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A MARTINGALE CENTRAL LIMIT THEOREM
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Abstract: The paper presents a martingale central limit theorem which connects the well-known result by McLeish (1974) with that one by Hall and Heyde (1980) and continues the research starting in [2].

Key words and phrases: A zero-mean martingale array, the central limit theorem, a uniform integrability.

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Let us formulate main results.

Theorem: Let \((S_{nk}, A_{nk}, k=1, \ldots, k_n, n \in \mathbb{N})\) be a zero-mean martingale array with differences \(X_{nk}\). Suppose that

(i) \(E \max \{|X_{nk}|; k=1, \ldots, k_n\} \rightarrow 0\)

(ii) \(U_n = \frac{k_{nv}}{k_{rv}} \chi_{nk}^2 \xrightarrow{d} \eta_2\), where \(\eta_2\) is an a.s. finite random variable,

(iii) \(\lim \limsup_{k \rightarrow +\infty} E[\exp(-tU_n) - E[\exp(-tU_n)/A_{nk}]] = 0\)

for every positive number \(t\).

Then \(S_{nk} \xrightarrow{d} S\), where the r.v. \(S\) has the characteristic function \(E \exp(-\frac{1}{2} t^2 \eta_2^2)\).

Proof: The proof has the same framework as the proof of 1) \(\xrightarrow{d}\) means the usual convergence in distribution.

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the theorem (2.3) in [3] and as the proof of the theorem 3.2 in [1], chapter 3, p. 58.

Put $M_n = \max \{|X_{nk}| \mid k=1,...,k_n\}$ and fix a real number $t$ and positive number $\varepsilon$. According to (iii) there are a natural number $j$ and a real number $D$ such that

1. $P(\gamma^2 \geq D) < \varepsilon$,

2. $\limsup_{t \to +\infty} E[\exp(-\frac{t^2}{2} U_n) - E[\exp(-\frac{t^2}{2} U_n)/A_n^j]] < \varepsilon \exp(-\frac{t^2}{2} D)$.

Define the following transformation

$$Y_{nk} = X_{nk} I \left( \sum_{k=1}^{k} x_n^2 \leq D \right),$$

$$J_n = \begin{cases} \max k |Y_{nk}^2 | \neq 0 & \text{if there is a natural number } k \text{ such that } Y_{nk}^2 \neq 0, \\ j & \text{if } Y_{nk}^2 = 0 \text{ for every } k=1,...,k_n. \end{cases}$$

$(Y_{nk}, A_{nk}, k=1,...,k_n, n \in \mathbb{N})$ is obviously an array of martingale differences.

Denote $I_{nk} = \mathbb{P}(tY_{nk}^2 \leq D, 1 + \sqrt{t}Y_{nk}^2), I_n = \sum_{k=1}^{k_n} I_{nk},$

$$W_n = \sum_{s=3}^{\infty} \frac{(-it)^{s-2}}{s-2} \mathbb{E}[Y_{nk}^s],$$

$$B_n = \left[ M_n \leq \frac{1}{2|\varepsilon|} \right], F_n = \left[ U_n \leq D \right], \mathbb{C}_n = B_n \cap F_n.$$

Now we can calculate

$$|W_n| \leq t^2 \sum_{s=3}^{\infty} \mathbb{E}[tM_n^{s-2}(Y_{nk}^s + \sum_{k=1}^{k_n} Y_{nk}^s \leq t^2(M_n^2 \leq D) \sum_{s=3}^{\infty} \mathbb{P}(tM_n \leq \frac{1}{2|\varepsilon|})^s).$$

Hence by (1)

3. $W_n I(\mathbb{C}_n)$ are uniformly bounded r.v's and $W_n I(\mathbb{C}_n) \to 0$.

We may derive an inequality for $I_{nk}$

$$|I_{nk}| \leq (1 + |t||Y_{nk}|^2) \sum_{k=1}^{k_n} (1 - t^2 Y_{nk}^2)^{j-1}.$$ (4.1)

$$\frac{1}{2} |tM_n| \exp \left( \frac{1}{2} t^2 D \right).$$
We shall use the following property.

**Lemma**: Let $f_n$ be complex functions which are $A_\infty$-measurable and uniformly bounded. Then $E(T_n-1)f_n \longrightarrow 0$.

