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CONVEX SETS AND HARNACK INEQUALITY
D. G. KESELMAN

Abstract: Given a locally compact part \mathcal{D} of a convex set, let $H(U)$ be a linear space of continuous locally affine functions on an open set $U \subset \mathcal{D}$. It is proved that the map $H:U \rightarrow H(U)$ is a harmonic sheaf of functions possessing the Brelot convergence property. Some properties of parts and faces of compact sets and of Choquet simplexes are discussed.

Key words: Faces and parts of convex sets, harmonic sheaf, Harnack inequality, Choquet simplex.

Classification: 31D05, 46A55

Introduction. For any convex set L with induced topology from locally convex Hausdorff space E we shall introduce the following notations: $A(L)$ - space of all continuous affine real-valued functions on L ;

$$A^+(L) = \{a \in A(L) : a \geq 0\};$$

face (x) will denote the smallest (not necessarily closed) face of L containing x ($y \in \text{face}(x) \iff \exists r > 0 : x + r(x-y) \in L$, it means that the point x is surrounded in the set $\text{face}(x)$).

Let $\mathcal{D} \subset L$ be a locally compact Gleason part (we shall note that in this locally compact topology every point $y \in \mathcal{D}$ will have the basis of compact convex neighbourhoods). The necessary and sufficient condition of the local compactness of \mathcal{D} which is contained in the convex compact S , is given in the proposition 7.

The condition is: \mathcal{D} must have at least one point which has

a compact neighbourhood belonging to \mathfrak{D} .

Definition. Let \mathcal{U} be an open subset of a part. The function $f: \mathcal{U} \rightarrow \mathbb{R}$ is called locally affine, if every point in \mathcal{U} has such a convex neighbourhood \mathcal{V} that the restriction $f|_{\mathcal{V}}$ is an affine function.

Let $H(\mathcal{U})$ be a linear space of all continuous and locally affine on \mathcal{U} functions (only the real-valued functions are considered in this paper).

It will be proved in Theorem 6 that the map $H: \mathcal{U} \rightarrow H(\mathcal{U})$ is a harmonic sheaf of functions possessing the Brelot convergence property. As shown in [1, p. 16] Harnack inequality will be valid for such sheaves, i.e. if \mathcal{U} is connected then for an arbitrary compact set $K \subset \mathcal{U}$ there exists a positive real number $\alpha_K \geq 1$ such that for any positive function $f \in H(\mathcal{U})$ and for any $x, y \in K$ we have $f(x) \leq \alpha_K f(y)$.

In the process of preparation for the construction of the sheaf we shall prove that if the point $x \in S$ has face (x) of the second category then for an arbitrary compact set $K \subset \text{face}(x)$ a positive real number $\alpha_K \geq 1$ exists such that for any positive function $f \in A^+(\text{face}(x))$ we have

$$\sup_{y \in K} f(y) \leq \alpha_K f(x).$$

From this we shall have that if $\text{face}(x)$ is a locally compact face then any lower semi-continuous affine function $f: S \rightarrow]-\infty; +\infty]$ has a continuous restriction on $\text{face}(x)$.

In the last paragraph of the paper it is proved that if S is Choquet simplex, for any of its part P the solving of the Dirichlet problem with an arbitrary continuous boundary function will be continuous on P in the part metric. Besides, it is proved that

for any point $x \in S$ any bounded set of the affine functions of the first Baire class contains a sequence which is on face (x) converging pointwise to some affine function $f: \text{face}(x) \rightarrow R$.

1. Properties of the faces of the second category of convex compacts

Proposition 1. Assume that L is bounded and the function $f: L \rightarrow]-\infty; +\infty]$ is a supremum of an increasing net of functions from $A(L)$. Assume that its effective set $\text{dom } f = \{x \in L: f(x) < +\infty\}$ is non-empty. Then:

- 1) $\text{dom } f$ is convex and is F_σ ;
- 2) if $\text{dom } f$ is a topological space of the second category, the restriction $f|_{\text{dom } f}$ is upper bounded.

Proof. We shall not prove the evident statement 1).

