Jan Reiterman

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ON LOCALLY SMALL BASED ALGEBRAIC THEORIES

J. REITERMAN

Abstract. Locally small based algebraic theories are well-known to include varietal theories (in particular, classical algebraic theories) as well as most non-varietal theories of nature. Some examples are presented to show that some theorems for varietal theories are no more valid for locally small based theories. E.g., a locally small theory cannot be in general reconstructed from its category of algebras, and the category of algebras for a locally small based theory need not be canonically algebraic.

Key words: Algebraic theory, varietal theory, locally small based theory, equational completeness, canonically algebraic categories, category of algebras.

Classification: 08A65, 08C05

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1. Introduction and preliminaries. The classical universal algebra deals with algebraic theories (Ω,E) where Ω is a set of finitary operation symbols and E is a set of equations for Ω-terms. We shall consider a more general case: both Ω and E are possibly proper classes and the operation symbols in Ω are possibly infinitary, i.e., the arities are arbitrary index sets. The most familiar examples of such theories are the theory of complete semilattices, of complete lattices and of complete Boolean algebras.

For instance, the theory of complete semilattices consists of an n-ary operation symbol $\bigvee_{i\in n}$ for every non-void set n and of equations of the form

$$\bigvee_{i \in k} x_{f(i)} = \bigvee_{j \in f[k]} x_j, \quad \bigvee_{i \in n} \Big(\bigvee_{j \in k_i} x_j \Big) = \bigvee_{j \in k} x_j, \quad \bigvee_{i \in \{0\}} x_i = x_0$$

where x_i are variables, $f: k \longrightarrow n$ is a map and $k = \bigcup_{i \in n} k_i$.

1.1. Terms and equations. If (Ω, E) is an algebraic theory, Ω-terms are defined recursively: For every set $\{x_i; i \in n\}$ of variables,

(i) each x_i is a term over $\{x_i; i \in n\}$,

(ii) if t_i $(i \in k)$ are terms over $\{x_i; i \in n\}$ and $\sigma \in \Omega$ has arity k then $\sigma(t_i)_{i \in k}$ is a term over $\{x_i; i \in n\}$.

Further, the class $\tilde{E}$ of equations which can be deduced from E is defined recursively, too:

(iii) $E \subset \tilde{E}$,

(iv) $t = t$ is in $\tilde{E}$ for every term t,

(v) if $t = t'$ and $t' = t''$ are in $\tilde{E}$ then $t = t''$ is in $\tilde{E}$,

(vi) if $t = t'$ is in $\tilde{E}$ for some terms t, t' over $\{x_i; i \in n\}$ and $t_i = t_i'$ is in $\tilde{E}$ for every $i \in n$ then $t(t_i)_{i \in n} = t'(t'_i)_{i \in N}$ is in $\tilde{E}$.

Here $t(t_i)_{i \in n}$ is the term obtained from t by substitution $x_i \longrightarrow t_i$ $(i \in n)$; the substitution is defined recursively in the obvious way.

If the equation $t = t'$ is in E, we also write $t = t'$ mod E.

1.2. Linton presentation of a theory. Given an algebraic theory (Ω, E), the Linton presentation of (Ω, E) [7] is the category $\mathbf{T}_{(\Omega,E)}$ the objects of which are sets, and morphisms from a set n to a set k are k-tuples of terms over $\{x_i; i \in n\}$, each term being taken mod E. The composition in the category is the substitution. Two theories (Ω, E), (Ω', E') are said to be equivalent if their Linton presentations are isomorphic.

1.3. Varietal theories. A theory (Ω,E) is varietal if $\mathbb{T}_{(\Omega,E)}$ is locally small, i.e., each $\mathbb{T}_{(\Omega,E)}(n,k)$ is a set (not a proper class). Of course, classical algebraic theories, such as the theory of groups, of rings, etc., are varietal. Complete semilattices provide an example of a non-classical theory which is varietal. Indeed, each term over $\{x_i; i\in n\}$ is equal, mod E, to some $\bigvee_{i\in m} x_i$ for some $m\subset n$.

