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A NONSTANDARD TREATMENT WITH QUANTITIES

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Abstract: The following assertion is consistent: If we compute by two different manners(using considerations of "integral calculus" with the infinitely small quantities) two values x_1, x_2 of a quantity and we use only internal means and the predicate "to be infinitely small" then $x_1/x_2 \doteq 1$. An example is given proving that the assertion does not hold if we use the predicate "to be a standard real number".

Key words: Standard, nonstandard, infinitely small, monad, indiscernibility equivalence.

Classification: Primary 03E70
Secondary 03H05

Introduction. Let us formulate the assertion from the abstract exactly. Let $x \doteq y$ denote ($\forall n$ standard natural number) $(|(x/y)-1|<1/n)$.

Assertion $\ast$: If F is a mapping having the following properties:

1) F is defined by a normal formula (only the quantification of internal sets is allowed) using the predicate "to be a standard natural number" and F is a part of a *finite set.

2) F is one-one.

3) $dom(F) \subset \{[a,b); [a,b) \subseteq [0,x_1)\} \,\&\, rng(F) \subset \{[c,d); [c,d) \subseteq [0,x_2)\} \,\&\, (\forall I_1, I_2 \in dom(F))(I_1 \cap I_2 = 0) \,\&\, (\forall I_1, I_2 \in rng(F))(I_1 \cap I_2 = 0)$.

4) $(\forall t \in [0, x_1))(\exists ! [a,b) \in dom(F))(t \in [a,b)) \,\&\, (\forall u \in [0,x_2))$

$(\exists! [c,d) \in rng(F))(u \in [c,d))$.

5) $(\forall a,b,c,d)([c,d) = F([a,b)) \Longrightarrow d-c \doteq b-a)$.

Then $x_1 \doteq x_2$ holds.

Remark: We express the assertion $*$ in other words. Let us have two partitions (they need not be internal sets) and a correspondence between these two partitions. The partitions and the correspondence are constructed using only internal means and the notion "to be an infinitely large natural number ". Moreover, we need the corresponding elements to be near (in its length). Then the basic intervals are also near (in the same sense).

Thus if we have two procedures how to compute a quantity as a sum of infinitely small parts and we are able to describe a one-to-one correspondence between terms of these two sums such that the corresponding parts are near (a different neglection), then we obtain the same (in standard part) result.

It is proved that if every projective part of real numbers is Lebesguely measurable, then the assertion $*$ is valid in enlargement. Especially we have the consistency of $*$ by Solovay's result. We have also the consistency of $*$ with AST (alternative set theory). The assertion may be also interesting from, the point of view of nonstandard models of arithmetic.

Technical means: We shall use the set theoretical apparatus. But compared with the classical set theory we add a new class FN = {n; n is a standard natural number} and all the classes definable from FN by normal formulas (the Gödelian closure of FN). Thus we shall work in a special version of the theory of semisets (see [VH] and cf. also [N],[Č],[BB]). The main difference in comparison with the classical set theory lies in the possibility of the existence of a subclass of a set not being a set (a proper semiset). The powerset is then the system of all the parts being sets and not the system of all the possible parts. We use notions and points of view from nonstandard analysis and alternative set theory. The proof of the consistency of $*$ will be done in two steps. We prove $*$ using some axioms in the first section; in the second one we discuss the consistency of the given axioms. To the end of the first section we construct (for any infinitely large α) a one-one mapping of α onto 2α using the predicate to be a standard real number (cf. also [ČV1]).

Let us stress the fact that the results of [Č5] are substantially used in the proof of the consistency of $*$.

§ 1. <u>The proof of $*$ from some special axioms</u>. We use three sorts of variables: $x,y,\ldots$ for sets, $X,Y,\ldots$ for designated classes and classes. Every set is a designated class and every designated class is a class. The fact that a class is a designated class is expressed by $\mathrm{Dsg}(X)$. If we restrict ourselves on sets and designated classes, then our axioms are all the axioms of a special case of G.B. set theory. The axioms of infinity, regularity and choice are irrelevant for our considerations. Let us note that for every designated class X we have $(\forall x)(\exists y)(X \cap x = y)$.

