \restriction u_1 = \pi \restriction u_1 = f \restriction u_1$, whence $f''\mathfrak{M} = F''\mathfrak{M} \cong \mathfrak{M}$.

Recall that $\mathfrak{M}$ is minimal (w.r.t. the ordering $\leq$) if for every function f with $\mathrm{dom}(f) \in \mathfrak{M}$ there is some $u \in \mathfrak{M}$ such that $f \restriction u$ is either constant or one-to-one.

The existence of minimal ultrafilters was shown in [T]. The following stronger result, however, can be proved, imitating the standard proof (cf. [B2], Th. 2).

Lemma 1.2. The class of minimal ultrafilters on Sd_V is uncodable.

Since every isomorphism class is codable, we have immediately that:

Corollary 1.3. The class of isomorphism classes of minimal ultrafilters is uncodable.

Next imitate the proof of Th. 6 of [B2] to get:

Lemma 1.4. There is no $\leq$ -maximal ultrafilter on Sd_V..

§ 2. ω -complete and rich ultrafilters. Recall that $\mathfrak{M}$ is ω -complete if for every sequence $\{u_n; n \in FN\} \subseteq \mathfrak{M}$, there is a $u \in \mathfrak{M}$ such that $u \subseteq \bigcap_n u_n$..

Let us call $\mathfrak{M}$ rich, if it contains all supersets of a countable class X. X is called a nucleus of $\mathfrak{M}$.

It is not hard to see that the classes of ω -complete and rich ultrafilters are disjoint.

Let us give some characterizations of them in terms of endomorphic universes.

Lemma 2.1. Let F, $\mathfrak{M}$,d be coherent, and F"V = A. Then

(i) $\mathfrak{M}$ is rich iff ($\exists$ countable $Y \subseteq A$) ($d \in E_A(Y)$)

(ii) $\mathfrak{M}$ is ω -complete iff ($\forall$ countable $Y \subseteq A$) ($E_A(Y) = E_{A[d]}(Y)$)

Proof. (i) If X is a nucleus of $\mathfrak{M}$ and Y = F"X, then obviously $d \in E_A(Y)$ and vice-versa.

(ii) Let $\mathfrak{M}$ be ω -complete and $Y = \{y_1, y_2, \ldots\} \subseteq A$.

Since $A[d] = \{f(d); f \in A\}$, it suffices to prove that for every $f \in A$ with $d \in dom(f)$ and $Y \in f(d)$, there is a $u \in A$ such that $Y \subseteq u \subseteq f(d)$.

Let f be such a function and f = F(g). Put $X = \{x_1, x_2, \dots\}$, where $y_n = F(x_n)$. Then $F(x_n) \in F(g)(d)$, for every $n \in FN$, whence, by coherence,

$$v_n = \{x; x_n \in g(x)\} \in \mathfrak{M}.$$

It follows from ω-completeness that there is a $v \in \mathfrak{M}$ such that $v \subseteq \bigcap_{\mathfrak{M}} v_n$. Then

$$x \in v \rightarrow X \subseteq g(x),$$

hence

(1) $$X \subseteq \bigcap \{g(x); x \in v\} = w$$

and $$v \subseteq \{x; w \subseteq g(x)\}.$$

Therefore $\{x : w \subseteq g(x)\} \in \mathfrak{M}$ or, by coherence,

(2) $$F(w) \subseteq f(d).$$

We have from (1) and (2) that

$$Y \subseteq F(w) \subseteq f(d).$$

Conversely, suppose $E_A(Y) = E_{A[d]}(Y)$ for all countable $Y \subseteq A$. Let $(v_n)_{n \in FN}$ be a subclass of $\mathfrak{M}$. We have to find $v \in \mathfrak{M}$ with $v \subseteq \bigcap_{\mathfrak{M}} v_n$. From $E_A(Y) = E_{A[d]}(Y)$ it follows that

(3) $(Y \subseteq A \,\&\, Y \subseteq f(d)) \rightarrow (\exists u \in A)(Y \subseteq u \subseteq f(d))$.

