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Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 2, 341--357

Persistent URL: http://dml.cz/dmlcz/106456

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DISCRETE SPECTRUM OF OPERATOR VALUED FRIEDRICHS MODELS
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Abstract: The operator valued Friedrichs model is studied. It is proved that there is only a finite number of eigenvalues outside the continuous spectrum.

Key words: Friedrichs model, Fredholm theory, Puiseux series.

Classification: 45B05, 81C10

Several problems of mathematical physics lead to the study of a spectrum of a self-adjoint operator (operator valued Friedrichs models) acting on the Hilbert space $L_2(S^\nu, \mathcal{H})$ according to the following formula

$$ (Hf)(x) = u(x)f(x) + \int_{S^\nu} K(x,y)f(y)dy, f \in L_2(S^\nu, \mathcal{H}) $$

Here $S^\nu$ is a $\nu$-dimensional torus, $\mathcal{H}$ is an $n$-dimensional complex Hilbert space, and the matrices

$$ u(x) = \begin{pmatrix} u_{11}(x) & \cdots & u_{1n}(x) \\ \vdots & \cdots & \vdots \\ u_{n1}(x) & \cdots & u_{nn}(x) \end{pmatrix} $$

and

$$ K(x,y) = \begin{pmatrix} K_{11}(x,y) & \cdots & K_{1n}(x,y) \\ \vdots & \cdots & \vdots \\ K_{n1}(x,y) & \cdots & K_{nn}(x,y) \end{pmatrix} $$

are self-adjoint. We shall suppose that $u_{ij}(x) = u_{ji}(x)$ and $K_{ij}(x,y) = K_{ji}(x,y)$.
A spectrum of operator of the form (1) was first investigated by Friedrichs [1] for \( u(x) = x \), and in [2] for an arbitrary real-analytic function \( u(x) \).

Here we shall give a more detailed description of the spectrum of operator (1), namely, we shall prove that there is only a finite number of eigenvalues outside the continuous spectrum.

Let us denote by \( \Sigma_{\text{cont}}(H) \) the continuous spectrum of the operator \( H \), and by \( \Gamma_x \) the set
\[
\Gamma_x = \{ z \in \mathbb{C}^1 : \sigma'(x,z) = 0 \},
\]
where \( \mathbb{C}^1 \) is the complex plane, \( \sigma'(x,z) \) is a determinant of \( u(x)-zE \).

It is well known that the self-adjointness of \( u(x), x \in S^\nu \) implies that \( \Gamma_x \subset \mathbb{R}^1 \), where \( \mathbb{R}^1 \) is the real line.

**Proposition 1.** It is
\[
\Sigma_{\text{cont}}(H) = \bigcup_{x \in S^\nu} \Gamma_x
\]

**Proof.** Let \( z \in \bigcup_{x \in S^\nu} \Gamma_x \), i.e. \( \sigma'(x,z) = 0 \) for some \( x \in S^\nu \). Then the operator \( u(x)-zE \), where \( E \) is the identity operator in \( \mathcal{H} \), is not invertible. Therefore, the operator
\[
[(H_0-zI)f](x) = (u(x)-zE)f(x), f \in L_2(S^\nu, \mathcal{H})
\]
where \( I \) is the identity operator in \( L_2(S^\nu, \mathcal{H}) \), is not invertible in the space of bounded operators on \( L_2(S^\nu, \mathcal{H}) \) i.e. \( z \in \Sigma_{\text{cont}}(H_0) \).

Since
\[
\int_{S^\nu \times S^\nu} \|K(x,y)\|^2 dy < \infty
\]
we infer that the operator
\[
[(H-H_0)f](x) = \int_{S^\nu} K(x,y)f(y)dy, f \in L_2(S^\nu, \mathcal{H})
\]
belongs to the class of Hilbert-Schmidt operators. Using the well-known theorem of H. Weyl (see [13]) we conclude that the continuous spectra of both $H$ and $H_0$ coincide. Thus $z \in \Sigma_{\text{cont}}(H)$.

Now let $z \in \Sigma_{\text{cont}}(H)$. Using again the mentioned theorem of H. Weyl we have also $z \in \Sigma_{\text{cont}}(H_0)$, and thus $z \in \Gamma_x$ for some $x \in S^\nu$, i.e. $z \in \bigcup_{x \in S^\nu} \Gamma_x$.