For classes we use the following axioms

1) $(\exists Y)(X \in Y) \Rightarrow (\exists x)(X = x)$ (Sets)

2) $(\forall x)(x \in X \equiv x \in Y) \Rightarrow X = Y$ (Extensionality)

3) If $\varphi(x,X_1,\ldots,X_n)$ is a formula in which only set variables are quantified (i.e. a normal formula), then

$(\forall X_1,\ldots,X_n)(\exists Y)(\forall x)(x \in Y \equiv \varphi(x,X_1,\ldots,X_n))$

(Schema of existence).

Theories of the given type have been introduced and investigated in [BB]. A special case of such a theory is AST (the system of axioms from [S1]), where the role of designated classes play

Sd_V classes or Sd_V^* classes.

Notation: We use the common notation. Moreover, we use α, β, γ,... for infinitely large natural numbers (introduced below) and σ, ρ, ξ, ζ,... for semisets (subclasses of sets). By Card(x) or $|x|$ we denote the cardinality of x (from the point of view of G.B. set theory). We also use $|x|$ for the absolute value in the context of the real number system. We use further N for the set or designated class of natural numbers, i.e. ordinal numbers less than the first limit ordinal in the view-point of G.B. set theory.

Axiom FN: $(\exists X \subsetneq N)(\forall n \in X)(n \subset X \& (\forall Y)(\exists y)(Y \cap n = y))$.

It appears that there is only one proper class X with the property described in this axiom and we denote this class by FN. The first part of the conjunction describes the completeness of FN and the second one may be understood as the standardness of elements of FN.

Definition 1.1: $IL(\alpha) \equiv \alpha \in N\text{-}FN$ (α is infinitely large).

Definition 1.2: $Fin(X) \equiv (\forall Y \subset X)(Y \in V)$.

Here we are not consistent with the Tarski's definition.

Definition 1.3: $Count(X) \equiv (\exists F)(F: X \longleftrightarrow FN)$ (X is countable).

Here we are also inconsistent with the commonly used definition. Our definitions of Fin(X) and Count(X) are consistent with the external meaning of these notions.

Axiom of prolongation: Every countable function is a part of a set function.

Axiom of weak choice: $(\forall R, dom(R)=FN)(\exists F, dom(F)=FN)$ $(F \subseteq R \& F$ is a function).

Let us note that this axiom is not a consequence of the axiom of choice for designated classes, as the well ordering of V given by this axiom is the well ordering only with respect to designated classes.

<u>Axiom měř</u>: $(\forall a)(\forall \wp \subseteq a)(\forall n \in FN)(\exists c,d \subseteq a)(c \subseteq \wp \subseteq d \subseteq a\ \&$ $\&\ |d-c|/|a| < 1/n)$.

The property described in this axiom is connected with the nonstandard treatment of measure due to Loeb and Anderson.

To describe another property (namely the reality) of classes we shall need some nonstandard topological notions (cf. [V]).

<u>Definition 1.4</u>: Let R be reflexive and symmetric. R is said to be compact iff $(\forall x \subseteq dom(R))(\neg Fin(x) \Longrightarrow (\exists y,\ z \in x)(y \neq z\ \&$ $\&\ \langle y,z\rangle \in R))$.

<u>Definition 1.5</u>: A) Let G be a designated system of equivalences (i.e. $Dsg(G)\ \&\ dom(G) \in N\ \&\ (\forall \alpha \in dom(G))(G''\{\alpha\}$ is an equivalence). G is said to be a generating system of a totally disconnected indiscernibility equivalence iff

1) $dom(G) \in N-FN$

2) $(\forall n \in FN)(G''\{n\}$ is compact)

3) $(\forall \alpha \in dom(G))(\alpha \neq 0 \Longrightarrow G''\{\alpha\} \subseteq G''\{\alpha - 1\})$.

