Commentationes Mathematicae Universitatis Carolinae

Constantin Zălinescu

On a class of convex sets

Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 3, 543--549

Persistent URL: http://dml.cz/dmlcz/106475

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
27,3 (1986)

ON A CLASS OF CONVEX SETS

Constantin ZALINESCU

Abstract: Let X be a real linear space, $\bar{x} \in X$ and $C \subset X$ be a convex set such that $X = C + R\bar{x}$. We give a characterization for this relation when C is a cone, and necessary conditions for the general case.

Key words: Convex sets, cone, separation theorem, natural topology.

Classification: 52A05

Gerstewitz and Iwanow [2] used the notion of directed linear spaces with respect to some convex subset ("... X sei bezüglich C gerichtet, d.h. $X = C + R\bar{x}$ für ein $\bar{x}$ ...") in order to construct some concave function defined on the whole space. In this short note we give a characterization of this notion when the convex subset is a cone, and necessary conditions in the general case.

Throughout the paper, X is a real linear space and X' is its algebraical dual. For the nonempty convex set $A \subset X$ we denote by ^sA, ^lA, ^iA, A^i, cone A, A_∞, $\bar{A}$ the linear hull, the affine hull, the intrinsic core, the core, the conic hull, the asymptotic cone and the closure in the natural topology, respectively (see [1] and [3]). We recall that for the convex set $A \subset X$

(1) $\bar{x} \in {}^iA \Longleftrightarrow \forall x \in A \ \exists \lambda > 0 : (1+\lambda)\bar{x} - \lambda x \in A,$

(2) $\bar{x} \in A^i \Longleftrightarrow \forall x \in X \ \exists \lambda > 0 : \bar{x} + \lambda x \in A,$

(3) $A^i \neq \emptyset \Longrightarrow \overline{A^i} = \bar{A},\ \bar{A}^i = A^i$,

while for a convex cone A, ${}^{l}A = A - A$ and

(4) $0 \in A^i \Longleftrightarrow A = X$.

We also use the notations R_+ and R_+^* for the sets of nonnegative reals and positive reals, respectively. The Greek letters denote always real numbers.

<u>Proposition 1</u>. Let $K \subset X$ be a convex cone and $\bar{x} \in X$. Then $X = K + R\bar{x}$ if and only if

a) K is a linear subspace of codimension 1 and $\bar{x} \notin K$, or

b) $\{\bar{x}, -\bar{x}\} \cap K^i \neq \emptyset$.

<u>Proof</u>. "$\Longleftarrow$": If a) holds, then, obviously, $X = K + R\bar{x}$. If b) holds, let us take the case $\bar{x} \in K^i$. If $x \in X$, then, by (2), there exists $\lambda > 0$ such that $\bar{x} + \lambda x \in K$, and so $x \in K + R\bar{x}$.

"$\Longrightarrow$": If $K = X$, then b) holds. Let us consider $K \neq X$ in the sequel. We have

$$X = K + R\bar{x} \subset (K - K) + R\bar{x} = {}^{s}K + R\bar{x}.$$

There are two possibilities: (i) ${}^{s}K \neq X$ and (ii) ${}^{s}K = X$. In the case (i) ${}^{s}K$ is a linear subspace of codimension 1 and $\bar{x} \notin {}^{s}K$. Let $u \in {}^{s}K \subset X$; then $u = y + \lambda\bar{x}$ for some $y \in K$ and $\lambda \in R$, and so $\lambda\bar{x} = u - y \in {}^{s}K$. Therefore $\lambda = 0$, whence $u \in K$. Hence $K = {}^{s}K$ and a) holds.

(ii) As $K - K = X$, $\bar{x} = \bar{x}_1 - \bar{x}_2$ with $\bar{x}_1, \bar{x}_2 \in K$. Let $\bar{y} = \bar{x}_1 + \bar{x}_2 \in K$. If $\lambda, \mu > 0$, then

$$K + \lambda\bar{x} = K + 2\lambda\bar{x}_1 - \lambda\bar{y} \subset K - R_+\bar{y},\ K - \mu\bar{x} = K + 2\mu\bar{x}_2 - \mu\bar{y} \subset K - R_+\bar{y}.$$

Therefore $X = K - R_+\bar{y}$. Let us show that $\bar{y} \in K^i$. Consider $y \in X$; then $y + \bar{y} \in X = K - R_+\bar{y}$, and so there exists $\lambda \geq 0$ such that $(1 + \lambda)\bar{y} + y \in K$. Hence, by (2), $\bar{y} \in K^i$. Let us show now that $\{\bar{x}, -\bar{x}\} \cap K^i \neq \emptyset$. Suppose, by way of contradiction, that $\{\bar{x}, -\bar{x}\} \cap K^i = \emptyset$. Then there exist $x_1', x_2' \in X' \setminus \{0\}$ such that