**Theorem 1.** The resolvent $R_z(H)$ of $H$ exists. It can be expressed by the formula

$$
(R_z f(x) = [u(x) - zE]^{-1} f(x) + \sum_{y \in S^\nu} K(x, y) f(y)dy
$$

for all $z \in \mathbb{C}$, $\text{Im } z \neq 0$ where $\Delta(z)$, and $\mathcal{D}(x, y; z)$ are defined below (in (11), (13)).

**Proof.** We shall find an explicit formula for $R_z(H)$ as the inverse of $H - zI$. Let for some $g \in L^2(S^\nu, \mathcal{H})$

$$
([H - zI] f)(x) = (u(x) - zE) f(x) + \int_{S^\nu} K(x, y) f(y)dy = g(x), \text{ } f \in L^2(S^\nu, \mathcal{H})
$$

Since $u(x)$ is self-adjoint in $\mathcal{H}$, the determinant $\mathcal{D}(x, z)$ of the matrix $u(x) - zE$ is nonvanishing for all $z \in \mathbb{C}$, $\text{Im } z \neq 0$, and hence the inverse operator

$$
(u(x) - zE)^{-1} = \frac{1}{\mathcal{D}(x,z)} \begin{pmatrix}
\mathcal{D}_{11}(x, z) & \cdots & \mathcal{D}_{n1}(x, z) \\
\vdots & \ddots & \vdots \\
\mathcal{D}_{1n}(x, z) & \cdots & \mathcal{D}_{nn}(x, z)
\end{pmatrix}
$$

exists. Here $\mathcal{D}_{ji}(x, z)$ denotes the signed minor of the element $u_{ij}(x, z)$ of the matrix $u(x) - zE$. Introducing the notation

$$
\hat{f}(x) = [u(x) - zE] f(x), \text{ } f \in L^2(S^\nu, \mathcal{H})
$$

we can write (4) as

$$
\hat{f}(x) + \int_{S^\nu} K(x, y) [u(y) - zE]^{-1} \hat{f}(y)dy = g(x), \text{ } \hat{f} \in L^2(S^\nu, \mathcal{H})
$$

which can be formulated as a system.
of integral equations. Here
\[ f(x) = (f_1(x), \ldots, f_n(x)), \quad g(x) = (g_1(x), \ldots, g_n(x)), \quad i = 1, 2, \ldots, n, \]
and \( L_2(S^\nu, C^1) \) is the Hilbert space of all square integrable complex functions defined on the \( \nu \)-dimensional torus \( S^\nu \), and.

We shall now rewrite (6) as an integral equation equivalent to the system (6). To this end we denote by \( M \) the union of disjoint copies of \( S^\nu \), i.e.
\[ M = \bigcup_{j=1}^{n} (S^\nu)_j, \quad (S^\nu)_j = S^\nu, \quad j = 1, 2, \ldots, n. \]

Define now a measure on \( M \) such that its restriction to each \( (S^\nu)_j = S^\nu, \quad j = 1, 2, \ldots, n \) coincides with the Lebesgue measure. For each \( z \in C^1 \), \( \text{Im} \ z \neq 0 \) we define the function (kernel)
\[ K(\lambda, \mu; z) = \left\{ \begin{array}{ll}
1 & z = 0 \\
\sum_{j=1}^{n} K_1(x,y) - \delta_{1j}(y,z). & z \neq 0
\end{array} \right. \]

Finally we define the following functions on \( M \):
\[ f(\lambda) = f_i(x), \quad g(\lambda) = g_i(x), \quad i = 1, 2, \ldots, n. \]
Then the system of integral equations (6) is equivalent with
\[ f(\lambda) + \int_M K(\lambda, \mu; z)f(\mu)d\mu = g(\lambda), \quad f, g \in L_2(M, C^1), \]
where \( L_2(M, C^1) \) is the Hilbert space of all square integrable complex valued functions on \( M \).

**Proposition 2.** Any \( z \in C^1 \setminus \Sigma_{\text{cont}}(H) \) is an eigenvalue of \( H \) if and only if the homogeneous equation
\[ f(\lambda) + \int_M K(\lambda, \mu; z)f(\mu)d\mu = 0 \]
has a nonzero solution \( f \in L^2(M, C^1) \).