B) If G is a generating system of a totally disconnected indiscernibility equivalence, then $\bigcap\{G''\{n\};n \in FN\}$ is said to be a totally disconnected indiscernibility equivalence.

In the paper we shall omit the words totally disconnected in the notion indiscernibility equivalence as we shall use (except of $\doteq$ used for other purpose) only totally disconnected indiscernibility equivalences.

<u>Definition 1.6</u>: Let $\cong$ be an indiscernibility equivalence.

1) $Fig_{\cong}(X) = (\cong)''X$. (The figure of X)

2) $\mathcal{Fig}_{\cong}(X) \equiv X = Fig_{\cong}(X)$. (X is a figure)

<u>Axiom of realness</u>: $(\forall X)(\exists \cong)(\cong$ is an indiscernibility

equivalence & $\mathcal{F}ig_{\leq}$ (X)). (Cf. [ČV1].)

Let us now prove a theorem of a set theoretical form playing a fundamental role in the proof of the main result.

Theorem 1.7 (on approximation): If $\alpha \in N$ and X has the following property $(\forall x \subseteq X)(|x| \leq \alpha)$, then $(\forall n \in FN)(\exists b)(X \subseteq b \ \& \ \& |b| < \alpha(1 + 1/n))$.

To prove the theorem we use the axioms of prolongation, of weak choice, měř and of realness. The theorem can be easily strengthened to the following less readable version:
$(\forall x \subseteq X)(\forall n \in FN)(|x| \leq \alpha(1+1/n)) \Longrightarrow (\forall n \in FN)(\exists b)(X \subseteq b \ \& |b| \leq \leq \alpha(1+1/n))$.

At first let us prove two lemmas. For the proofs let us fix α for a natural number such that $(\forall x \subseteq X)(|x| \leq \alpha)$ and let G denote the generating system for an indiscernibility equivalence in which X is a figure.

Lemma 1.8: $(\forall t \in X)(\exists n \in FN)\ |(G''\{n\})''\{t\}\ | \leq \alpha)$.

Proof: As $\mathcal{F}ig(X)$ we have $(\forall \sigma \in dom(G)-FN)(\forall t \in X)$ $((G''\{\sigma\})''\{t\} \subseteq X)$ and hence $|(G''\{\sigma\})''\{t\}\ | \leq \alpha$, now it suffices to use overspill for finishing the proof.

Lemma 1.9: A set sequence $\{y_i\}$ of disjoint sets such that y_i are equivalence classes of equivalences from G and $|y_i| \leq \alpha$ and $X \subseteq \bigcup\{y_i;\ i \in FN\}$ can be defined.

Proof: We choose successively y_i as the equivalence classes of equivalences from G (we start from the largest - the coarsest) which have the property $|y_i| \leq \alpha$ and which are disjoint with the formerly chosen. From the compactness we have that for any $n \in FN$ there is only a finite number of equivalence classes of $G''\{n\}$. Using L. 1.8 we know that for every $t \in X$ there is y_ξ such that $t \in y_\xi$ and $y_\xi = (G''\{n\})''\{t\}$ for a certain n. From these facts we have $X = \cup\{y_i; i\in FN\}$.

Remark: The lemma has a small incorrectness as it is possible that the sequence $\{y_i\}$ is finite. But the explicit expression of this possibility would make the assertion less readable.

