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Angelo Bella

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FREE SEQUENCES IN PSEUDO-RADIAL SPACES

Angelo BELLA

Abstract. It is proved that for a pseudo-radial regular space X the inequality $t(X)\leq F(X)$ holds. Sufficient conditions in order to have $t(X) = F(X)$ are given and some consequences derived. It is also proved that for a regular space X the inequality $q\chi(X)\leq F(X)$ holds.

Key words: Pseudo-radial space, tightness, free sequence.

Classification: Primary 54A25
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1. Introduction and basic definitions. Pseudo-radial or chain-net spaces were first introduced by Herrlich in 1967 (see [7]), as a generalization of sequential spaces. Recently many authors have studied this class of spaces particularly in connection with the theory of cardinal invariants. One of the results in this direction, obtained by Jané, Meyer, Simon, Wilson in 1981 (see [8]), is that if X is a pseudo-radial Hausdorff space then $t(X)\leq S(X)$, where t(X) and S(X) are respectively the tightness and spread of X.

In this paper we improve this result for a pseudo-radial regular space X, proving that $t(X)\leq F(X)$, where $F(X) = \sup\{\tau$ such that there exists in X a free sequence of length $\tau\}$. Then we give sufficient conditions in order to have $t(X) = F(X)$. As an application of these results we generalize a theorem, recently proved by Arhangel'skii-Isler and Tironi, concerning the relation between $t(X)$ and $q\chi(X)$, where $q\chi(X)$ is the quasi-character of the space X (see Def. 4 below). At the end of the paper it is also proved that for a regular space, without any assumption of pseudo-radiality, $q\chi(X) \leq F(X)$.

For terminology and definitions not explicitly mentioned here we refer to [5] and [9].

All cardinal numbers are supposed to be initial ordinals and τ^+ will denote the successor cardinal of the cardinal number τ. The cardinality of a set S is denoted by $|S|$. Given a topological space X, a set $A \subseteq X$ and a cardinal number τ we put $[A]_\tau = \bigcup \overline{B}$ such that $B \subseteq A$ and $|B| \leq \tau$. All spaces considered here are supposed to be T_1. We recall the following:

Definition 1. A space X is said to be pseudo-radial or chain-net, if for every non-closed set $A \subseteq X$ there exist a point $x \in \overline{A} \setminus A$ and a (transfinite) λ-sequence $(x_\alpha : \alpha < \lambda)$ in A converging to x.

Without any loss of generality we can always assume that, in the preceding definition, the length of the sequence is a regular cardinal.

There are other equivalent definitions of a pseudo-radial space. One of the most interesting, due to Arhangel´skii (see [2]), is the following:

Definition 1 . A space X is pseudo-radial if and only if for every non-closed set $A \subseteq X$ there exist a point $x \in \overline{A} \setminus A$ and a subset B of A of regular cardinality such that for every neighbourhood U of x $|B \setminus U| < |B|$. If for a topological space X every point $x \in \overline{A} \setminus A$ has the properties described in Def. 1 or 1´, the space is called radial or Fréchet chain-net.

In [4] Arhangel´skii, Isler and Tironi introduced a new class of spaces that lies between pseudo-radial and radial spaces. These spaces are called almost-radial and in certain sense, they generalize sequential spaces better than pseudo-radial spaces.

Definition 2. A space X is said to be almost-radial if for every non-closed set A there exist a point $x \in \overline{A} \setminus A$ and a (transfinite) λ-sequence $(x_\alpha : \alpha < \lambda)$ in A converging to x and such that x does not belong to the closure of every initial segment of the λ-sequence.

Definition 3. For a pseudo-radial space X , $\sigma_C(X)$ is the least cardinal number such that for every non-closed set A there is a sequence in A of length $\lambda \leq \sigma_C(X)$ converging to a point $x \in \overline{A} \setminus A$.

Definition 4. Let X be a topological space, $A \subseteq X$ and $x \in \overline{A} \setminus A$. We call the primitive quasi-character of x with respect to A, $pq\chi(x,A)$, the least cardinal number τ such that there exists a family γ of subsets of A such that $|\gamma| \leq \tau$, $x \notin \overline{B}$, for every B in γ , but $x \in \overline{\bigcup \gamma}$. We call the quasi-character at the point $x \in X$, the cardinal number $q\chi(x,X) = \sup \{ pq\chi(x,A) : A \subseteq X$ and $x \in \overline{A} \setminus A\}$ and the quasi-character of the space X, the cardinal number $q\chi(X) = \sup \{ q\chi(x,X) : x \in X\}$. For a discrete space X we put $q\chi(X) = 1$.