It is well known that some essential properties of classical theories remain true for varietal ones, e.g. the existence of free algebras, the Birkhoff variety theorem, operational stability, equational completeness and canonical algebraicity; the last three properties are (1),(2),(3) below; to explain them, we start with some definitions.

If $\underline{K}$ is a concrete category with an underlying functor $|\_|:\underline{K}\longrightarrow$ Set, then an implicit n-ary operation in $\underline{K}$ is a natural transformation $\sigma:|\_|^n\longrightarrow|\_|$, equivalently, a family ($\sigma_A:|A|^n\longrightarrow |A|)_A$ where A runs over all $\underline{K}$-objects such that $\sigma_B|h|^n=h\sigma_A$ for every $\underline{K}$-morphism $h:A\longrightarrow B$.

Given an algebraic theory (Ω,E), denote by (Ω,E)-alg the category of (Ω,E)-algebras (i.e., Ω-algebras satisfying all E-equations) and their homomorphisms. Further, let $\bar{\Omega}$ be the list of all implicit operations in (Ω,E)-alg and $\bar{E}$ the list of all equations for Ω-terms which are satisfied by all (Ω,E)-algebras. Each Ω-term induces an implicit operation. If an implicit operation is induced by an Ω-term then it is called explicit. Otherwise, it is called wild.

Following Linton [7], a varietal theory can be reconstructed from the category of its algebras, viz

(1) each implicit operation in (Ω,E)-alg is explicit

(operational stability)

(2) an equation for Ω-terms holds in all (Ω,E)-algebras iff it can be deduced from E

(equational completeness)

and the category of algebras of a varietal theory is canonically algebraic, i.e.,

(3) the natural comparison functor (Ω,E)-alg $\longrightarrow(\bar{\Omega},\bar{E})$-alg is an isomorphism

so that one can recognize whether a concrete category is equivalent to a category of algebras of a varietal theory.

Finally, notice that in general, the category of algebras for a theory can be illegitimate in the sense that the collection of all its algebras is bigger than (more precisely, is not equipotent with) any proper class. E.g., if $\Omega=\{c_i;i\in I\}$ is a proper class of nullary symbols and $E=\emptyset$ then algebras with a two-element underlying set are obviously in one-to-one correspondence with subclasses of I. However, it is well-known and not difficult to see that if (Ω,E) is a varietal theory then

(4) the category (Ω,E)-alg is legitimate.

Remark. If a theory is non-varietal then (1),(2),(3) need not hold (see [12],[4],[12] respectively). On the other hand, there are theories which are non-varietal and satisfy (1),(2),(3) [13].

1.4. Locally small based theories. The best known examples of non-varietal theories are those of complete lattices [3] and complete Boolean algebras [3],[1]. Notice also closure algebras [5], distributive complete lattices [2], frames [2] with countable meets and functor algebras [6]. All these are special cases of locally small based algebraic theories:

A theory (Ω, E) is locally small based if there exists a subcategory $\underline{B}$ of $\mathbb{T}_{(\Omega,E)}$ which is locally small (i.e., each class $\underline{B}(n,k)$ is a set), such that the only subtheory of $\mathbb{T}_{(\Omega,E)}$ containing $\underline{B}$ is all of $\mathbb{T}_{(\Omega,E)}$. In terms of (Ω, E) we have:

(Ω, E) is locally small based if (and, up to equivalence, only if) for every set n, the class of all terms of the form

$\sigma(x_{f(i)})_{i\in k}$ (k is a set, $f:k \longrightarrow n$ a map and $\sigma \in \Omega$ is k-ary),

taken mod E, is a set.

Remark [9]. If (Ω, E) is locally small based then the category (Ω, E)-alg is legitimate, in fact, small fibred (i.e., algebras with a given underlying set constitute a set).