Extend the sequence $(v_n)_{n \in FN}$ to a set $r = \{v_\beta ; \beta \leq \alpha\}$ and define the function $g : Ur \rightarrow P(r)$ as follows:

$$g(x) = \{v \in r; x \in v\}.$$

Then

$$(\forall v \in r)(x \in v \leftrightarrow v \in g(x)),$$

hence

$$v_n = \{x; v_n \in g(x)\}.$$

We get by coherence that for all $n \in FN$

$$F(v_n) \in F(g)(d),$$

that is, $\{F(v_1), F(v_2), \dots\} \subseteq F(g)(d)$.

By (3) there is a t such that

$$\{F(v_1),F(v_2),\dots\} \subseteq F(t) \subseteq F(g)(d),$$

thus $\{v_1,v_2,\dots\} \subseteq t$ and $\{x; t \subseteq g(x)\} \in \mathfrak{M}$.

Put $v = \{x; t \subseteq g(x)\}$. Then

$$x \in v \leftrightarrow t \subseteq g(x)$$

and, since $v_1,v_2,\dots \subseteq t$, we get

$$x \in v \rightarrow v_n \in g(x) \leftrightarrow x \in v_n$$

for all $n \in FN$; therefore $v \subseteq \bigcap_n v_n$ and this completes the proof.

Let F be an endomorphism and $\mathfrak{M}$ be an ultrafilter. F, $\mathfrak{M}$ are <u>compatible</u> if there is a d such that F, $\mathfrak{M}$, d are coherent.

The following is obvious.

<u>Lemma 2.2</u>. F, $\mathfrak{M}$ are compatible iff $\bigcap\{F(u); u \in \mathfrak{M}\} \neq \emptyset$ and F , $\mathfrak{M}$, d are coherent iff $d \in \bigcap\{F(u); u \in \mathfrak{M}\}$.

<u>Lemma 2.3</u>. Let $F''V = A$ and let F, $\mathfrak{M}_1$, d_1, and F, $\mathfrak{M}_2$, d_2 be coherent. Then

$$A[d_1] \subseteq A[d_2] \longrightarrow \mathfrak{M}_1 \leq \mathfrak{M}_2.$$

<u>Proof</u>. Suppose $A[d_1] \subseteq A[d_2]$. There is some $f \in A$ such that $f(d_2) = d_1$. Let $F(g) = f$. Then $\mathrm{dom}(g) \in \mathfrak{M}_2$ and for every $u \in \mathfrak{M}_2$ such that $u \subseteq \mathrm{dom}(g)$, we have $d_2 \in F(u) \subseteq \mathrm{dom}(f)$. Hence $d_1 \in f''F(u)$, therefore $g''u \in \mathfrak{M}_1$. This proves that $g''\mathfrak{M}_2 = \mathfrak{M}_1$.

<u>Lemma 2.4</u>. $\mathfrak{M} \leq \mathfrak{N}$ iff for every endomorphism F, F, $\mathfrak{M}$ compatible $\rightarrow$ F, $\mathfrak{N}$ compatible. Specifically, F, $\mathfrak{N}$, d coherent $\rightarrow$ $\rightarrow$ F, $g''\mathfrak{N}$, $F(g)(d)$ coherent.

<u>Proof</u>. Let $g''\mathfrak{N} = \mathfrak{M}$ and F, $\mathfrak{N}$, d be coherent. Then $\{x; \varphi(x,y_1,\dots,y_n)\} \in \mathfrak{M} \leftrightarrow g^{-1}{}''\{x; \varphi(x,y_1,\dots,y_n)\} \in \mathfrak{N} \leftrightarrow$ $\{x; \varphi(g(x),y_1,\dots,y_n)\} \in \mathfrak{N} \leftrightarrow \varphi(F(g)(d),F(y_1),\dots,F(y_n))$.

Conversely, suppose the condition is true. There are F, d such that F, $\mathfrak{M}$, d are coherent and $F''V[d] = V$ (cf. [S - V], last but two theorems). By assumption there is a d' such that F, $\mathfrak{M}$, d' are coherent. Since $F''V[d'] \subseteq F''V[d]$, it follows from the preceding lemma that $\mathfrak{M} \leq \mathfrak{N}$.

Lemma 2.5. Let $\mathfrak{M} \leq \mathfrak{N}$. If $\mathfrak{N}$ is ω-complete (rich) then $\mathfrak{M}$ is ω-complete (rich).

Proof. Straightforward.