2) It is clearly seen that $g = f|_{\text{dom } f}$ is a lower semicontinuous affine function and that is why it has an upper bound m in R on some non-empty intersection $\text{dom } f$ with the open subset $\mathcal{V} \subset E$ (the set of discontinuity points of g is of first category in $\text{dom } f$: by Osgood's Theorem and, by Baire's Theorem g has a continuity point in $\text{dom } f$). We shall choose a point y from this intersection. As $\text{dom } f$ is bounded in E , it can be absorbed by the neighbourhood $\mathcal{V} - y$ of zero point. That is why the inclusion

$$\text{dom } f \subset y + n(\mathcal{V} - y)$$

is valid for some natural n . As g is an affine function, it must be upper bounded on $\text{dom } f$. Indeed, let the point $z \in \text{dom } f$, then there exists a point $t \in \mathcal{V} \cap \text{dom } f$ such that

$$z = y + n(t - y),$$

then

$$t = \frac{1}{n} z + \frac{n-1}{n} y,$$

$$g(t) = \frac{1}{n} g(z) + \frac{n-1}{n} g(y)$$

and

$$g(z) = n g(t) + (1-n)g(y) \leq n m + (1-n)g(y).$$

Remark. The proof of the statement 2) is a repetition of the proof of the first part of the Choquet theorem for a case when $\text{dom } f$ is compact (see [3], Theorem 1.2.6).

Corollary. 1.1. If condition 2) of the proposition 1 is fulfilled, the effective set of the function f is closed.

Theorem 2 (Bear H.S. [2]). Let S be a convex compact set and we shall consider the sequence $\{a_n\}$ belonging to the space $A(S)$ of all continuous affine real-valued functions on S which satisfy the requirement $a_n \leq a_{n+1}$, $n \in \mathbb{N}$.

If $\{a_n(x)\}$ converges for some point $x \in S$ then $\{a_n(y)\}$ converges for all $y \in \text{face}(x)$.

Corollary 2.1. Let the sequence $\{h_n\}$ belong to the space $A(\text{face}(x))$ of all continuous affine real-valued functions on $\text{face}(x)$ and $h_n \leq h_{n+1}$ for all $n \in \mathbb{N}$. Let us consider the function $h = \sup h_n$:

If $h(x) < +\infty$ then the inequality $h(z) < +\infty$ is valid for any point $z \in \text{face}(x)$.

Proof. For a point $z \in \text{face}(x)$ we shall choose a point $y \in \text{face}(x)$, so that $x \in]y; z[$. As the sequence $\{h_n|_{[y; z]}\}$ satisfies the theorem 2 (the point x is surrounded in $]y; z[$ and $h(x) < +\infty$) then $h(z) < +\infty$.

Theorem 3. Assume that the point $x \in S$ has the face (x) of

the second category; then for any compact

$$K \subset \text{face}(x)$$

there exists such a number $\alpha_K \geq 1$ that

$$\sup_{y \in K} f(y) \leq \alpha_K f(x)$$

for all $f \in A^+(\text{face}(x))$.

Proof. Assume the contrary, then there exist two sequences of points $\{x_n\} \subset K$ and functions $\{f_n\} \subset A^+(\text{face}(x))$ such that $f_n(x_n) \geq n^3 f_n(x)$ for all $n \in \mathbb{N}$.

Consider the function

$$f(y) = \sum_{t=1}^{\infty} \frac{f_t(y)}{t^2 f_t(x)}$$

It is evident that the sequence of the continuous affine functions

$$f^{(m)} = \sum_{t=1}^m \frac{f_t}{t^2 f_t(x)}, \quad m \in \mathbb{N}$$

is increasing to f . As $f(x) < +\infty$ then by Corollary 2.1 $f(y) < +\infty$ for all $y \in \text{face}(x)$. However,

$$f(x) = \sum_{t=1}^{\infty} \frac{f_t(x_n)}{t^2 f_t(x)} \geq \frac{f_n(x_n)}{n^2 f_n(x)} \geq \frac{n^3 f_n(x)}{n^2 f_n(x)} = n.$$

But it contradicts Proposition 1.

Corollary 3.1. Assume that the point $x \in S$ has $\text{face}(x)$ of the second category. Consider the sequence $\{h_n\} \subset A(\text{face}(x))$ with the property $h_n \leq h_{n+1}$ for all $n \in \mathbb{N}$ and the function $h = \sup h_n$. So if $h(x) < +\infty$ then the sequence $\{h_n\}$ uniformly converges to h on every compact $K \subset \text{face}(x)$. In particular, if a set $\text{face}(x)$ is a metrizable or locally compact then $h \in A(\text{face}(x))$.