Let us now prove the theorem on approximation. Let $\{y_i\}$ denote the sequence from L. 1.9. Let us fix $n\in FN$. We know (axiom měř) that for every $i\in N$ there are $b_i\subseteq c_i\subseteq y_i$ such that $b_i\subseteq y_i\cap X\subseteq c_i\subseteq$ $\subseteq y_i$ & $|c_i-b_i|/\alpha < 1/(2n\cdot 2^i)$. Using weak choice and prolongation we obtain set sequences $\{b_\xi;\ \xi\in\beta\},\{c_\xi;\ \xi\in\beta\}$ having the properties $b_i\subseteq c_i\subseteq y_i$ & $|c_i-b_i|/\alpha < 1/(2n\cdot 2^i)$ also for infinite i. (Thus e.g. elements of the sequence $\{b_i\}$ are disjoint.) Let us put $d_i = \cup\{b_j; j\leq i\}$. For every $i\in FN$ we have $d_i\subseteq X$ and hence $|d_i|\leq\alpha$. Let us put $e_i = \cup\{c_j;\ j\leq i\}$. For every $i\in FN$ we have $|e_i| <$ $< |d_i|(1+(1/2n)\cdot(\sum_{j<i} 2^{-j}))\leq\alpha(1+1/n)$. To finish the proof it suffices now to use overspill and put $b = e_\delta$ for a suitable $\delta\in N-FN$.

The following example proves that the estimate of the cardinality of the approximating set b cannot be improved.

Example: Let $\delta\in N-FN$. Let us put $\alpha = 2^{\delta}$. We define $F:\alpha\longleftrightarrow\{\gamma;(\forall n\in FN)(\gamma<\alpha(1+1/n))\}$. Let F be such that the righthand side halfmonads of the numbers $1/2^i$ for $i\in FN$ & $i\neq 0$ translate in the righthand side direction. Formally: If $(\exists i\in FN, i\neq 0)(\beta>(\alpha/2^i)\ \&\ (\forall n\in FN)(\beta<(\alpha/2^i)\cdot(1+1/n)))$, then $F(\beta) = \beta+(\alpha/2^i)\cdot F(\beta) = \beta$ in other cases. Now, every set part of F has cardinality less than α (as $dom(F) = \alpha$). If $x\supseteq F$ then $(\forall\gamma\in N-FN)(|x|>\alpha(1+1/\gamma))$ as to properties of rng(F).

Now we prove the main result.

<u>Theorem 1.10</u>: Let F be a one-one mapping having the following properties:

1) dom(F) is a partition of the interval $[0,x_1)$

rng(F) is a partition of the interval $[0,x_2)$

(the partitions need not be sets)

2) If $[c,d) = F([a,b))$ then $(b-a) \doteq (d-c)$

3) $(\exists v)(F \subset v \,\&\, \mathrm{card}(v) \in N)$

Then $x_1 \doteq x_2$.

Proof: Due to the symmetry it suffices to prove only $(\forall n \in FN)(x_2 < x_1(1+1/n))$. We suppose (without loss of generality) $(d-c)/(b-a) < (1+1/4n)$ for every tuple of intervals $\langle [c,d),[a,b)\rangle \in v$ Choose γ infinitely large in such a manner that for every interval $[a,b)$ from dom(v), $\gamma\cdot(b-a)$ is infinitely large. Let $\bar{a}$, $\bar{b}$ be numbers of the form ξ/γ such that $\bar{a}$ is the least $\geq a_1$ and $\bar{b}$ is the largest $\leq b$ (where $[a,b)$ are intervals from dom(v)). Let $[\bar{c},\bar{d})$ denote an interval such that $[\bar{c},\bar{d}) \supseteq [c,d)$ and $\bar{d}-\bar{c} = (\bar{b}-\bar{a})(1+(1/2n))$ for every $\langle [c,d),[a,b)\rangle \in v$. Let us divide intervals $[\bar{a},\bar{b})$ and $[\bar{c},\bar{d})$ on $(\bar{b}-\bar{a})\cdot\gamma$ subintervals of the same length. From F, a one-one correspondence $\bar{F}$ can now be easily described such that the intervals from dom($\bar{F}$) have the same length $1/\gamma$, are disjoint and included in $[0,x_1)$; the intervals from rng($\bar{F}$) have the same length $(1/\gamma)\cdot(1+(1/2n))$ and cover $[0,x_2)$. The estimate $x_2 < x_1\cdot(1+(1/n))$ we now obtain easily from the theorem on approximation (Th.1.7).