Let X be any class. Put

$$X^{(2)} = \{\{x,y\}; x,y \in X \wedge x \neq y\}.$$

Let $P = \{P_1, P_2\}$ be a partition of $X^{(2)}$. A class $Y \subseteq X$ is homogeneous for P, or P-homogeneous if $Y^{(2)} \subseteq P_1$ or $Y^{(2)} \subseteq P_2$.

The proof of the following is the standard one.

Lemma 2.6 (Ramsey). Let X be an arbitrary infinite class and let $P = \{P_1, P_2\}$ be a partition of $X^{(2)}$. Then there is a countable P-homogeneous class $Y \subseteq X$.

Corollary 2.7. Let X be an infinite set-definable class and $P = \{P_1, P_2\}$ a set-definable partition of $X^{(2)}$. Then there is an infinite P-homogeneous set $u \subseteq X$.

Proof. Find by 2.6 a countable P-homogeneous $Y \subseteq X$ and, then, use the axiom of prolongation to find a P-homogeneous set u, such that $Y \subseteq u \subseteq X$.

An ultrafilter $\mathfrak{M}$ is called Ramsey if for every set-definable partition $P = \{P_1, P_2\}$ of $V^{(2)}$, there is a P-homogeneous set $u \in \mathfrak{M}$.

Lemma 2.8. Every Ramsey ultrafilter is minimal.

Proof. Let $\mathfrak{M}$ be Ramsey and let F be a set-definable func-

$P_1 = \{\{x,y\};F(x) \neq F(y)\}$, $P_2 = \{\{x,y\};F(x) = F(y)\}$.
Let $u \in \mathfrak{M}$ be homogeneous for $\{P_1,P_2\}$. If $u^{(2)} \subseteq P_1$ then $F \restriction u$ is one-to-one; if $u^{(2)} \subseteq P_2$ then $F \restriction u$ is constant.

Lemma 2.9. There is an uncodable class of (isomorphism classes of) ω-complete Ramsey ultrafilters.

Proof. Let $T = \bigcup\{2^{\alpha}; \alpha \in \Omega\}$, i.e., T is the complete binary tree of height Ω. Let w be an infinite set and let $(w_{\alpha})_{\alpha \in \Omega}$, be an enumeration of $P(w)$ and $(P_{\alpha} = \{P^0_{\alpha}, P^1_{\alpha}\})_{\alpha \in \Omega}$ be an enumeration of all set-partitions of $w^{(2)}$. We shall define a one-to-one mapping $H:T \rightarrow P(w)$ such that H is an embedding of $\langle T, \subseteq \rangle$ into $\langle P(w), \supseteq \rangle$ and

(i) $H(s^\frown 0) \cap H(s^\frown 1) = \emptyset$, $\forall s \in T$,

(ii) if $dom(s) = \alpha + 1$ then $H(s)$ is P_{α}-homogeneous and either $H(s) \cap w_{\alpha} = \emptyset$ or $H(s) \subseteq W_{\alpha}$.

The definition is by recursion on the levels $T_{\alpha} = \{s \in T; dom(s) = \alpha\}$ of the tree.

Put $H(\emptyset) = w$. At limit levels T_{α} and for $s \in T_{\alpha}$ choose an infinite $u \subseteq \bigcap\{H(s \restriction \beta), \beta < \alpha\}$ such that either $u \subseteq w_{\alpha}$ or $u \cap w_{\alpha} = \emptyset$ and put $H(s) = u$.

Now suppose $H(s)$ is defined and $dom(S) = \alpha$. Divide $H(s)$ into infinite sets u_0, u_1. Find $v_0 \subseteq u_0$, $v_1 \subseteq u_1$ such that $v_i \cap w_{\alpha} = \emptyset$ or $v_i \subseteq w_{\alpha}$ for $i = 0,1$ and choose $v'_i \subseteq v_i$ which are p_{α}-homogeneous.

Put $H(s^\frown 0) = v'_0$, $H(s^\frown 1) = v'_1$.

It is clear that conditions (i),(ii) are satisfied and every branch of $H''T$ is a base of an ω-complete Ramsey ultrafilter. The corresponding branch of T is a function $F: \Omega \rightarrow \{0,1\}$ and different branches produce different ultrafilters. But the class of all such F is uncodable and this finishes the proof.