As $h_{n+p} - h_n \in A^+(\text{face}(x))$, the proof follows from the fact that for any compact $K \subset \text{face}(x)$ we have the following inequality:

$$0 \leq h_{n+p} - h_n \leq \alpha_K (h_{n+p}(x) - h_n(x)).$$

Corollary 3.2. Let $f: S \rightarrow]-\infty; +\infty]$ be a lower semicontinuous affine function. Assume that the point $x \in S$ has a locally compact face $\text{face}(x)$. Then if $h(x) < +\infty$ then

$$f/\text{face}(x) \in A(\text{face}(x)).$$

Proof. By the corollary 1.1.4 [3], f is a pointwise limit of the increasing net $\{a_\alpha\} \subset A(S)$. by the corollary 3.1, for any increasing sequence $\{b_n\} \subset \{a_\alpha\}$ the following inclusion is valid:

$$\sup_n b_n/\text{face}(x) \in A(\text{face}(x)).$$

That is why by the topological lemma by A. Cornea (see [1], p.10) $f/\text{face}(x)$ will be a continuous function.

2. Locally compact Gleason parts and the harmonic Brelot's sheaves

Definition. Let x and y be two points of the convex set \mathcal{U} . It is said that the segment $[x;y]$ extends in \mathcal{U} beyond the point x by a positive number $r > 0$ if $x + r(x-y) \in \mathcal{U}$.

Theorem 4 (H.S. Bear [2]). Let x and y belong to S . The segment $[x;y]$ extends in S beyond the point x by the positive number $r > 0$ if and only if

$$a(y) \leq (1 + \frac{1}{r}) a(x)$$

for all $a \in A^+(S)$.

Definition. Two points x, y are said to be included in one part of the convex set \mathcal{U} , if $\text{face}(x) = \text{face}(y)$, or which is equivalent to the line segment $[x;y]$ extends in \mathcal{U} beyond x and y .

Proposition 5. Let \mathcal{U} now be a convex subset of E with an induced topology from E . Assume that $\mathcal{U} = \bigcup_{\alpha \in J} \mathcal{U}_\alpha$, where \mathcal{U}_α are convex open subsets of \mathcal{U} . Let $f: \mathcal{U} \rightarrow R$. Then from

$f|_{\mathcal{U}_\alpha} \in A(\mathcal{U}_\alpha)$, $\alpha \in J$ it follows $f \in A(\mathcal{U})$ (as in the introduction on $A(\mathcal{U}_\alpha)$, $A(\mathcal{U})$ denote the spaces of all continuous affine real-valued functions on \mathcal{U}_α and on \mathcal{U}).

Proof. Let us check that f is an affine function. Let $a, b \in \mathcal{U}$ and consider the affine isomorphism $\varphi: \alpha \rightarrow (1-\alpha)a + \alpha b$ where $\alpha \in [0;1]$. Then the sets of the affine functions on segments $[a;b]$ and $[0;1]$ are isomorphic. As a restriction on $[a;b]$ of a locally convex topology from E is the topology, defined by the image of the topology on $[0;1]$ after mapping φ , then the function $f \circ \varphi$ will be locally affine on $[0;1]$ and hence affine on $[0;1]$. Hence we have that f is an affine function on $[a;b]$.

Definition. Let Y be a topological space. A sheaf of functions on Y is the map \mathcal{F} defined on the set of open sets of Y such that:

- a) for any open set \mathcal{U} of Y $\mathcal{F}(\mathcal{U})$ is the set of functions on \mathcal{U} ;
- b) for any two open sets \mathcal{U}, \mathcal{V} of Y such that $\mathcal{U} \subset \mathcal{V}$ the restriction of any function from $\mathcal{F}(\mathcal{V})$ to \mathcal{U} belongs to $\mathcal{F}(\mathcal{U})$;
- c) for any family $(\mathcal{U}_i)_{i \in J}$ of open sets of Y a function on $\bigcup_{i \in J} \mathcal{U}_i$ belongs to $\mathcal{F}(\bigcup_{i \in J} \mathcal{U}_i)$ if for any $i \in J$ its restriction to \mathcal{U}_i belongs to $\mathcal{F}(\mathcal{U}_i)$.

Definition. A sheaf of functions H on a locally compact space Y is called a harmonic sheaf, if for any open set \mathcal{U} of Y $H(\mathcal{U})$ is a real vector space of real continuous functions on \mathcal{U} . A function defined on the set containing an open set \mathcal{U} is called an H -function on \mathcal{U} if its restriction to \mathcal{U} belongs to $H(\mathcal{U})$.

Definition. We shall say that a harmonic sheaf H on a locally compact space Y possesses the Brelot convergence property, if the

limit function of any increasing sequence of H-functions on any open connected set of Y is an H-function whenever it is finite at a point.

Let \mathcal{D} be a locally compact part of L . By the proposition 5 the map H , defined in the introduction on the set of open subsets \mathcal{D} satisfies the axioms of the harmonic sheaf.

Theorem 6. The sheaf H on part \mathcal{D} satisfies the Brelot convergence property.