**Proof.** Any \( z \in C^1 \setminus \Sigma_{\text{cont}}(H) \) is an eigenvalue of \( H \) iff. for some \( f \in L^2(S^y, \lambda \mathcal{E}) \) the following relation holds true:

\[
(9) \quad (u(x) - zE)f(x) + \int_M K(x, y)f(y)dy = 0.
\]

By the same argument as before it is possible to show that (9) is equivalent to the system of homogeneous integral equations

\[
(10) \quad \begin{cases}
  f_1(x) + \int_S \sum_{j=1}^n K_{1j}(x,y;z)f_j(y)dy = 0, \\
  \vdots \\
  f_n(x) + \int_S \sum_{j=1}^n K_{nj}(x,y;z)f_j(y)dy = 0.
\end{cases}
\]

Further, from the definition of \( L^2(M, C^1) \) and the kernel \( K(\lambda, \mu; z) \) it follows that for any \( z \in C^1 \setminus \Sigma_{\text{cont}}(H) \), the system (10) has a nonzero solution iff the homogeneous integral equation (8) has a nonzero solution from \( L^2(M, C^1) \).

To finish the proof of Theorem 1 we use the self-adjointness of \( H \) to infer from Proposition 2 that for each \( z \in C^1 \), \( \text{Im } z \neq 0 \) the homogeneous equation

\[
f(\lambda) + \int_M K(\lambda, \mu; z)f(\mu)d\mu = 0,
\]

has no nonzero solution. Besides, since

\[
\int_{M \times M} |K(\lambda, \mu; z)|^2 d\lambda d\mu = \sum_{i,j=1}^N \int_S \int_S |K_{ij}(x,y;z)|^2 dx dy < \infty
\]

it follows that the operator

\[
[K(z)f](\lambda) = \int_S K(\lambda, \mu; z)f(\mu)d\mu, \quad f \in L^2(M, C^1)
\]

is of Hilbert-Schmidt type. Therefore, it follows from Fredholm theorem (see [4]) that the equation (7) has a unique solution \( f \in L^2(M, C^1) \), for any \( g \in L^2(M, C^1) \). This solution can be expressed as

\[
f(\lambda) = g(\lambda) - \int_M \Theta(\lambda, \mu; z)g(\mu)d\mu,
\]

- 345 -
where $\Delta(z)$, and $\mathcal{D}(\lambda, \mu, z)$ denote the Fredholm determinant, and minor, respectively. Considering the restriction of $f$ on $(S^y)_1, i = 1, 2, \ldots, n$, we obtain the solution of the system (6) in the following form:

$$
\Omega_i(x) = g_i(x) + \sum_{\mu} \int_{M} \frac{\mathcal{D}_i(x, \mu; z)}{\Delta(z)} g(\mu) d\mu = g_i(x) + \sum_{\mu} \int_{S^y} \frac{\mathcal{D}_{i,j}(x, y; z)}{\Delta(z)} g_i(y) dy, \ i = 1, 2, \ldots, n.
$$

Here $\mathcal{D}_{i,j}(x, y; z)$ and $\Delta(z)$ are given by the following formulas:

$$
\mathcal{D}_{i,j}(x, y; z) = K_{i,j}(x, y; z) + \sum_{\mu=1}^{\infty} \frac{1}{\mu!} D_{i,j}(x, y; z),
$$

$$
d_{i,j}(x, y; z) = \sum_{\mu=1}^{\infty} \frac{1}{\mu!} D_{i,j}(x, y; z),
$$

$$
\Delta(z) = 1 + \sum_{\mu=1}^{\infty} \frac{1}{\mu!} d_{i,j}(z).
$$

Therefore it follows from the formula (11) and (5) that the resolvent of $H$ acts on $L^2(S^y, \mathcal{H})$ according to the formula

$$
[R_z f](x) = [u(x) - z\mathcal{E}]^{-1} f(x) - \frac{[u(x) - z\mathcal{E}]^{-1}}{\Delta(z)} \int_{S^y} \mathcal{D}(x, y; z) f(y) dy,
$$

where

$$
\mathcal{D}(x, y; z) = \begin{pmatrix}
\mathcal{D}_{1,1}(x, y; z) & \cdots & \mathcal{D}_{1,n}(x, y; z) \\
\cdots & \cdots & \cdots
\end{pmatrix},
$$

$$
\mathcal{D}_{n,1}(x, y; z) & \cdots & \mathcal{D}_{n,n}(x, y; z)
\end{pmatrix}.
$$
The boundedness of $R_z$ follows from the explicit formula (13). Thus, the theorem 1 is proved.