Let now $\mathfrak{M}$ be a rich ultrafilter with nucleus X. It is easy to see that the class

$$\mathfrak{M}_X = \{X \cap u; u \in \mathfrak{M}\}$$

is a non trivial ultrafilter on the countable class X. If Y is another countable class, $F: X \to Y$ is a bijection and $F \subseteq f$, then $f''\mathfrak{M}$ is rich with nucleus Y and $f''\mathfrak{M} \cong \mathfrak{M}$. Hence, we may assume that all rich ultrafilters have a common nucleus, say FN. Then we denote by $\dot{\mathfrak{M}}$ the ultrafilter $\{u \cap FN; u \in \mathfrak{M}\}$ on FN.

The mathematics we can do in AST on FN (or any countable class), is exactly the mathematics we can do in ZFC + CH on ω. This is easily seen by a simple comparison of the axioms of the two theories. In particular, all notions and facts developed for the ultrafilters on ω are reasonable and valid for the ultrafilters on FN. Therefore minimal and Ramsey ultrafilters not only are meaningfull for a countable class but moreover they coincide (cf. [Bl], § 10, Th. 7).

<u>Lemma 2.10</u>. Let $\mathfrak{M}$, $\mathfrak{N}$ be rich ultrafilters (with nucleus FN). Then

(i) $\dot{\mathfrak{M}} = \dot{\mathfrak{N}} \leftrightarrow \mathfrak{M} = \mathfrak{N}$

(ii) $\mathfrak{M} \leq \mathfrak{N} \leftrightarrow \dot{\mathfrak{M}} \leq \dot{\mathfrak{N}}$

(iii) $\mathfrak{M}$ is minimal (Ramsey) iff $\dot{\mathfrak{M}}$ is minimal (Ramsey). Hence if $\mathfrak{M}$ is rich, $\mathfrak{M}$ is minimal iff it is Ramsey.

<u>Proof</u>. (i) Obviously $\mathfrak{M} = \{u; (\exists Y \in \dot{\mathfrak{M}})(Y \subseteq u)\}$, i.e. $\dot{\mathfrak{M}}$ is a kind of base for $\mathfrak{M}$

(ii) By the convention that $\mathfrak{M}$, $\mathfrak{N}$ have common nucleus FN we may suppose that for every f with $dom(f) \in \mathfrak{M}$, $f''FN \subseteq FN$. If $f''\mathfrak{N} = \mathfrak{M}$ and $F = f \restriction FN$ then $F''\dot{\mathfrak{N}} = \dot{\mathfrak{M}}$. Conversely if $F''\dot{\mathfrak{N}} = \dot{\mathfrak{M}}$

and $F \subseteq f$, then $f''\mathfrak{N} = \mathfrak{M}$.

(iii) By the prolongation axiom the properties "minimality" and "to be Ramsey" can be transferred easily from countable classes to sets extending them and vice-versa.

It follows from the preceding lemma that the class of rich ultrafilters ordered by $\leq$ is isomorphic to the class $\beta\omega$ of nontrivial ultrafilters on ω ordered by $\leq$. A thorough study of the latter can be found in [B1].

An interesting subclass of $\beta\omega$ are the so-called P-points. For the ordering of P-points see [B2].

An ultrafilter $\mathfrak{M}$ on FN is a <u>P-point</u>, if for every $F: FN \longrightarrow FN$, there is a class $Y \in \mathfrak{M}$ such that $F \restriction Y$ is either constant or finite-to-one.

(It is easy to see that this definition transferred to ultrafilters on Sd_V is equivalent to the definition of minimal ultrafilters.)

Let $\mathfrak{M}$ be rich. We call $\mathfrak{M}$ P-point if $\dot{\mathfrak{M}}$ is a P-point. Since every minimal $\dot{\mathfrak{M}}$ is a P-point , clearly every minimal (rich) ultrafilter is a P-point. There are P-points on FN which are not minimal (cf. [B2] Th.9).

<u>Lemma 2.11</u>. Let $\mathfrak{M}$ be rich. If $\dot{\mathfrak{M}}$ is a proper (not minimal) P-point then $\mathfrak{M}$ is a proper P-point.

