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TWO EXAMPLES OF PSEUODO-RADIAL SPACES
Petr SIMON*, Gino TIRONI**)

Abstract: Using an Ostaszewski-type construction, we prove in ZFC the existence of
a) Hausdorff pseudo-radial space of countable tightness which is not sequential,
b) Hausdorff pseudo-radial space in which tightness and quasi-character differ.

Key words and phrases: Pseudo-radial space, sequential space, tightness, quasi-character.

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Secondary 54G20, 54D55

Introduction. Pseudo-radial or chain-net spaces were introduced by H. Herrlich in 1967 [H]. They are a natural characterization of both linearly ordered and sequential spaces. (See for example [A],[MW].) Recall that a space X is pseudo-radial, if for each non-closed \( M \subseteq X \) there is some \( x \in \overline{M} - M \) and a (countable or transfinite) sequence \( \{ x_\alpha : \alpha < \omega \} \subseteq M \) converging to \( x \), i.e. each neighbourhood of \( x \) contains all \( x_\alpha \)'s beginning from some \( \omega \) on.

If "there is some \( x \in \overline{M} - M \)" is replaced by "for each \( x \in \overline{M} - M \)" in the above definition, then the space is called radial or Fréchet chain-net.

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When investigating the cardinality properties of pseudo-radial and radial spaces, A.V. Arhangel’skii, R. Isler and G. Tironi introduced a new cardinal invariant, so-called quasi-character, as follows.

\[ q^\mathbb{Q}(X) = \min \{ \tau : (\forall A \subseteq X)(\forall \sigma \in \mathcal{P}(A))(|\mathcal{G}| \leq \tau \land \forall x \in \bigcup\mathcal{G} \land (\forall F \in \mathcal{G})(x \notin \overline{F})) \}. \]

They also proved that for \( T_1 \) radial spaces, \( q^\mathbb{Q}(X) = t(X) \), leaving the case of pseudo-radial spaces open. The best result in this direction says that \( q^\mathbb{Q}(X) = t(X) \) for a pseudo-radial space \( X \) provided \( t(X) \) is a successor cardinal and GCH is assumed ([AIT]).

We shall construct assuming ZFC only a Hausdorff pseudo-radial space \( Z \) with \( \omega = q^\mathbb{Q}(X) < t(X) \).

Using essentially the same construction, we shall also disprove the old conjecture that a pseudo-radial Hausdorff space with countable tightness is necessarily sequential. Here, of course, a plenty of counterexamples was published by various authors before ([0], [JKR] to mention few), but as far as we know—all of them depended on some additional axiom of set-theory.

The construction. Let \( \mathfrak{a}_0 \) be a cardinal number, define by induction \( \mathfrak{a}_{n+1} = 2^{\mathfrak{a}_n} \), \( \mathfrak{a} = \sup \{ \mathfrak{a}_n : n \in \omega \} \). Equip each \( \mathfrak{a}_n \) with the discrete topology and denote by \( M \) the Tychonoff product \( \prod_{n \in \omega} \mathfrak{a}_n \). Then \( M \) is a complete metric zero-dimensional space, \( w(M) = \mathfrak{a} \), \( |M| = 2^\mathfrak{a} = \mathfrak{a}^\omega \). Further, if \( C \subseteq M \) and \( |C| > \mathfrak{a} \), then \( |\overline{C}| = 2^\mathfrak{a} \).

The last assertion needs, perhaps, a proof.