The fact that non-varietal theories of nature are locally small based and the preceding remark suggest the idea that the consideration of general algebraic theories should be restricted to locally small based ones. Simultaneously, this raises the following question: do locally small based theories satisfy (1),(2), (3) ? The aim of the current paper is to show that it is not the case. Notice that none of the counterexamples mentioned in 1.3, Remark, is locally small based.

The preliminary version of the paper (without proofs) appeared as [10].

2. The ordinal theory [14] is obtained from the theory of complete semilattices (see above) by adding a unary operation α with $\alpha x \geq x$. It is obviously locally small based.

2.1. Claim. The ordinal theory does not satisfy (1), i.e., the category of its algebras admits a wild operation.

Proof. Let A be an (ordinal theory) algebra. For any non-void index set m, put

$$\sigma_A(a_i)_{i\in m} = \bigvee B \quad (a_i \in A)$$

where B is the subalgebra generated by $\{a_i; i\in m\}$. It is easy to see that $\sigma_B h^m = h\,\sigma_A$ for every homomorphism $h:A\longrightarrow B$. Hence $\sigma = (\sigma_A)_A$ is an implicit operation. To prove that σ is wild, define an ordinal valued function ϑ on the class of terms as follows

(a) If x_i is a variable then $\vartheta(x_i) = 0$.

(b) If t is any term then $\vartheta(\alpha t) = \vartheta(t) + 1$.

(c) $\vartheta(\bigvee_{i\in m} t_i) = \bigvee_{i\in n}\vartheta(t_i)$.

Further, define algebras A_n ($n\in \mathrm{Ord}$) as follows. The underlying set of A_n is n+1, i.e., the set of all ordinals 0,1,...,n. The complete semilattice structure of A_n is the usual one and $\alpha_{A_n}(a) = a+1$ for $a\neq n$ and $\alpha(n) = n$.

Now, it is not difficult to prove by induction that if σ is an explicit operation of arity m induced by a term t then

(d) $\sigma_{A_n}(a_i)_{i\in m} = \min(n, \bigvee_{i\in m} a_i + \vartheta(t))$.

In particular, $\sigma_{A_n}(0,\dots,0,\dots)\leq \vartheta(T)$. On the other hand, for $n > \vartheta(t)$ we have $\sigma_{A_n}(0,\dots,0,\dots) = n > \vartheta(t)$. It follows $\sigma\neq\sigma$ and so σ is wild. The proof is concluded.

We are going to prove that the category of algebras for the ordinal theory is not canonically algebraic. To end this, consider the collection $\bar{\Omega}$ of all implicit operations in the category of ordinal theory algebras and the collection $\bar{E}$ of all equations for $\bar{\Omega}$-terms which are satisfied by all ordinal theory algebras. Then $\bar{E}$ is obviously generated by equations of the form

(e) $\rho(x_j)_{j\in k} = \sigma(\sigma_i(x_j)_{j\in m_i})_{i\in m}$

where σ is an m-ary implicit operation, σ_i is an m_i-ary implicit operation for every $i \in m$, $k = \bigcup_{i \in m} m_i$ and ρ is the implicit operation defined by

$$\rho_A(a_j)_{j \in k} = \sigma_A(\sigma_{iA}(a_j)_{j \in m_i})_{i \in m} \quad (a_j \in A) \quad ,$$

for every algebra A. We shall make use of the following

2.2. Lemma [13]. Each implicit operation σ is locally explicit, i.e., for every algebra A there exists an explicit operation ρ such that $\sigma_A = \rho_A$.

2.3. Claim. The category of algebras for the ordinal theory does not satisfy (3), i.e., it is not canonically algebraic.

Proof. Every ordinal theory algebra may be viewed as an $(\overline{\Omega},\overline{E})$-algebra and our task is to construct an $(\overline{\Omega},\overline{E})$-algebra which does not arise in this way. Consider the ordinal theory algebras A_n from the proof of 2.1. Notice that for every explicit operation σ of arity m we have

(f) $\quad \sigma_{A_n}(a_i)_{i \in m} \geq \bigvee_{i \in m} a_i$.