<u>Proof</u>. By assumption for every $F: FN \longrightarrow FN$ there is a $Y \in \dot{\mathfrak{M}}$ such that $F \restriction Y$ is either constant or finite-to-one and there is some $G: FN \longrightarrow FN$ such that for all $Y \in \dot{\mathfrak{M}}$ $G \restriction Y$ is neither constant nor one-to-one.

Let $G \subseteq g$. Then $g \restriction u$ is not constant for any $u \in \mathfrak{M}$ (otherwise $G \restriction u \cap FN$ would be constant) nor one-to-one (otherwise $G \restriction u \cap FN$

would be one-to-one).

Some interesting facts (established in [B1],[B2]) concerning the ordering of ultrafilters and P-points and which might be related to analogous facts for endomorphic universes, are the following:

Fact 1. Every increasing sequence of ultrafilters in ω has an upper bound.

Fact 2. Every decreasing sequence of P-points has a lower bound which is a P-point.

Fact 3. There is a P-point such that no minimal ultrafilter lies below it. (This is announced in [B2] to have been proved independently by R.A. Pitt and M.E. Rudin.)

Let A be an endomorphic universe. A universe B is said to be a successor universe of A if $A \subsetneqq B$ and there is no universe C such that $A \subsetneqq C \subsetneqq B$. It follows from Lemma 1.4 of [T] that B is a successor universe of A iff there are $d \in B - A$ and $\mathfrak{M}$ minimal such that $A[d] = B$ and F, $\mathfrak{M}$, d are coherent, where $F''V = A$.

Fact 3 implies the following.

Lemma 2.12. There is a universe A having no successor universe.

Proof. By fact 3 there is a (rich) ultrafilter $\mathfrak{M}$ having no minimal ultrafilter below it. There are d, F such that F, $\mathfrak{M}$, d are coherent and $F''V[d] = V$ (cf. [S - V]). Let $A = F''V$ and suppose B is the successor for A. Then, there are $d_1 \in B - A$ and $\mathfrak{M}_1$ minimal such that $A[d_1] = B$ and F, $\mathfrak{M}_1$, d_1 are coherent. It follows from 2.3 that $\mathfrak{M}_1 \leq \mathfrak{M}$ and this is a contradiction.

An immediate corollary is the following.

Corollary 2.13. There is a class of endomorphic universes linearly and densely ordered by inclusion.

The following is a generalization of the last but two theorems of [S - V] which wr repeatedly refer to.

Theorem 2.14. Let A be an end.universe and let $\mathfrak{M}$, d be such that $d \in A$ and 0, $\mathfrak{M}$, d are coherent. Then there is an endomorphism F such that F, $\mathfrak{M}$, d are coherent and F"V[d] = A.

Proof. Let F_0 be an endomorphism such that F_0, $\mathfrak{M}$, d are coherent and let F_1, G be such that F_1"V = F_0"V[d] and G"V = A. The elements $F_1^{-1}(d)$, $G^{-1}(d)$ are connected by the similarity $G^{-1} \circ F_1$, hence there is an automorphism F_2 such that $F_2(F_1^{-1}(d)) = G^{-1}(d)$. Put $H = G \circ F_2 \circ F_1^{-1}$ and $F = H \circ F_0$. Then one easily checks that F, $\mathfrak{M}$, d are coherent and F"V[d] = A.

Corollary 2.15. For every endomorphic universe A there is a $B \subsetneqq A$ such that A is a successor of B. More generally, for every A there is a decreasing sequence of universes $(A_n)_{n \in FN}$ such that $A_0 = A$ and for every n, A_n is a successor universe of A_{n+1}.

Proof. In view of 2.14 and 1.4 of [T] given A it suffices to find $d \in A$ and minimal $\mathfrak{M}$ such that 0, $\mathfrak{M}$, d are coherent. The latter means that d belongs to all classes of $\mathfrak{M}$ defined by formulas of FL (i.e. parameter-free). If $\varphi_n(x)$ is an enumeration of all these formulas then $(\exists x)(\forall n)\varphi_n(x)$. Since A is a universe, we get $(\exists x \in A)(\forall n)\varphi_n(x)$. This proves the first claim, from which the second comes immediately.

Corollary 2.16. There is a maximal universe A such that $E_A(X) = X$ for every countable $X \subseteq A$.