Denote by \( A_n \) the set \( \{ \xi \in \mathfrak{a}_n : |\varpi^{-1}(\xi) \cap \overline{C}| > \mathfrak{a} \} \). Then

\[ C = \bigcup_{n \in \omega} \varpi^{-1}(\mathfrak{a}_n - A_n) \cup \bigcap_{n \in \omega} A_n \cap \overline{C} \] Since for each \( n \in \omega \), \( |\varpi^{-1}(\mathfrak{a}_n - A_n) \cap \overline{C}| \leq \mathfrak{a}_n \cdot \mathfrak{a} = \mathfrak{a} \), we have \( |\bigcap_{n \in \omega} A_n \cap \overline{C}| > \mathfrak{a} \). But this means that for each \( \mathfrak{a} < \mathfrak{a} \) there is some \( n \in \omega \) with \( |A_n| > \mathfrak{a} \), otherwise \( |\bigcap_{n \in \omega} A_n \cap \overline{C}| < \mathfrak{a} \) would contradict the assumption.
\(|c| > \aleph . So we have proved the following:

If \( \kappa < \aleph \), and if \(|c| > \aleph \), then there is some \( n \in \omega \) such that

\[ |\{ \xi \in \omega_n : |\pi_n^{-1}(\{\xi\}) \cap c| > \aleph \}| > \kappa . \]

The standard branching argument works now: for each \( n \in \omega \) and for each \( \varphi \in \prod \{ \omega_i \cup \{ \aleph \} \} \), there is a closed \( C_{\varphi} \subseteq c \) such that

\[ |C_{\varphi}| > \aleph , \ C_{\varphi} \neq c \quad \text{if and only if} \quad \varphi \not< \psi , \ C_{\varphi} \cap \psi = \emptyset \ \text{if and only if there is} \ i \in \text{dom} \ \varphi \cap \text{dom} \ \psi \ \text{such that} \ \varphi(i) \not= \psi(i). \]

Indeed, if \( \varphi \in \prod \{ \omega_i \cup \{ \aleph \} \} \) and \( C_{\varphi} \) is known (\( C_{\varphi} = c \) of course), then there is some \( \kappa \) with

\[ |\{ \xi \in \omega_n : |\pi_n^{-1}(\{\xi\}) \cap C_{\varphi}| > \aleph \}| > \aleph_n . \]

So we can select \( C_{\varphi} \cap \eta_i \) for \( \eta_i \in \omega_n \) to be a member of the disjoint family

\[ \{ \pi_m^{-1}(\{\xi\}) \cap C_{\varphi} : \xi \in \omega_m \ \& \ |\pi_m^{-1}(\{\xi\}) \cap C_{\varphi}| > \aleph \} . \]

Since, obviously, for each \( f \in \prod \{ \omega_i \cup \{ \aleph \} : \cap \text{dom} \ \varphi \cap \text{dom} \ \psi \neq \emptyset \), we have \(|c| \geq \aleph^\omega \) and, by our choice of \( \aleph \), \( \aleph^\omega = 2^\aleph \).

The above are the properties of \( M \) which we shall need further.

Denote by \( \varphi \) the metric topology of \( M \) and fix some clopen base \( \mathcal{B} \) for \( M \), \( |\mathcal{B}| = \aleph . \)

Enumerate all subsets of \( M \) of cardinality \( \aleph \) the closure of which is of cardinality \( 2^\aleph \) as \( \{ t_\kappa : \kappa < 2^\aleph \} \) in such a way that each set is listed \( 2^\aleph \) times. Then for each \( t_\kappa \) select a point \( x_\kappa \in \overline{t_\kappa} \) and a convergent sequence \( S_\kappa \), such that \( \lim S_\kappa = x_\kappa \), \( S_\kappa \subseteq \overline{t_\kappa} \) and for \( \alpha \not= \beta \), \( x_\alpha \neq x_\beta \). This is clearly possible for, by the previous choice, each \( t_\kappa \) has \( 2^\aleph \) accumulation points, so there is still one among them distinct of all \( x_\beta \), \( \beta < \kappa . \)

Let \( X = \{ x_\kappa : \kappa < 2^\aleph \} \) and denote again by \( \varphi \) the original topology of \( M \) restricted to \( X . \)