**Theorem 2.** The operator (1) has only a finite number of eigenvalues not belonging to the continuous spectrum.

We shall restrict ourselves to the case $\nu = 1$ and $u_{1\nu}(x) = 0$ for $i \neq j$ to avoid certain technical difficulties of the general case. In addition, without loss of generality we can assume that $u_j(x) \equiv u_{jj}(x)$ and $K_{1j}j_2(x, y), j, j_1, j_2 = 1, 2, \ldots, n$ are $2\pi$ periodic functions defined on $[0, 2\pi]$ and $[0, 2\pi] \times [0, 2\pi]$, respectively. We notice that in this special case the continuous spectrum of $H$ consists of

$$
\Sigma_{\text{cont}}(H) = \bigcup_{j=1}^{\infty} [A_j, B_j],
$$

where $A_j = \inf_{x \in \mathbb{R}} u_{jj}(x)$, $B_j(x) = \sup_{x \in \mathbb{R}} u_{jj}(x)$, and the function $d_q(z), q = 1, 2, \ldots$ from (12) can be written as

$$
d_q(z) = \sum_{j_1, \ldots, j_q = 1} d_{j_1 j_2} \ldots j_q(z) = \sum_{j_1, \ldots, j_q = 1}^{2\pi}
$$

$$
\int_0^{2\pi} \ldots \int_0^{2\pi} \left| \begin{array}{c}
K_{j_1 j_1}(t_1, t_1) \ldots K_{j_1 j_2}(t_1, t_q') \\
\ldots \\
K_{j_q j_1}(t_q, t_1) \ldots K_{j_q j_q}(t_q, t_q')
\end{array} \right| \frac{dt_1 \ldots dt_q}{(u_{1j_1}(t_1) - z) \ldots (u_{jj}(t_q) - z)}
$$

The following lemma plays a crucial role in the proof of Theorem 2.

**Lemma 1.** Let $A' \in \Sigma_{\text{cont}}(H)$ and $u_{j}^{-1}(A') = \{x_{j}, x_{j_2}, \ldots, x_{jm_j} \}$, $j = 1, 2, \ldots, n$. Then there is an $\varepsilon$-neighborhood $V_\varepsilon(A') = \{z \in \mathbb{C}^1 : 0 < |z - A'| < \varepsilon \}$ of $z = A'$ such that the restriction $\Delta(z)/C_+^1$ of the $\Delta(z)$, where $C_+^1 = \{z \in \mathbb{C}^1: \text{Im} z > 0 \}$ is the half-plane, has an analytic continuation onto $V_\varepsilon(A')$. This analytic continuation $\Delta^\varepsilon(z)$ is a multivalued function with the branching
point \( z = A' \) and can be in \( V'_e(A') \) expanded into the series

\[
\Delta^s(z) = \sum_{A \in \mathbb{Z}} F_{A', s}(K)(z-A') \frac{P(z-A')}{P}, \quad z \in V'_e(A').
\]

Here

\[
Q = \prod_{j=1}^{n} \prod_{A \in \mathbb{Z}} \frac{R_{js} - 1}{R_{js}}
\]

and \( R_{js} - 1 = R(x_{js}) - 1 \) denote the multiplicity of the root \( x = x_{js} \) of the function \( u_j(x) \), \( j = 1, 2, \ldots, m_j \), \( P \) is the lowest common multiple of the numbers

\[
\{ R_{11}, \ldots, R_{1m_1}, \ldots, R_{n1}, \ldots, R_{nm_n} \}.
\]

The proof of this lemma is based on Lemma 2 which we shall prove first.

**Lemma 2.** Let \( A' \in \Sigma_{\text{cont}}(H) \). Then for any \( q = 1, 2, \ldots \) there is a neighborhood \( V'_e(A') \) of \( z = A' \), and a function \( d^q(z) \) defined on it, such that

\[
d^q(z)/V'_e(A') \cap C^1 = d^q(z)/V'_e(A') \cap C^1,
\]

where \( C^1 = \{ z \in C^1 : \text{Im } z > 0 \} \). The function \( d^q(z) \) is a multivalued function with the branching point \( z = A' \).