(5) $$\langle \bar{x}, x_1' \rangle \leq 0 \leq \langle x, x_1' \rangle \quad \forall x \in K,$$

(6) $$\langle -\bar{x}, x_2' \rangle \leq 0 \leq \langle y, x_2' \rangle \quad \forall y \in K.$$

If $\langle \bar{x}, x_1' \rangle = 0$, then, by (5), $0 \leq \langle \lambda\bar{x} + x, x_1' \rangle$ for every $\lambda \in R$ and $x \in K$, and so $x_1' = 0$, a contradiction. Hence, $\langle \bar{x}, x_1' \rangle < 0 < \langle \bar{x}, x_2' \rangle$. Therefore there are $\alpha, \beta > 0$ such that $\langle \bar{x}, \alpha x_1' + \beta x_2' \rangle = 0$. From (5) and (6) we obtain that $0 \leq \langle x, \alpha x_1' + \beta x_2' \rangle$ for every $x \in K$, so that, as above, we obtain $\alpha x_1' + \beta x_2' = 0$. We may take $\alpha = \beta = 1$, whence $x_2' = -x_1'$. From (5) and (6) we obtain that $0 \leq \langle x - y, x_1' \rangle$ for all $x, y \in K$. As $K - K = X$, we obtain once again $x_1' = 0$, a contradiction.

Corollary 2. Let $K \subset X$ be a convex cone and $\bar{x} \in K \setminus \{0\}$. Then $K + R\bar{x} = X$ if and only if $\bar{x} \in K^i$.

Proof. The sufficiency is proved in the preceding proposition. If $\bar{x} \in K \setminus \{0\}$ and $X = K + R\bar{x}$, then the statement a) or b) of Proposition 1 holds. As a) is impossible in our hypotheses, we have $\{\bar{x}, -\bar{x}\} \cap K^i \neq \emptyset$. If $-\bar{x} \in K^i$, as $x \in K \setminus \{0\} \subset K$, we obtain $0 = \bar{x} - \bar{x} \in K + K^i = K^i$, so that $K = X$. Hence $\bar{x} \in K^i$.

Proposition 3. Let $C \subset X$, $C \neq X$, be a convex set and $\bar{x} \in X$. If $X = C + R\bar{x}$, then one and only one of the following assertions holds:

a) there exists a linear subspace $X_0 \subset X$ of codimension 1 such that $C = c + X_0$ and $\bar{x} \notin X_0$;

b) there exists a linear subspace $X_0 \subset X$ of codimension 1 and $\alpha, \beta \in R$, $\alpha < \beta$, such that $\bar{x} \notin X_0$ and $C^i =]\alpha, \beta[\bar{x} + X_0$;

c) $\bar{C} + R_+^* \bar{x} \subset C^i$;

d) $\bar{C} - R_+^* \bar{x} \subset C^i$.

Proof. It is clear that in our hypothesis, at most one of the conditions a) - d) can take place. Let us show that at least one of them holds. We may suppose, without loss of generality (w.l.g.), that $0 \in C$. For this aim, let $K = \text{cone}\, C$. Then $X = K + R\bar{x}$, so that, by Proposition 1, we have

(i) K is a linear subspace of codimension 1 and $\bar{x} \notin K$ and, therefore $\bar{x} \notin C$, or

(ii) $\{\bar{x}, -\bar{x}\} \cap K^i \neq \emptyset$.

If (i) holds, as above, we obtain that C is a linear subspace. Indeed, if $x \in {}^sC = K$, then $x \in C + R\bar{x}$, i.e. $x = y + \lambda\bar{x}$ for some $y \in C$ and $\lambda \in R$. Therefore $\lambda\bar{x} = x - y \in K$. Hence $\lambda = 0$ and $x = y \in C$.

Suppose now that (ii) holds and, w.l.g., $\bar{x} \in K^i \subset K$. It follows that there exist $\bar{\lambda} > 0$ and $\bar{u}$ such that $2\bar{u} \in C$ and $\bar{x} = \bar{\lambda}\bar{u}$. We intend to show that $\bar{u} \in C^i = {}^iC$, as ${}^sC = {}^1C = X$. Let $x \in C$; as $2x \in K$ and $\bar{u} \in K^i$, there exists $\mu > 0$ with $(1 + \mu)\bar{u} - 2\mu x \in K$, and so there exist $\eta \geq 1$, $v \in C$ such that $(1 + \mu)\bar{u} - 2\mu x = \eta v$. Let us take $\alpha = \eta/2\eta + \mu - 1) \in]0,1[$ and $\lambda = 2\alpha\mu/\eta$. Then $(1 + \lambda)\bar{u} - \lambda x = \alpha v + (1 - \alpha)2\bar{u} \in C$, so that $\bar{u} \in C^i$, by (1).