We shall construct a new topology \( \tau \) on \( X \) in Ostaszewski
style. Let \( X_{\alpha} = \{ x_\beta : \beta < \alpha \} \) for \( \alpha < 2^\omega \). Define \( \tau_{\alpha} \) to be the discrete topology on \( X_{\alpha} \). Suppose \( (X_{\gamma}, \tau_{\gamma}) \) have been defined for all \( \alpha < \beta \) where \( \beta < 2^\omega \). The inductive assumptions are as follows:

(i) For each \( \alpha < \gamma < \beta \), \( (X_{\alpha}, \tau_{\alpha}) \) is a subspace of \( (X_{\gamma}, \tau_{\gamma}) \).

(ii) For each \( \alpha < \gamma < \beta \), \( X_{\alpha} \) is an open subset of \( (X_{\gamma}, \tau_{\gamma}) \).

(iii) Each \( (X_{\alpha}, \tau_{\alpha}) \) is first-countable, locally compact, locally countable.

(iv) The topology \( \tau_{\alpha} \) is finer than \( \mathcal{P}(X_{\alpha}) \), for each \( \alpha < \beta \).

If \( \beta \) is a limit cardinal, let \( \tau_{\beta} = \bigcup \{ \tau_{\alpha} : \alpha < \beta \} \). Obviously \( (X_{\beta}, \tau_{\beta}) \) again satisfies (i) - (iv).

If \( \beta = \alpha + 1 \), we are to find a neighbourhood basis of \( x_{\infty} \).

There are two possibilities:

If \( |S_{\infty} \cap X_{\infty}| < \omega \), let \( x_{\infty} \) be isolated in \( X_{\beta} \), i.e. a neighbourhood basis of \( x_{\infty} \) is \( \{ x_{\infty} \} \).

If \( |S_{\infty} \cap X_{\infty}| = \omega \), select some clopen base of \( X_{\infty} \) in \( (M, \mathcal{P}) \), say \( \{ B_0 \supseteq B_1 \supseteq \ldots \supseteq B_n \supseteq \ldots \} \) such that for each \( n \), \( S_{\infty} \cap X_{\infty} \cap (B_n - B_{n+1}) \neq \emptyset \), and select \( y_n \in S_{\infty} \cap X_{\infty} \cap (B_n - B_{n+1}) \).

Since, by our assumption, \( \tau_{\infty} \) is finer than \( \mathcal{P} \), \( B_n - B_{n+1} \) is an open neighbourhood of \( y_n \), so we can find a countable compact neighbourhood of \( y_n \), say \( U_n \), with \( U_n \subseteq B_n - B_{n+1} \). Fix this choice of \( U_n \)'s and define the neighbourhood base at \( x_{\infty} \) as

\[ \{ x_{\infty} \} \cup \bigcup \{ U_n \cap x_{\infty} : k \in \omega \} \].

It is again clear that (i) - (iv) hold for \( (X_{\beta}, \tau_{\beta}) \).

As might be expected, the desired topology \( \tau \) for \( X \) is

\[ \bigcup_{\alpha < 2^\omega} \tau_{\alpha} \].

Clearly, \( (X, \tau) \) is first-countable, locally compact, locally countable. The next property, being crucial, has to be proved: if \( C \) is closed in the topology \( \tau \) for \( X \), then either \( |C| \leq \omega \) or
$|C| = 2^\omega$.

Indeed, suppose $|C| > \omega$. Then $|\overline{C^0}| = 2^\omega$ and, since $w(M) = \omega$, there is a subset $T \subseteq C$, $|T| = \omega$, such that $\overline{T^0} = \overline{C^0}$. In particular, $|\overline{T^0}| = 2^\omega$.

Since $|T| = \omega$, there is some $\gamma < 2^\omega$ such that $T \subseteq X_\gamma$. The set $T$ appears $2^\omega$ times in our list, and in each occurrence $\omega > \gamma$ with $T_\omega = T$, the point $x_\omega$ belongs to $\overline{T^\omega}$. So $|\overline{T^\omega}| = 2^\omega$ and since $C$ was assumed to be closed in $\mathcal{C}$, $C \supseteq \overline{T^\omega}$.