**Proof of Lemma 2.** For any \( A' \in \Sigma_{\text{cont}}(H) = \bigcup_{j=1}^{n} [A_j, B_j] \)
we denote by \( u_j^{-1}(A') \subset [0, 2\pi] \), \( j = 1, 2, \ldots, m' \) its pre-image with respect to the mapping \( u_j \). It is obviously finite, i.e. we can write \( u_j^{-1}(A') = \{ x_j \}, \ldots, x_{m_j} \}. \) Let us denote by \( u_j(x_j) \) and \( K_{j_1 j_2}(x_{1}, x_{2}) \) the analytic continuations of \( u_j(x) \), and \( K_{j_1 j_2}(x_{1}, x_{2}) \), into \( Q \subset C^1 \), and \( Q \times Q \subset C^2 \), respectively, where \( Q \subset C^1 \) is some complex neighborhood of the segment \([0, 2\pi]\).

Because \( u_j(x_j) \), \( j = 1, 2, \ldots, n \) is regular in \( x = x_j \), \( \forall = 1, 2, \ldots, m_j \) there are some \( \varepsilon > 0 \) and \( \delta' > 0 \) (in the following we shall assume that these numbers are sufficiently small) such that
for each \( z \in V_e(A') \) the equation
\[
u_j(\xi) - z = 0
\]
has exactly \( R_{j\nu} \), \( \nu = 1,2,\ldots,m_j \) solutions in the disc
\(|x_{j\nu} - \xi| < \delta' \). These solutions are branches of some \( R_{j\nu} \)-valued
analytical functions, whose branching point \( z = A' \) has an order
\( R_{j\nu} \) and can be in \( V_e(A') \) expanded into the series
\[
(14) \quad \psi_j(z) = x_{j\nu} + c_{j1}^\nu (z-A')^{1/R_{j\nu}} + c_{j2}^\nu (z-A')^{2/R_{j\nu}} + \ldots,
\]
where
\[
(15) \quad c_{j1}^\nu = \left[ \frac{R_{j\nu}}{u_j(x_{j\nu})} \right]^{1/R_{j\nu}}, \quad \nu = 1,2,\ldots,m_j.
\]

This statement follows from the theorem about inverse function of an analytic function (see [5]).

We put
\[
(z-A')^\nu = |z-A'|^{1/R_{j\nu}} \exp \left\{ i \arg (z-A') + \frac{2\pi i}{R_{j\nu}} \right\}.
\]
s = 0,1,\ldots,\( R_{j\nu} - 1 \) and call this value the s-th value of the root
\( (z-A')^{1/R_{j\nu}} \). Correspondingly we call the
\[
\psi_{j\nu}(z) = x_{j\nu} + c_{j1}^\nu (z-A')^{1/R_{j\nu}} + c_{j2}^\nu [(z-A')^{1/R_{j\nu}}]^{2} + \ldots
\]
the s-th value of the multivalued function \( \psi_j(z) \).

**Proposition 3.** For any \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that
for each \( z \in V_e(A') \), \( \text{Im } z > 0 \) the number of values \( \psi_{j0}(z), \psi_{j1}(z), \ldots, \psi_{jR_{j\nu}-1}(z) \) which belong to \( \{ \xi \in C^1: \text{Im } \xi > 0, |\xi - x_{j\nu}| < \delta \} \), equals to \( P_{j\nu} \). Here, \( P_{j\nu} \) is the integer part of
\[
\frac{1}{2} \{ R_{j\nu} + [\text{sgn } u_j(x_{j\nu})] \} R_{j\nu} \}.
\]

**Proof.** Since \( \text{Im } z > 0 \) we observe that \( P_{j\nu} \) values of \( (z-A')^{1/R_{j\nu}} \),

where $P_{j\nu}$ is the integer part of $\frac{1}{2}(R_{j\nu} + 1)$, belong to the upper half-plane. From this fact and from (15) it follows that $P_{j\nu}$ values

$$\psi_{j1}^\nu(z-A')_0, d_{j1}^\nu(z-A')_1, \ldots, d_{j1}^\nu(z-A')_{p_{j\nu}-1}$$

belong to the upper half-plane. The smallness of $\varepsilon > 0$ then implies that $P_{j\nu}$ values

$$\psi_{j0}^\nu(z), \psi_{j1}^\nu(z), \ldots, \psi_{j\nu-1}^\nu(z)$$

belong to $\{\zeta \in \mathbb{C} : \text{Im}\zeta > 0, |\zeta - x_{j\nu}| < \sigma\}$.