Assume now that c) and d) do not hold. Then $I = \{\lambda \in R : \lambda\bar{x} \in C\}$ is a bounded interval with nonempty interior. We have only to show that I is bounded above. Suppose that $R_+\bar{x} \subset C$. As $\bar{C} + R_+^*\bar{x} \not\subset C^i$, there exist $\bar{c} \in \bar{C}$ and $\bar{\mu} > 0$ such that $\bar{c} + \bar{\mu}\bar{x} \notin C^i$. By a separation theorem we get $x' \in X' \setminus \{0\}$ such that

$$(7) \qquad \langle \bar{c} + \bar{\mu}\bar{x}, x' \rangle \leq \langle x, x' \rangle \qquad \forall x \in \bar{C}.$$

Taking $x = \bar{c}$ in (7) we get $\langle \bar{x}, x' \rangle \leq 0$. If $\langle \bar{x}, x' \rangle = 0$ then, as in the proof of Proposition 1, we obtain $x' = 0$. Thus $\langle \bar{x}, x' \rangle < 0$. Taking now $\lambda\bar{x}$ instead of x, with arbitrary $\lambda \in R_+^*$, in (7) we get that $\langle \bar{x}, x' \rangle \geq 0$, a contradiction. Hence there are $\bar{\alpha}, \bar{\beta} \in R$, $\bar{\alpha} \leq 0 < \bar{\beta}$ (since $\bar{u} \in C^i$ and $0 \in C$) such that $]\bar{\alpha}, \bar{\beta}[\subset I \subset [\bar{\alpha}, \bar{\beta}]$. Moreover $]\bar{\alpha}, \bar{\beta}[\, \bar{x} \subset C^i$ and $\bar{\alpha}\bar{x}, \bar{\beta}\bar{x} \notin C^i$. Once again, by a separation theorem, we get $x_1', x_2' \in X' \setminus \{0\}$ such that

(8) $\langle \bar{\alpha}\bar{x}, x_1' \rangle \le \langle x, x_1' \rangle \quad \forall x \in C,$

(9) $\langle \bar{\beta}\bar{x}, x_2' \rangle \ge \langle x, x_2' \rangle \quad \forall x \in C.$

If $\langle \bar{x}, x_1' \rangle = 0$ or $\langle \bar{x}, x_2' \rangle = 0$, from (8) or (9), as above, we get $x_1' = 0$ or $x_2' = 0$, a contradiction. As $0 \in C$ and $\bar{u} \in C^i$, we may consider that

(10) $$\langle \bar{x}, x_1' \rangle = \langle \bar{x}, x_2' \rangle = 1.$$

Let us assume that x_1' and x_2' are linearly independent. Then there exists $\bar{v} \in X$ such that

(11) $$\langle \bar{v}, x_1' \rangle = \bar{\alpha} - 1, \quad \langle \bar{v}, x_2' \rangle = \bar{\beta} + 1.$$

As $\bar{v} \in C + R\bar{x}$, we have $\bar{v} = \tilde{v} + \lambda\bar{x}$ for some $\tilde{v} \in C$ and $\lambda \in R$. Therefore, by (8) - (11), we have

$$\bar{\alpha} \le \langle \tilde{v}, x_1' \rangle = \langle \tilde{v} - \lambda\bar{x}, x_1' \rangle = \bar{\alpha} - 1 - \lambda,$$

$$\bar{\beta} \ge \langle \tilde{v}, x_2' \rangle = \langle \tilde{v} - \lambda x, x_2' \rangle = \bar{\beta} + 1 - \lambda,$$

which yield a contradiction. Hence x_1' and x_2' are linearly dependent, i.e. $x_2' = \gamma x_1'$ for some $\gamma \in R$. We have $\gamma = 1$ by (10). Thus (8), (9) and (10) can be written together as

(12) $\bar{\alpha} \le \langle x, x' \rangle \le \bar{\beta} \quad \forall x \in C; \quad \langle \bar{x}, x' \rangle = 1.$

Let now $\tilde{x} \in X$ be such that $\bar{\alpha} < \langle \tilde{x}, x' \rangle < \bar{\beta}$. Suppose that $\tilde{x} \notin C^i$; then there is some $x^* \in X' \setminus \{0\}$ such that $\langle \tilde{x}, x^* \rangle = \langle x, x^* \rangle$ for every $x \in C$. Once again $\langle \bar{x}, x^* \rangle \ne 0$; one may take $\langle \bar{x}, x^* \rangle \in \{-1, 1\}$. Assume that x' and x^* are linearly independent and take $\langle \bar{x}, x^* \rangle = 1$. There exists $\bar{v} \in X$ such that

$$\langle \bar{v}, x^* \rangle = \langle \tilde{x}, x^* \rangle, \quad \langle \bar{v}, x' \rangle = \bar{\beta} + 1.$$

