Petr Lachout

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Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 2, 371--375

Persistent URL: http://dml.cz/dmlcz/106458

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
27,2 (1986)

A MARTINGALE CENTRAL LIMIT THEOREM

Petr LACHOUT

Abstract: The paper presents a martingale central limit theorem which connects the well-known result by McLeish (1974) with that one by Hall and Heyde (1980) and continues the research starting in [2].

Key words and phrases: A zero-mean martingale array, the central limit theorem, a uniform integrability.

Classification: Primary 60F05
Secondary 60G42

Let us formulate main results.

Theorem: Let $(S_{nk}, A_{nk}, k=1,\dots,k_n, n \in N)$ be a zero-mean martingale array with differences X_{nk}. Suppose that

(i) $E \max \{|X_{nk}| \mid k=1,\dots,k_n\} \longrightarrow 0$

(ii) $U_n = \sum_{k=1}^{k_n} X_{nk}^2 \xrightarrow{d} \eta^2$ [1], where η^2 is an a.s. finite random variable,

(iii) $\lim_{k \to +\infty} \limsup_{n \to +\infty} E|\exp(-tU_n) - E[\exp(-tU_n)/A_{nk}]| = 0$

for every positive number t.

Then $S_{nk_n} \xrightarrow{d} S$, where the r.v. S has the characteristic function $E \exp(-\frac{1}{2} t^2 \eta^2)$.

Proof: The proof has the same framework as the proof of

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1) $\xrightarrow{d}$ means the usual convergence in distribution.

the theorem (2.3) in [3] and as the proof of the theorem 3.2 in [1], chapter 3, p. 58.

Put $M_n = \max\{|X_{nk}| \mid k=1,\dots,k_n\}$ and fix a real number t and positive number ε. According to (iii) there are a natural number j and a real number D such that

(1) $P(\eta^2 \geq D) < \varepsilon$,

(2) $\limsup\limits_{n \to +\infty} E|\exp(-\frac{t^2}{2} U_n) - E[\exp(-\frac{t^2}{2} U_n)/A_{nj}]| < \varepsilon \exp(-\frac{t^2}{2} D)$.

Define the following transformation

$$Y_{nk} = X_{nk}\, I\left(\sum_{s=1}^{k-1} X_{ns}^2 \leqq D\right),$$

$$J_n = \begin{cases} \max\{k \mid Y_{nk} \neq 0\} & \text{if there is a natural number } k \text{ such that } Y_{nk} \neq 0, \\ j & \text{if } Y_{nk} = 0 \text{ for every } k=1,\dots,k_n. \end{cases}$$

$(Y_{nk}, A_{nk}, k=1,\dots,k_n, n \in N)$ is obviously an array of martingale differences.

Denote $T_{nk} = \prod_{s=1}^{k} (1 + itY_{ns})$, $T_n = T_{nk_n}$,

$$W_n = \sum_{s=3}^{+\infty} \frac{(-it)^s}{s} \sum_{p=1}^{k_n} Y_{np}^s,$$

$$B_n = [M_n \leqq \frac{1}{2|t|}],\quad F_n = [U_n \leqq D],\quad C_n = B_n \cap F_n.$$

Now we can calculate

$$|W_n| \leqq t^2 \sum_{s=3}^{+\infty} (|t| M_n)^{s-2} (Y_{nJ_n}^2 + \sum_{k=1}^{J_n-1} Y_{nk}^2) \leqq t^2 (M_n^2 + D) \sum_{s=3}^{+\infty} (|t| M_n)^{s-2}.$$

Hence by (i)

(3) $W_n I(B_n)$ are uniformly bounded r.v.'s and $W_n I(B_n) \xrightarrow{d} 0$.

We may derive an inequality for T_{nk}

$$|T_{nk}| \leqq (1 + |t|\, |Y_{nJ_n}|) \prod_{s=1}^{J_n-1} (1 + t^2 Y_{nk}^2)^{\frac{1}{2}}$$

(4) $|T_{nk}| \leqq (1 + |t| M_n) \exp(\frac{1}{2} t^2 D)$.

We shall use the following property.

<u>Lemma</u>: Let f_n be complex functions which are A_{nj}-measurable and uniformly bounded. Then $E(T_n-1)f_n \longrightarrow 0$.

<u>Proof</u>: $E\ T_n f_n = E\{T_{nj} f_n\ E[\prod_{k=j+1}^{j_n} (1+itY_{nk})/A_{nj}]\} = E\ T_{nj} f_n$.