2. Free sequences in pseudo-radial spaces.

Theorem 1. If X is a pseudo-radial regular space then $t(X) \leq F(X)$.

Proof. Let $\tau < t(X)$. By the definition of tightness there exists a set $A \subseteq X$ such that $[A]_\tau = A \neq \overline{A}$. Since X is pseudo-radial there exist a point $p \in \overline{A} \setminus A$ and a linearly ordered set $S \subseteq A$ such that every neighbourhood of p contains a final segment of S. For every ordinal $\alpha < \tau^+$ we construct transfinite sequences of points

$x_\alpha \in S$ and open sets V_α satisfying the following properties:

(i) if $\alpha < \beta < \tau^+$ then $x_\alpha < x_\beta$;

(ii) for every $\alpha < \tau^+$ $\overline{\{x_\beta : \beta < \alpha\}} \subseteq V_\alpha$;

(iii) $S^{x_\alpha} \cap V_\alpha = \emptyset$, where S^{x_α} denotes the final segment of S with initial element x_α.

We proceed by transfinite induction. Let us suppose we have just constructed x_β and V_β for every $\beta < \alpha$, where $\alpha < \tau^+$. Since $|\{x_\beta : \beta < \alpha\}| \leq \tau$ we have $p \notin \overline{\{x_\beta : \beta < \alpha\}}$, and then there exists a closed neighbourhood U of p such that $U \cap \overline{\{x_\beta : \beta < \alpha\}} = \emptyset$. U contains a final segment of S. Let x_α be the initial element of this segment and put $V_\alpha = X - U$. It is easy to verify that x_α and V_α have the required properties. Now we show that the sequence $(x_\alpha)_{\alpha < \tau^+}$ is free. From the construction made above it follows that for every $\alpha < \tau^+$, $\overline{\{x_\beta : \beta < \alpha\}} \subseteq V_\alpha$ and $\{x_\beta : \beta \geq \alpha\} \subseteq S^{x_\alpha}$. From (iii) we have $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset$ and this proves that the sequence is free. By the definition of F(X) we have $\tau^+ \leq F(X)$ and, since τ is an arbitrary cardinal number less than t(X), we can conclude that $t(X) \leq F(X)$.

Corollary 1. If X is an almost-radial regular space then $\sigma_C(X) \leq F(X)$.

Proof. It follows from Theorem 1 and from the fact that, for an almost radial space, $\sigma_C(X) = t(X)$ (see [4] Th. 2.9).

Corollary 2. If X is an almost-radial regular space and $F(X) \leq \aleph_0$ the space X is sequential.

Question. Does Theorem 1 or Corollary 1 hold for Hausdorff spaces?

For a topological space it is important to know when $t(X) = F(X)$. A wellknown case is when the space is compact Hausdorff (see [1]).

Lemma 1. Let X be a topological space. If $L(X) < F(X)$, then $F(X) \leq t(X)$.

Proof. On the contrary, let us suppose $t(X) < F(X)$. Let $\lambda = \max\{t(X), L(X)\}$, we have $\lambda^+ \leq F(X)$, then there exists a free sequence $S = (x_\alpha)_{\alpha < \lambda^+}$ of length λ^+. Since $L(X) < \lambda^+$ and λ^+ is regular there must exist a complete accumulation point p for the set S. But $t(X) \leq \lambda$, so there exists a set $B \subseteq S$ such that $p \in \overline{B}$ and $|B| \leq \lambda$.

Since λ^+ is regular there exists $\alpha_0 < \lambda^+$ such that $B \subseteq \{x_\beta : \beta < \alpha_0\}$. Then, $p \in \overline{\{x_\beta : \beta < \alpha_0\}} \subseteq X - \overline{\{x_\beta : \beta \geq \alpha_0\}}$ and this contradicts the fact that p is a complete accumulation point of S.

Theorem 2. If X is a pseudo-radial regular space and $L(X) < F(X)$ then $t(X) = F(X)$.