**Proposition 4.** The function $[u_j^\nu(\psi_j^\nu)]^{-1}$ can be in $V_\varepsilon(A')$ expressed in the form

$$[u_j^\nu(\psi_j^\nu)]^{-1} = \frac{(R_{j\nu} - 1)!}{u_j^\nu(R_{j\nu})} \prod_{j=1}^{\nu} \left[1 + \sum_{k=1}^{\infty} c_k(z-A')^{k/R_{j\nu}}\right]$$

and the function $K_{j1j2}^\nu(\psi_{j1}^\nu, \psi_{j2}^\nu)$ in the form

$$K_{j1j2}^\nu(\psi_{j1}^\nu, \psi_{j2}^\nu) = K_{j1j2}^\nu(x_{j1}^\nu, x_{j2}^\nu) + \sum_{s_1=1, s_2=1}^{\infty} c_{s_1s_2}(z-A')^{s_1/R_{j1}+s_2/R_{j2}}.$$

**Proof.** Since $\zeta = x_{j\nu}$, $\nu = 1, 2, \ldots, m_j$ is a zero point of the order $R_{j\nu} - 1$ of the function $u_j^\nu(\zeta^\nu)$, we can expand this function into the following series:

$$u_j^\nu(\zeta^\nu) = \sum_{n=0}^{R_{j\nu}-1} \frac{(R_{j\nu})^n}{(R_{j\nu} - 1)!} (\zeta - x_{j\nu})^n + \frac{(R_{j\nu+1})}{R_{j\nu}!} (\zeta - x_{j\nu})^{R_{j\nu}+1} + \ldots$$

Substituting (14) into (16) we obtain

$$u_j^\nu(\psi_j^\nu) = \sum_{n=0}^{R_{j\nu}-1} \frac{(R_{j\nu})^n}{(R_{j\nu} - 1)!} (\zeta - x_{j\nu})^n + \frac{(R_{j\nu+1})}{R_{j\nu}!} (\zeta - x_{j\nu})^{R_{j\nu}+1} + \ldots =$$

- 350 -
For a sufficiently small \( \varepsilon > 0 \) and \( z \in \mathcal{V}_\varepsilon'(A') \) we have an inequality
\[
|C_{j2}^\varepsilon(z-A')|^{1/R_j^\varepsilon} < 1.
\]
This inequality, together with (17), implies the statement of Proposition 4 for the function \( [u_j^\psi(\psi^\psi)]^{-1} \). The second assertion of Proposition 4 is proved in an analogous way.

Coming back to the proof of Lemma 2 let \( \varepsilon > 0 \) and \( \sigma' > 0 \) be such that the segments
\[
(x_{j1} - \sigma', x_{j1} + \sigma'), \ldots, (x_{jm_j} - \sigma', x_{jm_j} + \sigma'), j = 1, 2, \ldots, n
\]
do not intersect and
\[
u_j^{-1}(\mathcal{V}_\varepsilon'(A')) \subset \bigcup_{j=1}^m \{ \xi \in \mathbb{C}^1_+ : |\xi - x_j| < \sigma' \}.
\]
Investigate the function
\[
d_j(z) = \frac{\pi}{2\pi i} \int_{0}^{2\pi} \frac{K_{j1}(\xi, \xi)}{u_j(\xi)^{-1}} d\xi, \quad j = 1, 2, \ldots, n
\]
which is regular in \( \mathcal{V}_\varepsilon'(A') \cap \mathbb{C}^1_+ \). We can write the function \( d_j(z) \) in terms of its residua as follows:
\[
= 2\pi i \left[ \sum_{j=1}^m \frac{\pi}{2\pi} \int_{0}^{2\pi} \frac{K_{j1}(\psi^\psi, \psi^\psi)}{u_j(\psi^\psi)^{-1}} + \int_{\Gamma_j} \frac{K_{j1}(\xi, \xi)}{u_j(\xi)^{-1}} d\xi \right]
\]
Here, \( \Gamma_j \) is the contour, coinciding with \( [0, 2\pi] \) outside of all intervals
\[
(x_{j1} - \sigma', x_{j1} + \sigma'), \ldots, (x_{jm_j} - \sigma', x_{jm_j} + \sigma')
\]
and containing all the half-circles.
\{ \xi \in \mathbb{C}^1 : |\xi - z_0| = \sigma, \text{ Im } \xi \geq 0 \}.