Proof. Take by 2.9 an ω-complete minimal $\mathfrak{M}$ and F, d such that F, $\mathfrak{M}$, d are coherent and F"V[d] = V. Then the universe

$A = F''V$ is maximal (by 1.4 of [T]) and from 2.1 (ii) we have

$$E_A(X) = E_V(X) = X \text{ for all countable } X \subseteq A.$$

§ 3. More on Ramsey ultrafilters. Given an ultrafilter $\mathfrak{M}$, put

$$\mathfrak{M}^{(2)} = \{u^{(2)}; u \in \mathfrak{M}\}.$$

Since $u^{(2)} \cap v^{(2)} = (u \cap v)^{(2)}$, $\mathfrak{M}^{(2)}$ is a filter-base on Sd_V. The following characterization of Ramsey ultrafilters is immediate from the definition.

Lemma 3.1. $\mathfrak{M}$ is Ramsey iff $\mathfrak{M}^{(2)}$ is an ultrafilter-base.

Fix a definable linear ordering $\prec$ of V and identify each two-element set $\{x,y\}$ with the pair $\langle x,y \rangle$ such that $x \prec y$. Let

$$\Delta = \{\langle x,x \rangle; x \in V\}, \quad A = \{\langle x,y \rangle; x \prec y\}, \quad B = \{\langle x,y \rangle; y \prec x\}.$$

Then $X^{(2)} = X^2 \cap A$, hence

$$\mathfrak{M}^{(2)} = \{u^2 \cap A; u \in \mathfrak{M}\}.$$

Lemma 3.2. For every ultrafilter $\mathfrak{M}$, the filter generated by the base $\mathfrak{M} \times \mathfrak{M} = \{u^2; u \in \mathfrak{M}\}$ is contained in at least three ultrafilters. It is contained in exactly three iff $\mathfrak{M}$ is Ramsey.

Proof. The three classes $\mathfrak{M} \times \mathfrak{M} \cup \{\Delta\}$, $\mathfrak{M} \times \mathfrak{M} \cup \{A\}$, $\mathfrak{M} \times \mathfrak{M} \cup \{B\}$ can be extended to non-trivial ultrafilters which are apparently distinct. Now $\mathfrak{M} \times \mathfrak{M} \cup \{\Delta\}$ is an ultrafilter-base since it generates the ultrafilter $F''\mathfrak{M}$ where $F(x) = \langle x,x \rangle$ $\forall x \in V$.

On the other hand, $\mathfrak{M} \times \mathfrak{M} \cup \{A\}$ generates $\mathfrak{M}^{(2)} = \{u^2 \cap A; u \in \mathfrak{M}\}$ which, as we remarked earlier, is an ultrafilter iff $\mathfrak{M}$ is Ramsey. Similar considerations hold for $\mathfrak{M} \times \mathfrak{M} \cup \{B\}$ if we identify the set $\{x,y\}$ with the pair $\langle x,y \rangle$, $y \prec x$.

Lemma 3.3. Let $\mathfrak{M}$ be Ramsey. Then (i) $\mathfrak{M}^{(2)}$ is not minimal.

(ii) If $\mathfrak{M}$ is ω-complete (rich), then $\mathfrak{M}^{(2)}$ is ω-complete (rich).

Proof. (i) Since $\mathfrak{M}^{(2)} = \{u^2 \cap A; u \in \mathfrak{M}\}$, we have $\mathfrak{M} \times \mathfrak{M} \subseteq \mathfrak{M}^{(2)}$ and if P_1 is the projection to the first coordinate, then $P_1''(\mathfrak{M} \times \mathfrak{M}) = P_1'' \mathfrak{M}^{(2)} = \mathfrak{M}$. This means that $\mathfrak{M} \leq \mathfrak{M}^{(2)}$. Moreover for every $u \in \mathfrak{M}$ P_1 cannot be 1 - 1 on $u^2 \cap A$, hence $\mathfrak{M} < \mathfrak{M}^{(2)}$.