By the virtue of 2.2, the same is valid for all implicit operations $\sigma \in \overline{\Omega}$. Let Σ be the collection of all $\sigma \in \overline{\Omega}$ with $\sigma_{A_n}(0,\ldots,0,\ldots) = n$ for all n. Then Σ is non-void for it contains the wild operations δ from the proof of 2.1.

Let A be the ordinal theory algebra with the underlying set $\{0,1\}$ and with $0 \leq 1$, $\alpha_A(0) = 0$, $\alpha_A(1) = 1$. Then

(g) $\quad \sigma_A(0,\ldots,0,\ldots) = 0$

for every explicit and, consequently (cf. 2.2), for every implicit operation $\sigma \in \overline{\Omega}$.

We are going to define an $(\overline{\Omega},\overline{E})$-algebra B. Its underlying set is the same as that of A, and

(h) $\sigma_A = \sigma_B$ if $\sigma \in \overline{\Omega} - \Sigma$,

(i) σ_B is constant with value 1 if $\sigma \in \Sigma$.

Then (f),(h),(i) yield

(j) $\sigma_B(a_i)_i \geq \bigvee_i a_i$

for all $\sigma \in \overline{\Omega}$. Of course, we have to prove that B is indeed an $(\overline{\Omega},\overline{E})$-algebra, i.e., that it satisfies all equations (e).

Case I. $\sigma_i \in \Sigma$ for some i. Using (j) we see that $\sigma_B(\sigma_{iB}(a_j)_{j\in m_i})_{i\in m} = 1$. As A_n satisfies (f),(e), we conclude that $\rho \in \Sigma$, too. Then $\rho_B(a_j)_{j\in k} = 1$ and (e) holds.

Case II. $\sigma \in \Sigma$. By (f), $\sigma_{A_n}(\sigma_{iA_n}(a_j)_{j\in m_i})_{i\in m} = n$. It follows by (e) that $\rho \in \Sigma$ and then (e) is satisfied by B.

Case III. $\sigma, \sigma_i \in \overline{\Omega} - \Sigma$ for all i. We have

$\sigma_{A_n}(0,\dots,0,\dots) = k$ for some n and $k < n$.

For any $s > n$, consider the homomorphism $h: A_s \longrightarrow A_n$ defined by $h(x) = \min(x,n)$. Then

$\sigma_{A_s}(0,\dots,0,\dots) = k$ for all $s > n$.

Indeed, $k = \sigma_{A_n}(0,\dots,0,\dots) = \sigma_{A_n}(h(0),\dots,n(0),\dots) = h\,\sigma_{A_s}(0,\dots,0,\dots)$. Then $\sigma_{A_s}(0,\dots,0,\dots) = k$ for $h(x) = k$ implies $x = k$. Using (d) we see that any explicit operation τ_s with $\tau_{s\,A_s} = \sigma_{A_s}$ is induced by a term t with $\vartheta(t) = k$ which, in turn, implies

$\sigma_{A_s}(a_i)_{i\in m} \leq \bigvee_{i\in m} a_i + k$ for all sufficiently large s and $a_i \in A_s$.

Analogously, there are k_i $(i \in m)$ with

$\sigma_{i\,A_s}(a_j)_{j\in m} \leq \bigvee_{j\in m} a_j + k_i$.

This all together yields $\sigma_{A_s}(0,\dots,0,\dots) =$

$= \sigma_{A_s}(\sigma_i\, _{A_s}(0,\dots,0,\dots))_{i\in m} \leq \bigvee_{i\in m} \sigma_i\, _{A_s}(0,\dots,0,\dots) + k \leq$

$\leq \bigvee_{i\in m} k_i + k$ for all sufficiently large s. Then $\varphi \in \bar{\Omega} - \Sigma$, too, and (e) holds in B. We have proved that B is an $(\bar{\Omega},\bar{E})$-algebra.