To the end of the section let us give an example proving that by using a predicate choosing from every monad of the interval $[0,1)$ one element, we can define, by a normal formula, for every infinitely large α a one-one mapping $\bar{F}: \alpha \longleftrightarrow 2\alpha$ (cf. [ČV 1]). From $\bar{F}$ we can easily define a one-one correspondence between the partitions of $[0,x)$ and of $[0,2x)$. (But the idea of the construction leads in the opposite direction.) Remember that a monad is the class of elements being infinitely close one to the other and that we can obtain the choosing predicate from the predicate "to be a standard real number". We define $\bar{F}$ as follows:

For $0 < x < 1$ where x is the chosen element of a monad, let $\bar{x}$ be the least $\beta < \alpha$ such that $\beta/\alpha \geq x$.

For β such that $\beta/\alpha \doteq x \,\&\, 0 < x < 1$ put $\bar{F}(\beta) = \bar{x} + \beta$,

for β such that $\beta/\alpha \doteq 0$ put $\bar{F}(\beta) = \beta$ and

for β such that $\beta/\alpha \doteq 1$ put $\bar{F}(\beta) = \alpha + \beta$.

Note that FN used for the definition of $\doteq$ can be obtained by a normal formula from the choosing predicate V(x) e.g. by

$n \in FN \equiv n \in N \,\&\, (\exists x_1, x_2)(V(x_1) \,\&\, V(x_2) \,\&\, x_1 \neq x_2 \,\&\, |x_1 - x_2| < 1/n)$.

§ 2. <u>The consistency of axioms</u>. In this section we discuss the consistency of the axioms used in the first section.

The axiom FN: FN can be interpreted as the external set of standard natural numbers (e.g. constants for natural numbers in the ultrapower).

The axiom of prolongation: This axiom can be obtained from some forms of saturatedness of the model (see [PS]), or from the axiom of realness and the weak form of prolongation (see [Č 3]). The weak form of prolongation needs the existence of the prolonging function only for functions from FN to FN and thus it is fulfilled e.g. in enlargements (the enlargement of the function restricted on an infinitely large natural number). For nonstandard models of arithmetic (interpreting FN as ω) we obtain the weak form of prolongation e.g. when $\mathcal{P}(\omega)$ is the standard system of the model. In the ultrapower we obtain the prolongation e.g. by the following considerations: For the ultrapower with ω as the index set let us note at first that F (the function to be prolonged) can be understood as a mapping from ω to ${}^{\omega}V$ (where V denotes the universe of the basic model). Let us put $f(n) = \{\langle (F(1))(n), 1\rangle, \langle (F(2))(n), 2\rangle, \ldots, \langle (F(n))(n), n\rangle\}$ to obtain the prolonging function. For the ultrapower with $\mathcal{P}_{fin}(A)$ as an index set (the ultrapower constructing the enlargement) we proceed analogously. We only use the cardinality of the corresponding set of the index set for the number of elements of the n -tuple (i.e. for $a \in \mathcal{P}_{fin}(A)$ we put $f(a) = \{\langle (F(1))(a),1\rangle, \langle (F(2))(a),2\rangle, \ldots, \langle (F(n))(a),n\rangle\}$, where $|a| = n$).

The weak axiom of choice: In the ultrapower we use an analogous consideration as for the prolongation axiom using a suitable form of the axiom of choice in the basic model. In the framework of nonstandard models of arithmetic we can obtain the weak axiom of choice from the "second order scheme of choice". In AST we obtain the weak choice from the axiom of choice.

The axioms of realness and měř: In [Č 5] it is proved that for the classes defined by normal formulas from FN the axiom of realness holds and also the consistency of the axiom měř for these classes is proved there.

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