Since \( \xi \in \Gamma_\sigma \) we conclude that

\[ u_j(\xi) \in V_\varepsilon(A') = \{ z \in \mathbb{C}^1 : |z - A'| < \varepsilon \} \cdot \]

Therefore, the function \[ \int_{\Gamma_\sigma} \frac{K_j(\xi, \xi')}{u_j(\xi') - z} d\xi \]
is regular in \( V_\varepsilon(A') \). Using the representation (18) of \( d_j(z) \), \( j = 1, 2, \ldots, n \), the existence of an analytical continuation of \( d_j(z) \) into the region \( V_\varepsilon(A') \) through the interval \((A' - \varepsilon, A')\) and also through \((A', A' + \varepsilon)\) follows. Both these analytical continuations coincide. We denote by \( d_j^*(z) \) the analytical continuation of \( d_j(z) \). From the proposition 4 it follows that \( d_j^*(z) \) is a multivalued function with the branching point \( z = A' \), expressed in the Puiseux series in the powers of \( z - A' \).

Consider now the function

\[
d_{j,j_2}(z) = \int_0^{2\pi} \frac{1}{u_{j_1}(\xi_1) - z} \left\{ 2 \left| \begin{array}{c} K_{j_1j_2}(\xi_1, \xi_1)K_{j_1j_2}(\xi_1, \xi_2) \\ K_{j_2j_1}(\xi_2, \xi_1)K_{j_2j_1}(\xi_2, \xi_2) \\ \end{array} \right| \right\} \frac{d\xi_1}{u_{j_1}(\xi_1) - z} \frac{d\xi_2}{u_{j_2}(\xi_2) - z}, \]

Using the "theorem about resida" to the function \( d_{j,j_2}(z) \) several times, we obtain:

\[
(19) \quad d_{j,j_2}(z) = \int_0^{2\pi} \frac{1}{u_{j_1}(\xi_1) - z} \left\{ 2 \left| \begin{array}{c} K_{j_1j_1}(\xi_1, \xi_1)K_{j_1j_2}(\xi_1, \xi_2) \\ K_{j_2j_1}(\xi_2, \xi_1)K_{j_2j_2}(\xi_2, \xi_2) \\ \end{array} \right| \right\} \frac{d\xi_1}{u_{j_1}(\xi_1) - z} \frac{d\xi_2}{u_{j_2}(\xi_2) - z}
\]

\[
= (2\pi)^2 \left\{ \begin{array}{c} m_{j_1}^1 \sum_{\nu_1 = 1} m_{j_2} \sum_{\nu_2 = 1} \frac{1}{u_{j_1}(\xi_1) - z} \\ \end{array} \right\} \frac{d\xi_1}{u_{j_1}(\xi_1) - z} \]

\[
= \sum_{\nu_1 = 1} m_{j_1}^1 \sum_{\nu_2 = 1} m_{j_2} \sum_{\nu_2 = 1} \frac{1}{u_{j_1}(\xi_1) - z} \frac{d\xi_1}{u_{j_1}(\xi_1) - z} \frac{d\xi_2}{u_{j_2}(\xi_2) - z} = -352 -
\]
Using analogical considerations as in the case of $d_j(z)$, $j = 1, 2, \ldots, n$ it follows from the representation (19) that the function $d_{j_1j_2}(z)$ which is regular on the upper half-plane, has an analytical continuation into $V_\mathcal{E}^\mathcal{S}(A')$, through the segments $(A'-\varepsilon, A')$ and $(A', A'+\varepsilon)$. Both these analytical continuations coincide.

From Proposition 4 it follows that this continuation is a multi-valued function with the branching point $z = A'$.

Now we investigate the function $d_{j_1j_2\ldots j_q}(z)$, defined by the formula
\[d_{j_1 \ldots j_q}(z) = \int_0^{2\pi} \ldots \int_0^{2\pi} \prod_{i=1}^q K_{j_1 j_1}(s_i, s_i) \ldots K_{j_q j_q}(s_q, s_q) \times \]
\[\prod_{i=1}^q \frac{d\xi_i}{(u_{j_1}(s_i) - z) \ldots u_{j_q}(s_q) - z)}\]