Now the fact that B is not an $(\bar{\Omega},\bar{E})$-interpretation of an ordinal theory algebra A' follows by the observation that, because of (h), the only candidate is A' = A which is impossible by (i), see also (g). The proof of 2.3 is finished.

2.4. Claim. The ordinal theory is equationally stable, i.e., it satisfies (2).

Proof. Let m be an arbitrary index set. Let A be an m-complete semilattice (one in which every m-indexed family admits a join) equipped with an unary operation α with $\alpha x \geq x$. Then A can be embedded into an ordinal theory algebra $\tilde{A}$ such that m-indexed joins and α are preserved. Indeed, let $\tilde{A}$ be the complete semilattice of ideals in A that are closed under m-indexed joins. For $J \in \tilde{A}$, let αJ be the least ideal from A which contains all αx $(x \in J)$. Now we use the following

2.5. Lemma. Let (Ω,E) be a theory such that for every n, the class Ω_n of n-ary symbols in Ω is a set. For every n, let E_n be the class (in fact: the set) of all equations for Ω_n-terms with variables from $\{x_i; i \in n\}$ which can be deduced from E. If every (Ω_n,E_n)-algebra can be embedded into an (Ω,E)-algebra such that all Ω_n-operations are preserved then the theory (Ω,E) is equationally stable.

Proof. We have to prove that if an equation t = t' for Ω-terms t, t' holds in all (Ω,E)-algebras then it can be deduced from E. Suppose the contrary. Then t = t' is an equation for Ω_n-

terms for a suitable n which cannot be deduced from E_n. As the theory (Ω_n, E_n) is varietal, there exists an (Ω_n, E_n)-algebra which does not satisfy the equation $t = t'$. Then the assumptions of the lemma yield an (Ω, E)-algebra not satisfying this equation, a contradiction.

3. <u>The power set theory</u> corresponds to the power set functor $P: \mathrm{Set} \longrightarrow \mathrm{Set}$ in such a way that its algebras can be described by maps $PX \longrightarrow X$. It is generated by operation symbols σ_n where n runs over all sets and by all equations of the form

$$\sigma_k(x_{f(i)})_{i\in k} = \sigma_{f[k]}(x_j)_{j\in f[k]}$$

where $f: k \longrightarrow n$ is a map.

Using this theory, we shall show another illegitimacy phenomenon which may occur.

3.1. <u>Claim</u>. The power set theory is locally small based and its cateoory of algebras admits more than a proper class of wild operations.

<u>Proof</u>. To be more precise, we shall construct, for every proper class $C \subset \mathrm{Ord}$, a wild operation δ^C such that $C \neq C'$ implies $\delta^C \neq \delta^{C'}$.

Let $\pi^{(i)}$ ($i \in \mathrm{Ord}$) be unary explicit operations defined by

$$\pi_A^{(0)}(a) = a, \quad \pi_A^{(i)}(a) = \sigma_A(\pi_A^{(j)}(a))_{j<i} \quad (i > 0)$$

for every algebra A and $a \in A$. Here and in what follows we omit the subscripts indicating the arity of σ. Let $\delta^{i,C}$ be explicit operations,

$$\delta_A^{i,C}(a) = \sigma_A(\pi_A^{(j)}(a))_{j<i,\, j\in C}$$

for every algebra A and $a \in A$. For every algebra A there exists an ordinal n_A such that, for every $a \in A$, the sets $\{\pi_A^{(j)}(a);$ $j < i,\ i \in C\}$ are the same for all $i > n_A$ and so are the values $\sigma_A^{i,C}(a)$ $(i > n_A)$. Put

$$\sigma_A^C = \sigma_A^{i,C} \text{ where } i > n_A \text{ is arbitrary.}$$

Then $\sigma^C = (\sigma_A^C)_A$ is an implicit operation. Indeed, given a homomorphism $h: A \longrightarrow B$, we may take a sufficiently large i with $\sigma_A^C = \sigma_A^{i,C}$, $\sigma_B^C = \sigma_B^{i,C}$ and then $h\sigma_A^{i,C} = \sigma_B^{i,C}h$ for $\sigma^{i,C}$ is an explicit operation.