Proof. It follows immediately from Theorem 1 and Lemma 1.

Theorem 3. If X is a pseudo-radial Lindelöf non discrete space, then $t(X) = F(X)$.

Proof. Since X is non discrete, we have $t(X) \geq \aleph_0$ and then if $t(X) < F(X)$, we must have $L(X) < F(X)$ in contrast with Lemma 1, so $F(X) \leq t(X)$ and the result follows from Theorem 1.

We recall that, as usual, a Lindelöf space is assumed to be regular.

As a consequence of Corollary 2 and Theorem 3 we obtain a characterization of sequential spaces in the class of almost-radial Lindelöf spaces.

Corollary 3. An almost-radial Lindelöf space X is sequential if and only if $F(X) \leq \aleph_0$.

In [3] Arhangel'skii, Isler and Tironi have studied relations between $t(X)$ and $q\chi(X)$ for a pseudo-radial space X. While it is known that, for almost-radial spaces, $t(X) = q\chi(X)$ (see [4] Th. 2.7), it is not clear if the same relation holds in general for pseudo-radial spaces.

In [3] several partial answers have been given. One of them is that under G.C.H. if X is a pseudo-radial compact Hausdorff space, then $t(X) = q\chi(X)$.

In this theorem (see [3] Th. 2.9), compactness needs only to guarantee that $t(X) = F(X)$, so we immediately obtain the following:

<u>Theorem 4</u>. Under G.C.H. if X is a pseudo-radial Lindelöf space then $t(X) = q\chi(X)$.

Proof. If X is not discrete, the theorem follows from Theorem 3 and Theorem 2.9 in [3], while if X is discrete, we have $t(X) = q\chi(X) = 1$.

To conclude, we give a result similar to Theorem 1 for the quasi-character that holds without any assumption of pseudo-radiality.

<u>Theorem 5</u>. If X is a regular space, then $q\chi(X) \leq F(X)$.

Proof. Let A be a non-closed subset of X, and $p \in \overline{A} - A$ such that $pq\chi(p,A) = \tau$. We construct for every ordinal $\alpha < \tau$ transfinite sequences of points $x_\alpha \in A$ and open sets V_α satisfying the following properties

(i) for every $\alpha < \tau$, $\overline{\{x_\beta : \beta < \alpha\}} \subseteq V_\alpha$;

(ii) for every $\alpha < \tau$ $\{p, x_\alpha\} \cap \bigcup_{\beta \leq \alpha} \overline{V}_\beta = \emptyset$.

We proceed by transfinite induction. Let $\alpha < \tau$ and suppose we have just constructed x_β and V_β for every $\beta < \alpha$. Since

$|\alpha| < \tau$, $p \notin \overline{V}_\beta$, for every $\beta < \alpha$, and $pq\chi(p,A) = \tau$, it follows that $p \notin \overline{\bigcup_{\beta<\alpha} \overline{V}_\beta}$. Then there exists a closed neighbourhood U of p such that $U \cap \overline{\bigcup_{\beta<\alpha} \overline{V}_\beta} = \emptyset$. Choose $x_\alpha \in \mathring{U} \cap A$ and put $V_\alpha = X-U$. It is easy to verify that x_α and V_α have the required properties. Now we show that the sequence $(x_\alpha)_{\alpha<\tau}$ is free. From the construction made above it follows that for every $\alpha < \tau$ $\overline{\{x_\beta: \beta < \alpha\}} \subseteq V_\alpha$ and $\{x_\beta: \beta \geq \alpha\} \cap \overline{V}_\alpha = \emptyset$. Thus, we have $\overline{\{x_\beta: \beta < \alpha\}} \cap \overline{\{x_\beta: \beta \geq \alpha\}} = \emptyset$ and this means that the sequence is free. We have $\tau \leq F(X)$ and by the choice of τ we can conclude that $q\chi(X) \leq F(X)$.

Remark. Theorems 1 and 5 together with Proposition 2.1 in [3] and Proposition 7 in [10] suggest that the behaviour of the quasi-character in a certain class of topological spaces C is similar to the tightness in the subclass of all pseudo-radial spaces belonging to C.

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Dipartimento di Matematica, Università di Catania, Città Universitaria, Viale A. Doria 6, Catania, Italy

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