(which is regular on the upper half-plane). Finally, using the mentioned "theorem about resida" repeatedly many times we can write down the following expression:

\[d_{j_1 \ldots j_q}(z) = (2\pi i)^q \prod_{i=1}^q \frac{m_{j_1} \ldots m_{j_q}}{\gamma_1, \gamma_2, \ldots, \gamma_q = 1} \prod_{s_1=0}^p \frac{P_{j_1, \gamma_1} - 1}{s_1=0} \ldots \prod_{s_q=0}^q \frac{P_{j_q, \gamma_q} - 1}{s_q=0}\]

\[= \left( \prod_{i=1}^q K_{j_1 j_1}(s_i, s_i) \ldots K_{j_q j_q}(s_q, s_q) \right) \times \]
\[\prod_{i=1}^q \frac{1}{u_{j_1}(s_i) \ldots u_{j_q}(s_q)} + \ldots + \]
\[+(2\pi i)^{q-q'} \prod_{i=1}^{m_{j_{q'-1}}} \prod_{s_{q'-1}=0}^{\gamma_{q'-1}} \frac{\prod_{j_{q'-1} \ldots j_q} P_{j_{q'-1}, \gamma_{q'-1}} - 1}{s_{q'-1}=0} \ldots \prod_{s_{q'}=0}^{\gamma_{q'}} \frac{P_{j_q, \gamma_q} - 1}{s_{q'}=0}\]

- 354 -
Using the same arguments as above we infer from (20) that the function has an analytical continuation into the \( \varepsilon \)-neighborhood \( V_\varepsilon'(A') \) of \( z = A' \) through the intervals \( (A' - \varepsilon, A') \) and \( (A', A' + \varepsilon) \).

We denote by \( d_{j_1 \ldots j_q}(z) \) the analytical continuation of 
\[ d_{j_1 \ldots j_q}(z) / \varepsilon \]
into the region \( V_\varepsilon'(A') \). From the proposition 4 it follows that the function is, generally speaking, a multivalued
function with the branching point \( z = A' \).

Let us denote by \( d_q^*(z), q = 1,2,... \) the function

\[
d_q^*(z) = \frac{1}{j_1,j_2,...,j_q} d_q^* \left( J_1 J_2 \cdots J_q(z), z \in V_e'(A') \right)
\]

which is again a multivalued one on \( V_e'(A') \), with a branching point \( z = A' \). From the Puiseux theorem and the expression (20) it follows that the function \( d_q^*(z), z \in V_e'(A') \) expands into the following series:

\[
d_q^*(z) = \sum_{s=-q}^{\infty} F_{s,q}(K)(z-A')^{s/p}, \quad z \in V_e'(A')
\]

Thus the proof of Lemma 2 is completed.

**Proof of Lemma 1.** Denote by

\[
\Delta^*(z) = \sum_{q=1}^{\infty} \frac{1}{q^p} d_q^*(z), \quad z \in V_e'(A').
\]

Then from (21) for \( d_q^*(z), q = 1,2,... \) and from the Hadamard inequality for determinants (see [4]) it follows that the series (22) converges absolutely in \( V_e'(A') \) and defines a multivalued analytical function with a unique branching point \( z = A' \). Therefore, from the Puiseux theorem (see [6]) and the expansion (22) used for \( d_q^*(z), q = 1,2,... \) we obtain the statement of Lemma 1.

**Proof of Theorem 2.** Following Proposition 2 and the Fredholm theorem a point \( z \in \mathcal{C}^{-1} \setminus \Sigma_{cont}(H) \) is an eigenvalue of \( H \) iff \( \Delta(z) = 0 \).

Because of the self-adjointness of \( H \), it is sufficient to show that the function \( \Delta(z) \) has only a finite number of real zeros not belonging to the continuous spectrum. We shall only show that \( \Delta(z) \) may have a finite number of real zeros greater than \( A \), \( A = \sup_{\lambda \in \Sigma_{cont}(H)} \). The remaining intervals of the complement of the continuous spectrum may be investigated in an analogous way. It
follows from Lemma 1 that the function $\Delta(z)$ can be expressed in the $\varepsilon$-neighborhood $V_\varepsilon(A') \setminus \{-\infty, A\}$ of 

$$z = A'$$

by the following series:

$$
\Delta(z) = \sum_{s=-\infty}^{\infty} F_{s,A}(K)(z-A)^{s/p}, \quad z \in V_\varepsilon(A') \setminus (-\infty, A].
$$

Therefore, $A$ cannot be a limit point for the set 

$$\{z \in \mathbb{R}: \Delta(z) = 0, z > A\}.$$ 

On the other hand, the function $\Delta(z)$ is regular in $C^1 \setminus \Sigma_{\text{cont}}(H)$ and $\Delta(z) \to 1$ for $|z| \to \infty$, $\text{Im} \ z = 0$, and thus it has only a finite number of zeros belonging to $(A + \varepsilon, \infty)$ for any $\varepsilon > 0$.

References


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(Obitatum 18.9. 1985)