To prove that σ^C is wild, define algebras A_m for all ordinals m. The underlying set of A_m is the set of all ordinals $\leq m+1$. The values $\sigma_{A_m}(a_i)_i$ of operations are defined as follows:

a) if $a_i \leq m$ for all i then $\sigma_{A_m}(a_i)_i$ is the least k such that $k > a_i$ for all i;

b) if $m+1 \in \{a_i\}_i$ and $m \in \{a_i\}_i$ then $\sigma_{A_m}(a_i)_i = m+1$;

c) if $m+1 \in \{a_i\}_i$ and $m \notin \{a_i\}_i$ then $\sigma_{A_m}(a_i)_i = 0$.

We see that $\pi_{A_m}^{(i)}(0) = i$ for $i \leq m$ and $\pi_{A_m}^{(i)}(0) = m+1$ for $i > m$. Hence

$$\sigma_{A_m}^C(0) = m+1 \text{ if } m \in C \text{ and } \sigma_{A_m}^C(0) = 0 \text{ if } m \notin C.$$

Using the same argument as in the proof of 2.1, we see that for every explicit operation τ there exists k such that

$$\tau_{A_m}(0) \leq k \text{ for all } m$$

where k depends on the complexity of the term inducing τ. Hence σ^C is wild. To prove that $C \neq C'$ implies $\sigma^C \quad \sigma^{C'}$, pick an m with, say, $m \notin C$ and $m \in C'$. Then $\sigma_{A_m}^C(0) = 0$ while $\sigma_{A_m}^{C'}(0) = m+1$.

3.2. Remark. The power set theory is equationally stable by 2.5. One can show, using methods similar to those used for the ordinal theory, that the category of its algebras is not canonically algebraic.

4. A degenerating theory. We are going to construct a theory which is locally small based but not equationally stable. In fact, the only algebras for this theory will be the trivial ones, in other words, they all will satisfy the equation $x = y$, but this equation will not be derivable from the axioms of the theory. Notice that our construction is a modification of that from [4] which was not locally small based.

Let Ω consist of n-ary symbols σ_n, ρ_n for every ordinal n where the σ_n's are to satisfy the equations of the power set theory (see above) and the ρ_n's are subject to equations

(1) $\rho_n(\ldots,x,\ldots,x,\ldots) = \rho_0$

(meaning that if we substitute one variable for two distinct ones then the result is equal to the nullary symbol ρ_0), and

(2) $\rho_n(p_i(x))_{i<n} = x$

where $p_i(x)$ are terms defined by

$p_0(x) = x$, $p_i(x) = \sigma_i(p_j(x))_{j<i}$.

Let A be an algebra for this theory. If n is sufficiently large and $a \in A$ is arbitrary then necessarily $p_{jA}(a) = p_{kA}(a)$ for some $j, k < n$, $j \neq k$, and hence $\rho_{nA}(p_{iA}(a))_{i<n}$ is equal to ρ_0 by (1) and, simultaneously, to a by (2). Thus, $a = \rho_0$, i.e. A is a one-element algebra.

To prove that the equation $x = y$ cannot be deduced from E, we shall construct "a large (Ω,E)-algebra" which is nontrivial. Let A be the class of all ordinals. Put

$\sigma_{nA}(a_i)_{i<n} = \min\{k; k > a_i \text{ for all } i<n\}$,

$\rho_{nA}(a_i)_{i<n} = \min\{a_i; i<n\}$ if $n>0$ and if the a_i's are pairwise distinct,

$\rho_{nA}(a_i)_{i<n} = \rho_0 = 0$ otherwise.

Then the operations σ_{nA}, ρ_{nA} satisfy the equations of the theory but not the equation $x = \rho_0$.

5. <u>A well-behaved theory</u>. Our intention is to show that (1)+(2)+(3) do not characterize varietal theories among the locally small based ones.