Proof. Let \mathcal{U} be an open and connected subset of \mathcal{D} . Each point $x \in \mathcal{U}$ has a convex compact neighbourhood $\mathcal{V} = \mathcal{V}(x) \subset \mathcal{U}$. That is why, if the increasing sequence of functions from the space $A(\mathcal{V})$ of all continuous affine functions on \mathcal{V} is bounded even in one interior point $y \in \mathcal{V}$ then by the corollary 3.1, its limit will belong to $A(\mathcal{V})$.

Now let $\{h_n\}$ be an increasing sequence of H-functions on \mathcal{U} , $h = \sup h_n$. Assume that in the point $x \in \mathcal{U}$ the function $h(x) < +\infty$. Let us prove that $\mathcal{U} = \text{dom } h$. If it is not so, then by the theorem 2 the set $T = \{y \in \mathcal{U} : h(y) = +\infty\}$ is open. By connection \mathcal{U} at the bound of the set $\text{dom } h$ there exists a point, the convex compact neighbourhood of which has a non-empty intersection both with T and with $\text{dom } h$, which is impossible by the theorem 2. So we have proved that h is bounded in every point $z \in \mathcal{U}$. Besides, we have proved earlier that h is a locally affine and continuous function. From this we have that $h \in H(\mathcal{U})$. A necessary and sufficient condition that the part $\mathcal{D} \subset S$ is locally compact, will be obtained in the following proposition.

Proposition 7. Assume that some point $x \in \mathcal{D} \subset S$ has a compact neighbourhood $K(x) \subset \mathcal{D}$, then \mathcal{D} is locally compact.

Proof. Let $y \in \mathcal{D}$. We can consider the neighbourhood $K(x)$ convex without limiting generality. The point x is surrounded in $K(x)$ (i.e. for every point $y \in K(x)$ and $y \neq x$ the segment $[x; y]$ may be extended in $K(x)$ beyond the point x). By the theorem 3 there exists such a number $\alpha > 1$ that the following inequalities are fulfilled:

$$\sup_{z \in K(x)} a(z) \leq \alpha a(x) \quad (\forall a \in A^+(K(x)))$$

and

$$\sup_{z \in K(x)} a(z) \leq \alpha a(y) \quad (\forall a \in A^+(S)).$$

Let $r > 0$ be such a number that $\alpha = 1 + \frac{1}{r}$. By the theorem 4 ($\forall z \in K(x)$) we obtain the following inclusions:

$$x + r(x-z) \in K(x), \quad y + r(y-z) \in S.$$

Let $t \in]0; r[$. It is obvious that ($\forall z \in K(x)$) we have the following inclusions:

$$x + t(x-z) \in K(x), \quad y + t(y-z) \in \mathcal{D}.$$

We shall consider the map

$$\varphi: z \rightarrow \tilde{z} = y + t(y-z), \quad z \in K(x).$$

It is clearly seen that φ is continuous and hence $K(\tilde{x}) = \varphi(K(x))$ is a compact neighbourhood of the point \tilde{x} . We shall consider the set

$$V(y) = K(\tilde{x}) + y - \tilde{x}.$$

If we can prove the inclusion $V(y) \subset \mathcal{D}$ then $V(y)$ will be a compact neighbourhood of the point y .

Indeed, let $k \in K(\tilde{x}) + y - \tilde{x}$, then

$$k = \tilde{z} + y - \tilde{x} = y + t(x-z).$$

We remark that for every $z \in K(x)$ we have the equalities

$$\tilde{z} - \tilde{x} = t(x-z), \quad \tilde{x} + (x-z)t = \tilde{z}, \quad \frac{\tilde{z}}{1+t} + \frac{t}{1+t} z = y.$$

Now we have

$$\begin{aligned}
 k &= y + (x-z)t = \frac{1}{1+t} [\tilde{x} + (x-z)t] + \frac{t}{1+t} [x + t(x-z)] = \\
 &= \frac{1}{1+t} + \frac{t}{1+t} [x + t(x-z)] \in \mathcal{D}.
 \end{aligned}$$

The proof is complete.