(ii) Let $\mathfrak{M}$ be ω-complete and $(u_n^{(2)})_n$ be a sequence of elements of elements of $\mathfrak{M}^{(2)}$. Then there is some $u \in \mathfrak{M}$ such that $u \subseteq \bigcap_{\mathfrak{M}} u_n$, whence

$$u^{(2)} \subseteq (\bigcap u_n)^{(2)} = \bigcap_{\mathfrak{M}} u^{(2)}.$$

Let $\mathfrak{M}$ be rich with nucleus X and let $p \supseteq X^{(2)}$. Consider the partition $\{p, V^{(2)} - p\}$. There is a $u \in \mathfrak{M}$ such that either $u^{(2)} \subseteq p$ or $u^{(2)} \subseteq V^{(2)} - p$. If $u^{(2)} \subseteq V^{(2)} - p$ then $u^{(2)} \cap X^{(2)} = \emptyset$, hence $u \cap X \mathrel{\hat{\lesssim}} 1$ which is a contradiction. Therefore $u^{(2)} \subseteq p$ and this means that $p \in \mathfrak{M}^{(2)}$. Thus $X^{(2)}$ is a nucleus for $\mathfrak{M}^{(2)}$.

Corollary 3.4. Let $\mathfrak{M}$ be Ramsey and F be an endomorphism such that F, $\mathfrak{M}^{(2)}$ are compatible. Then the universe F"V = A has at least two successor universes B_1, B_2. Moreover $B_1 \cap B_2 = A$.

Proof. Let F, $\mathfrak{M}^{(2)}$, $\{d_1, d_2\}$ be coherent. If P_1, P_2 are the projections to the first and second coordinate then $P_1'' \mathfrak{M}^{(2)} = P_2'' \mathfrak{M}^{(2)} = \mathfrak{M}$. Hence F, $\mathfrak{M}$, d_1, F, $\mathfrak{M}$, d_2 are coherent. Put $B_1 = A[d_1]$, $B_2 = A[d_2]$, B_1, B_2 are successors because $\mathfrak{M}$ is minimal and $B_1 \cap B_2 = A$ because $d_1 \mathrel{\overline{\overline{\wedge}}} d_2$ (cf. [T], 2.3).

In ZF the properties "to be minimal" and "to be Ramsey" are equivalent only for uniform ultrafilters.

Since the ultrafilters considered here contain sets which are not cofinal to the universe, it is likely that there exist minimal non-Ramsey ultrafilters.

The question is open to us but we can find some conditions implying the existence of such ultrafilters.

Let B be a (non-trivial) filter-base or subbase on Sd_V. We say that a class Z <u>extends</u> B if $B \cup Z$ still generates a non trivial filter.

With no loss of generality we suppose that for every set-definable F, dom(F) = V.

B is called <u>minimal</u> if for every $F \in Sd_V$ there is a set u such that $\{u\}$ extends B and either $F \upharpoonright u$ is 1 - 1 or F"u is finite.

<u>Lemma 3.5</u>. Let B be a minimal subbase. If Z is an at most countable class and extends B then $B \cup Z$ is minimal.

<u>Proof</u>. Let $Z = \{u_1, u_2, \dots\}$ and $B_1 = B \cup Z$. Pick a function $F \in Sd_V$. By assumption there is u such that $\{u\}$ extends B and $F \upharpoonright u$ is 1 - 1 if F"u is finite. We have to find v with $\{v\}$ extending B_1 and $F \upharpoonright v$ 1 - 1 or F"v finite.

<u>Case 1</u>. F"u is finite. If $u \cap (\bigcap Z)$ is infinite then clearly u extends B_1. Suppose $u \cap (\bigcap Z)$ is finite. Without loss of generality assume that $u \cap (\bigcap Z) = \emptyset$.

If $(\exists y)(F^{-1}(y) \cap (\bigcap Z)$ is infinite), then there is some $u' \subseteq \subseteq \bigcap Z$ such that $F \upharpoonright u'$ is constant. Put then $v = u \cup u'$.

Suppose $(\forall y)(F^{-1}(y) \cap (\bigcap Z)$ is finite. It follows easily from the prolongation axiom that

(1) $$(\exists n, k \in FN)(\forall y)(F^{-1}(y) \cap u_1 \cap \dots \cap u_n \precsim k).$$

Let n, k be the natural numbers asserted by (1). Put $w = u_1 \cap \cap \dots u_n$. Clearly w extends B_1. The sets $F^{-1}(y) \cap w$ form a partition of w into sets, each containing at most k elements. Decompose w into at most k sets $v_1, \dots, v_k$ such that $F \upharpoonright v_i$ is 1 - 1

$\forall i = 1,\dots,k$. Then some of the v_i extends B_1 and this is as required.