Then $E(T_n-1)f_n \longrightarrow 0$ since $T_{nj} \xrightarrow{d} 1$. □

Notice that

$$E[T_n \exp(-\frac{t^2}{2} U_n)\ I(F_n)] - E\exp(-\frac{t^2}{2} U_n) =$$

$$= E[T_n(\exp(-\frac{t^2}{2} U_n) - E[\exp(-\frac{t^2}{2} U_n)/A_{nj}])] +$$

$$+ E\{(T_n-1)E[\exp(-\frac{t^2}{2} U_n)/A_{nj}]\} - E[T_n \exp(-\frac{t^2}{2} U_n) I(U_n > D)].$$

Using (1),(2),(4) and the previous lemma we obtain

$$(5)\quad \limsup_{n\to+\infty} E|T_n \exp(-\frac{t^2}{2} U_n)\ I(F_n) - E\exp(-\frac{t^2}{2} U_n)| \leqq$$

$$\leqq 2\varepsilon + 2|t| \exp(\frac{t^2}{2} D) \limsup_{n\to+\infty} EM_n = 2\varepsilon .$$

Now we may write

$$E\exp(it\sum_{k=1}^{k_n} X_{nk}) - E\exp(-\frac{t^2}{2}\eta^2) = E[\exp(it\sum_{k=1}^{k_n} X_{nk})(1-I(C_n))] +$$

$$+ E\{[\exp(it\sum_{k=1}^{k_n} X_{nk}) - T_n \exp(-\frac{t^2}{2} U_n + W_n)]\ I(C_n)\} +$$

$$+ E[T_n\exp(-\frac{t^2}{2} U_n)(\exp W_n-1)I(C_n)] + E[T_n\exp(-\frac{t^2}{2} U_n)(I(C_n)-I(F_n))] +$$

$$+ \{E[T_n\exp(-\frac{t^2}{2} U_n)I(F_n)] - E\exp(-\frac{t^2}{2} U_n)\} +$$

$$+ \{E\exp(-\frac{t^2}{2} U_n) - E\exp(-\frac{t^2}{2}\eta^2)\}.$$

Noting that the second term of the right hand side of the equality is vanishing, we can see

$$|E\exp(it\sum_{k=1}^{k_n} X_{nk}) - E\exp(-\frac{t^2}{2}\eta^2)| \leqq P(M_n > \frac{1}{2|t|}) + P(U_n > D) +$$

$$+ E[|T_n|\ |\exp W_n-1|\ I(B_n)] + E\ |T_n|\ I(M_n > \frac{1}{2|t|}) +$$

$+ |E[T_n \exp(- \frac{t^2}{2} U_n) I(F_n)] - E \exp(- \frac{t^2}{2} U_n)| +$

$+ |E \exp(- \frac{1}{2} t^2 U_n) - E \exp(- \frac{1}{2} t^2 \eta^2)|.$

Using (i),(ii),(1),(3),(4) and (5) we obtain that

$$\limsup_{n \to +\infty} |E \exp(it \sum_{k=1}^{k_n} X_{nk}) - E \exp(- \frac{1}{2} t^2 \eta^2)| \leqq 3\varepsilon .$$

Now, it is clear that $S_{nk_n} \xrightarrow{d} S$, where the r.v. S has the characteristic function $E \exp(- \frac{1}{2} t^2 \eta^2)$. □□

Finally, let us remark that each of the following conditions implies the condition (iii).

(6) For every positive numbers ε, t there are a natural number j and functions f_n that are A_{nj}-measurable, $n \in N$, such that $\limsup_{n \to +\infty} E|\exp(-tU_n) - f_n| < \varepsilon$.

(7) Let ε be a positive number and $B_n \in \sigma(U_n)$, $n \in N$. Then there are a natural number j and sets $C_n \in A_{nj}$, $n \in N$, such that $P(B_n \triangle C_n) < \varepsilon$ for any $n \in N$.

(8) η^2 is a nonnegative constant a.s.

(9) The martingale array is defined on a common probability space, $U_n \xrightarrow{p} \eta^2$ and the σ-fields A_{nk} are nested (i.e. $A_{nk} \subset A_{n+1,k}$ for $k=1,\dots,k_n, n \in N$).

Note that (8) is the assumption (c) of the theorem (2.3) in [3] and (9) are the assumptions (3.19) and (3.21) of the theorem 3.2 in [1].

References

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Matematicko-fyzikální fakulta, Univerzita Karlova, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(Oblatum 7.11. 1985)