Consider the theory generated by n-ary operation symbols σ_n where n runs over all ordinals with the system of equations

$$\sigma_n(\ldots,x,\ldots,x,\ldots) = \sigma_0.$$

5.1, <u>Claim</u>. The theory is locally small based but not varietal.

<u>Proof</u>. The former is clear. As for the latter, consider algebras A_n ($n \in \mathrm{Ord}$) with underlying sets $\{0,1,\ldots,n\}$ and with operations $\sigma_{0\,A_n} = 0$ and σ_n ($n>0$) defined by

$$\begin{aligned}\sigma_n(a_i)_{i<n} &= \min\{k; k<n,\ k>a_i \text{ for all } i\},\ \text{if } a_i<n \text{ for all } i<n, \text{ and } a_i \neq a_j \text{ if } i\neq j,\\ &= n \quad \text{if } a_i = n \text{ for some } n \text{ and } a_i \neq a_j \text{ if } i \neq j,\\ &= 0 \text{ otherwise.}\end{aligned}$$

Then each A_n is clearly generated by 0. If a theory is varietal, then it admits a free algebra on one generator the cardinality of which is an upper bound for cardinalities of algebras on one generator. This proves that our theory is non-varietal.

5.2. <u>Claim</u>. The category of algebras of the theory does not admit any wild operation.

<u>Proof</u>. Call a term <u>normal</u> if each of its subterms is either a variable or σ_0 or of the form $\sigma_n(t_i)_{i<n}$ where the t_i's are pairwise distinct. It is easy to see that every term is equal, mod E, to some normal term. Let A_n be the set of normal terms with variables in $\{x_i;\ i<n\}$ involving only σ_i with $i\leq n$. Define operations on A_n by

$$\sigma_{i\,A_n}(t_j)_{j<i} = \sigma_i(t_j)_{j<i} \text{ if } i\leq n \text{ and if the terms } t_j \text{ are pairwise distinct,}$$

$$= \sigma_0 \text{ otherwise.}$$

One can prove easily by induction that if τ is an n-ary explicit operation induced by a normal term t which involves the σ_i's with $i\leq n$ only then

a) $\tau_{A_n}(x_i)_{i<n} = t$

It follows that

b) if σ is an implicit operation then either σ_{A_n} is constant with value σ_0 or σ_{A_n} is induced by a unique normal term $t\in A_n$.

Now suppose σ is an implicit operation such that for some algebra A, σ_A is not the constant map with value σ_0. We are going to prove that then σ is explicit. Put n = card A and choose a bijection $h:\{x_i;i<n\}\longrightarrow A$. Then h can be clearly extended to a map $h:A_n\longrightarrow A$ which is compatible with all operations σ_i $(i\leq n)$. As both σ_{iA} and σ_{iA_n} are constant maps to σ_0 for every $i>n$, $h:A_n\longrightarrow A$ is a homomorphism. It follows that also σ_{A_n} is not the constant map with value σ_0. Then, by b) above, σ_{A_n}

is induced by a unique normal term $t \in A_n$. Now consider an arbitrary algebra B. There exists an explicit operation τ induced by a term t' such that $\sigma_{B\times A_n} = \tau'_{B\times A_n}$. By a projection argument, $\sigma_B = \tau'_B$, $\sigma_{A_n} = \tau'_{A_n}$. Thus σ_{A_n} is induced by t', too, hence $t = t'$. We conclude that σ_B is induced by t for all algebras B, hence σ is explicit.

5.3. Corollary. The category of algebras for the theory is canonically equational.

5.4. Claim. The theory is equationally complete.

Proof. Lemma 2.5.

6. Open problems. Operational stability (1) implies canonical algebraicity (3); we conjecture that the converse is not true. It would be interesting to know whether complete lattices and complete Boolean algebras are operationally stable and canonically algebraic (both are equationally complete by 2.5).

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Math.Dept., Tech. University, Břehová 7, 115 19 Prague, Czechoslovakia

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