**Proof**: $E(T_n f_n) = E(T_n f_n - E(T_n f_n | A_n)) + E(T_n f_n | A_n)$

Then $E(T_n-1)f_n \longrightarrow 0$ since $T_n \to 1$. □

Notice that

$$E[T_n \exp(- \frac{1}{2} U_n) I(F_n)] - E \exp(- \frac{1}{2} U_n) =$$

$$= E[T_n \exp(- \frac{1}{2} U_n) - \exp(- \frac{1}{2} U_n)/A_n)] +$$

$$+ E(T_n-1)E[\exp(- \frac{1}{2} U_n/A_n)] - E[T_n \exp(- \frac{1}{2} U_n) I(U_n > D)].$$

Using (1), (2), (4) and the previous lemma we obtain

$$\limsup_{n \to +\infty} E|T_n \exp(- \frac{1}{2} U_n) I(F_n) - \exp(- \frac{1}{2} U_n)| \leq$$

$$\leq 2 \varepsilon + \frac{1}{2} t \exp(\frac{1}{2} D) \limsup_{n \to +\infty} E M = 2 \varepsilon.$$ 

Now we rewrite

$$E \exp(it \sum_{k=1}^{A_n} \chi_{nk}) - \exp(- \frac{1}{2} \gamma^2) = E[\exp(it \sum_{k=1}^{A_n} \chi_{nk}) (1 - I(C_n))] +$$

$$+ E \exp(it \sum_{k=1}^{A_n} \chi_{nk}) - T_n \exp(- \frac{1}{2} U_n + W_n)] I(C_n)] +$$

$$+ E[T_n \exp(- \frac{1}{2} U_n) \exp W_n) I(C_n)] + E[T_n \exp(- \frac{1}{2} U_n) (I(C_n) - I(F_n))] +$$

$$+ [E[T_n \exp(- \frac{1}{2} U_n) I(F_n)] - E \exp(- \frac{1}{2} U_n) I(F_n)] +$$

$$+ |E \exp(- \frac{1}{2} U_n) - \exp(- \frac{1}{2} \gamma^2)|.$$ 

Noting that the second term of the right hand side of the equality is vanishing, we can see

$$|E \exp(it \chi_{nk}) - \exp(- \frac{1}{2} \gamma^2) - P M_n - \frac{1}{2 \pi |1|} + P(U_n > 0) +$$

$$+ E|T_n| |\exp W_n - 1| I(B_n)] + E |T_n| I(M_n > \frac{1}{2 \pi |1|}) +$$

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\[ + |E[T_n \exp\left(-\frac{t^2}{2} U_n\right) I(F_n)] - E \exp\left(-\frac{t^2}{2} U_n\right)| + \\
+ |E \exp\left(-\frac{1}{2} t^2 U_n\right) - E \exp\left(-\frac{1}{2} t^2 \eta^2\right)|. \]

Using (i), (ii), (1), (3), (4) and (5) we obtain that

\[ \limsup_{n \to +\infty} |E \exp\left(it \sum_{k=1}^{n} X_{nk}\right) - E \exp\left(-\frac{1}{2} t^2 \eta^2\right)| \leq 3 \epsilon. \]

Now, it is clear that \( S_{nk} \xrightarrow{d} S \), where the r.v. \( S \) has the characteristic function \( E \exp\left(-\frac{1}{2} t^2 \eta^2\right) \). □ □

Finally, let us remark that each of the following conditions implies the condition (iii).

(6) For every positive numbers \( \epsilon, t \) there are a natural number \( j \) and functions \( f_n \) that are \( A_{nj} \)-measurable, \( n \in \mathbb{N} \), such that

\[ \limsup_{n \to +\infty} E |\exp(-tU_n) - f_n| < \epsilon. \]

(7) Let \( \epsilon \) be a positive number and \( B_n \in \sigma(U_n), n \in \mathbb{N} \). Then there are a natural number \( j \) and sets \( C_n \in A_{nj}, n \in \mathbb{N} \), such that

\[ P(B_n \Delta C_n) < \epsilon \quad \text{for any} \quad n \in \mathbb{N}. \]

(8) \( \eta^2 \) is a nonnegative constant a.s.

(9) The martingale array is defined on a common probability space, \( U_n \xrightarrow{p} \eta^2 \) and the \( \sigma \)-fields \( A_{nk} \) are nested (i.e. \( A_{nk} \subset A_{n+1,k} \) for \( k=1, \ldots, k_n, n \in \mathbb{N} \)).

Note that (8) is the assumption (c) of the theorem (2.3) in [3] and (9) are the assumptions (3.19) and (3.21) of the theorem 3.2 in [1].

References


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