3. On some characteristics of parts and faces of simplexes.

Let S be now Choquet simplex, $E(S)$ will denote the extreme boundary, i.e. the set of the extreme points of S , μ_x may denote the boundary measure (see [3]) which represents $x \in S$. On the linear space $C(\overline{E(S)})$ of all continuous real-valued functions on $\overline{E(S)}$ we define the Dirichlet operator $f \rightarrow u_f$, where

$$u_f(x) = \mu_x(f), \quad x \in S.$$

Let us consider the part P of S and any two points $x, y \in P$. For the part P of S we can define a function $\alpha_P: P \times P \rightarrow [1; +\infty[$ as follows:

$$\alpha_P(x; y) = \inf \{1 + \Delta^{-1} : [x; y] \text{ extends by } \Delta \text{ in } S\}.$$

Let us define as in [3] the part metric on P

$$\rho(x; y) = \ln \alpha_P(x; y).$$

Proposition 8. The affine function $u_f|_P$ is continuous in the part metric.

Proof. Let us consider the σ -neighbourhood of the point x .
 $V(x; \sigma) = \{y \in P : \rho(x; y) < \sigma\} = \{y \in P : \alpha_P(x; y) < e^\sigma\}.$

There exists such a number $\Delta > 0$ that $1 + \Delta^{-1} < e^\sigma$ and the segment $[x; y]$ may be extended in S by the number Δ , that is why by the theorem 11.5.25 [3] the following inequality will be valid:

$$\mu_y \leq (1 + \Delta^{-1}) \mu_x.$$

for a positive function $f \in C(\overline{E(S)})$ we obtain

$$u_f(y) - u_f(x) = \mu_y(f) - \mu_x(f) \leq \Delta^{-1} \mu_x(f) \leq \Delta^{-1} \|f\| < e^\sigma - 1) \|f\|.$$

As

$$u_f(x) - u_f(y) < (e^{\sigma} - 1) \|f\|$$

then

$$|u_f(x) - u_f(y)| < (e^{\sigma} - 1) \|f\|.$$

Let now be $\varepsilon > 0$, we consider the number $\sigma = \ln\left(\frac{\varepsilon}{\|f\|} + 1\right)$. It is obvious that as soon as $\varphi(x; y) < \sigma$ the inequality $|u_f(x) - u_f(y)| < \varepsilon$ will be fulfilled. If $f \neq 0$ we take such a number $c > 0$ that $f + c > 0$. In this case

$$\mu_y(f) - \mu_x(f) = \mu_y(f+c) - \mu_x(f+c)$$

and $\sigma = \ln\left(\frac{\varepsilon}{\|f+c\|} + 1\right)$. The proposition is complete.

Theorem 9. Let $\{f_n\}$ be a sequence of the affine functions of the first Baire class defined on S and $\|f_n\| < c$, $n \in \mathbb{N}$ for some number $c > 0$. Let $x \in S$, then the sequence $\{f_n\}$ has the subsequence $\{f_{n_k}\}$ which on face $\text{face}(x)$ converges pointwise to some affine function $f: \text{face}(x) \rightarrow \mathbb{R}$.

Proof. The sequence $\{f_n\}$ may be considered as the bounded sequence of continuous linear functionals on the space $L_2 = L_2(\mu_x)$ as

$$\left| \int f_n h \, d\mu_x \right| \leq c \sqrt{\int |h|^2 \, d\mu_x} = c \|h\|_{L_2}.$$

As the unit ball in L_2 is weakly compact then from $\{f_n\}$ we may choose the subsequence $\{f_{n_k}\}$ which converges on every function $h \in L_2$, i.e.

$$\int f_{n_k} h \, d\mu_x \rightarrow \cdot$$

For every point $y \in \text{face}(x)$ the measure μ_y is absolutely continuous with respect to the measure μ_x ($\mu_y \ll \mu_x$) and by the theorem 11.5.15 [3] we have

$$\left\| \frac{d\mu_y}{d\mu_x} \right\|_{L^\infty} \leq \text{const.}$$

Therefore the density function $\frac{d\mu_y}{d\mu_x} \in L_2(\mu_x)$. By the Choquet theorem [3, p. 16] for every function $f_n, n \in \mathbb{N}$ the barycentric formulae are valid:

$$\mu_y(f_n) = f_n(y).$$

Hence

$$f_{n_k}(y) = \int f_{n_k} d\mu_y = \int f_{n_k} \left(\frac{d\mu_y}{d\mu_x} \right) d\mu_x \rightarrow$$

It is obvious that the limit function is affine.

Let S be a metrizable simplex, B is a set of all bounded measurable Borel functions defined on $E(S)$. As above on B we define the Dirichlet operator. Let us denote by A_0 the set of continuity points of u_f for all $f \in B$. As follows from [4] if $A_0 \neq \emptyset$ then with every point x the set A_0 contains $\text{face}(x)$. If $\{f_n\}$ is a sequence function from the theorem 9 then the limit function f for its subsequence $\{f_{n_k}\}$ will belong to the first Baire class.

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