Case 2. $F \upharpoonright u$ is 1 - 1. The non-trivial subcase is again when $u \cap (\bigcap Z) = \emptyset$. If $F''(\bigcap Z)$ is finite, by the prolongation axiom we have that for some n $F''(u_1 \cap \dots \cap u_n)$ is finite and the set $v = u_1 \cap \dots \cap u_n$ extends B_1.. Suppose $F''(\bigcap Z)$ is infinite. Let $E = F''(\bigcap Z)$. There is some $X \in Sd_V$ such that $E \cap X$, $E \cap (V - X)$ are both infinite. The sets $u_1 = u \cap F^{-1}{}''X$, $u_2 = u \cap F^{-1}{}''(V - X)$ is a partition of u. Hence some of them, say u_1, extends B. Put $Y = (\bigcap Z) \cap F^{-1}{}''(V-X)$. Then $F''Y = E \cap (V - X)$ and $F''Y \cap F''u_1 = \emptyset$. Thus $F''Y$ is infinite and Y is a π-class, hence we can find $w \subseteq Y$ such that $F \upharpoonright w$ is 1 - 1. Since $F''w \cap F''u_1 = \emptyset$, $F \upharpoonright w \cup u_1$ is 1 - 1 and $\{w \cap u_1\}$ extends B_1.

Corollary 3.6. A filter-base B containing sets is minimal iff it can be extended to a minimal ultrafilter.

Proof. " $\Leftarrow$ " is trivial. Suppose B is minimal . Then B can be extended by transfinite induction to filter bases $\mathfrak{M}_\alpha$ such that $\mathfrak{M}_0 = B$ and $\mathfrak{M}_\alpha$ is taken by $\bigcup\{\mathfrak{M}_\beta ; \beta < \alpha\}$ by adding a set u_α extending $\bigcup\{\mathfrak{M}_\beta ; \beta < \alpha\}$ and such that $F_\alpha \upharpoonright u_\alpha$ is 1 - 1 or $F_\alpha'' u_\alpha$ is finite. By the previous lemma each $\mathfrak{M}_\alpha$ is minimal and this guarantees the induction step of the construction.

Let $P = \{P_1, P_2\}$ be a partition of $V^{(2)}$. We are interested in partitions with the following property:

(A) For any finite sequence of set-definable classes $X_1,\dots,X_n$, such that $X_1 \dots X_n = V - u$, u finite, some of the X_i is not P-homogeneous. Given P, let $B_p = \{X; V - X$ is P-homogeneous$\}$.

The following is obvious:

Lemma 3.7. The partition P satisfies (A) iff B_p generates a non-trivial filter on Sd_V. Then every ultrafilter extending B_p is non-Ramsey.

From 3.6 and 3.7 we get immediately:

Corollary 3.8. There exists a non-Ramsey minimal ultrafilter iff there exists a partition P such that P satisfies (A) and B_p is minimal.

It is not hard to find partitions satisfying (A) (e.g. $P_1 = \{\{x,y\}; x \cap y = 0\}$, $P_2 = \{\{x,y\}; x \cap y \neq 0\}$ is such) but checking of minimality of B_p seems really hard. However, the following holds.

Lemma 3.9. Any partition for which no proper class is homogeneous satisfies the conditions of 3.7.

Proof. Let P be such a partition. Property (A) is obviously true for P.

Let F be a function with dom(F) = V. We claim that there is a proper class $X \in Sd_V$ such that $F \upharpoonright X$ is constant or 1 - 1 . In fact, if F is not constant on any proper class, then all $F^{-1}(y)$ are sets forming a partition of V. Choose a set-definable selector Y for the classes $F^{-1}(y)$. Then Y is proper and $F \upharpoonright Y$ is 1 - 1.

By assumption B_p contains only cosets, hence the proper class X extends B_p.

We close by stating the questions for the existence of

1) non-Ramsey minimal (ω-complete ?) ultrafilters

2) partitions for which no proper classes are homogeneous.

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