Having passed the difficult part of the construction, choose a point $\omega$ not belonging to $X$ and define $Z = X \cup \{\omega\}$ with the neighbourhood base at $\omega$ consisting of all sets $\{\omega\} \cup (X - A)$, where $A \subseteq X$, $A$ is closed in $\mathcal{C}$, $|A| \neq \omega$. The space $Z$ is Hausdorff. This is trivial, since each point of $X$ has a countable compact neighbourhood.

The space $Z$ is pseudo-radial. Indeed, let $W \subseteq Z$, $\overline{W} \neq W$. If there is some $x \in X, x \in \overline{W} - W$, then there is a convergent sequence in $W$ with $x$ as its limit, by the first-countability of $(X, \mathcal{C})$. Otherwise $\overline{W} - W = \{\omega\}$, hence $W$ is closed in $(X, \mathcal{C})$ and $\omega$ is its accumulation point. According to our definition of topology on $Z$, $|W| > \omega$, hence $|W| = 2^\omega$ and for each neighbourhood $U$ of $\omega$, $|W - U| \leq \omega$. So any subset of $W$ of cardinality $\omega^+$ converges to $\omega$.

The tightness of $Z$ equals $\omega$. Indeed, if $W \subseteq X$ and $\omega \in \overline{W}$ then $|W| > \omega$. There is a set $T \subseteq W$, $|T| = \omega$ such that $\overline{T^0} \supseteq W$. But this implies that $|\overline{T^0}| = 2^\omega$, therefore $|\overline{T^\omega}| = 2^\omega$, too. But then $\omega \in \overline{T^\omega}$, therefore $t(Z) \leq \omega$. (Other points than $\omega$ are, of course, uninteresting.) On the other hand, $t(Z) \geq \omega$ for the trivial reason that if $W \subseteq X$, if $|W| < \omega$ then $|\overline{W^0}| < \omega$, too, so $\{\omega\} \cup (X - \overline{W^0})$ is a neighbourhood of $\omega$ disjoint with $W$.

It remains to consider two special cases.

1. Let $\omega_0 = 2$. In this case, the starting metric space is
nothing else than the Cantor set and the final space $Z$ is pseudo-radial, Hausdorff and $t(Z) = \omega$.

$Z$ is not sequential. Consider $\overline{xZ} - X$. This set contains the point $\infty$ only, and there is no sequence $\{s_n:n \in \omega \}$ converging to $\infty$ : notice that $\{s_n:n \in \omega \}$ should be a closed discrete subset of $(X,\tau)$ then, but in this case, $\{\infty\} \cup (X - \{s_n:n \in \omega \})$ is a neighbourhood of $\infty$ disjoint with it.

2. Let $\omega_0 = \omega$. We have $\omega > \omega$ in this case, and $Z$ is pseudo-radial, Hausdorff and $t(Z) = \omega$.

Yet $q\omega(Z) = \omega$. This is clear if one considers points from $X$ by the 1st countability of $(X,\tau)$.

Let us discuss the case $W \subseteq X$, $\infty \in \overline{W}$. Since $t(Z) = \omega$, there is some $T \subseteq W$, $|T| = \omega$, $\infty \in \overline{T}$. Making use of the fact that $\omega$ is a singular cardinal, find some $T_n \subseteq T$ such that $T = \cup \{T_n:n \in \omega \}$, and for each $n$, $|T_n| < \omega$. Then for each $n$, $|T_n| < \omega$, too, so $\infty \notin \overline{T_n}$. So $q\omega(Z) = \omega$.

Added in proof. After this paper was completed we learned from I. Juhász that he and W. Weiss found independently examples of pseudo-radial spaces with similar properties. We do not know any details of their proof.

References


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