As $\bar{v} = \tilde{v} + \lambda x$ for some $\tilde{v} \in C$ and $\lambda \in R$, we have, by (12),

$$\bar{\beta} + 1 = \langle \bar{v}, x' \rangle = \langle \tilde{v} + \lambda\bar{x}, x' \rangle = \langle \tilde{v}, x' \rangle + \lambda\langle \bar{x}, x' \rangle \le \bar{\beta} + \lambda,$$

$$\langle \tilde{x}, x^* \rangle = \langle \bar{v}, x^* \rangle = \langle \tilde{v} + \lambda\bar{x}; x^* \rangle = \langle \tilde{v}, x^* \rangle + \lambda\langle \bar{x}, x^* \rangle \le \langle \tilde{x}, x^* \rangle + \lambda,$$

which yield a contradiction. We obtain similarly a contradiction if $\langle \bar{x}, x^* \rangle = -1$. Therefore $x^* = \bar{\gamma}x'$ with $\bar{\gamma} \in \{-1, 1\}$. If $\bar{\gamma} = 1$ then $\langle \tilde{x}, x' \rangle = \langle x, x' \rangle$ for every $x \in C$, so that $\langle \tilde{x}, x' \rangle = \bar{\alpha}$, a contradiction. If $\bar{\gamma} = -1$ we obtain the contradiction $\langle \tilde{x}, x' \rangle \geq \bar{\beta}$. Therefore $\tilde{x} \in C^i$. Hence

$$\{x \in X: \bar{\alpha} < \langle x, x' \rangle < \bar{\beta}\} \subset C^i \subset C \subset \{x \in X: \bar{\alpha} \leq \langle x, x' \rangle \leq \bar{\beta}\},$$

which shows that b) holds with $X_0 = \{x \in X: \langle x, x' \rangle = 0\}$.

Remark. If the statement a) or b) of Proposition 3 holds, we obtain easily $X = C + R\bar{x}$. Simple examples show that this is not true if c) or d) holds.

Indeed, take $A = \{(x,y): x \in]-1,1[, \ y \geq 1/(1 - x^2)\} \subset R^2$. A is a closed convex set and $\bar{A} + R_+^* \bar{x} \subset A^i$ for $\bar{x} = (0,1) \in R^2$, but $A + R\bar{x} =]-1,1[\times R \neq R^2$.

Concerning the condition c) of Proposition 3 we have the following

Proposition 4. Let $C \subset X$ be a nonempty convex set and $\bar{x} \in X$.

(i) $\bar{C} + R_+^* \bar{x} \subset C^i$ if and only if $\bar{x} \in C_\infty$, $C^i \neq \emptyset$ and $\bar{C} + R\bar{x} = (\bar{C} + R\bar{x})^i$.

(ii) If $\bar{x} \in \bar{C}_\infty^i$ and $C^i \neq \emptyset$ then $C + R\bar{x} = X$.

Proof. (i) "$\Longrightarrow$": It is evident that $\bar{x} \in \bar{C}_\infty$ and $C^i \neq \emptyset$. Let $x \in \lambda\bar{x} + \bar{C}$ for some $\lambda \in R$. Then $x \in (\lambda - 1)\bar{x} + \bar{x} + \bar{C} \subset (\lambda - 1)\bar{x} + C^i \subset C^i + R\bar{x} = (\bar{C} + R\bar{x})^i$. Therefore $\bar{C} + R\bar{x} \subset (\bar{C} + R\bar{x})^i$.

"$\Longleftarrow$": Suppose that $\bar{c} + \bar{\lambda}\bar{x} \notin C^i$ for some $\bar{c} \in \bar{C}$ and $\bar{\lambda} > 0$. Then there exists $x' \in X' \setminus \{0\}$ such that

$$\langle \bar{c} + \bar{\lambda}\bar{x}, x' \rangle \leq \langle c, x' \rangle \quad \forall c \in \bar{C}.$$

As $\bar{x} \in \bar{C}_\infty$, we obtain that $\langle \bar{x}, x' \rangle = 0$. Therefore $\langle \bar{c}, x' \rangle \leq \langle x, x' \rangle$ for every $x \in \bar{C} + R\bar{x}$. As $\bar{C} + R\bar{x}$ is algebraically open, it follows that $x' = 0$, a contradiction. Hence $\bar{C} + R_+^* \bar{x} \subset C^i$ (in fact we have equality).

(ii) Suppose that $\tilde{x} \notin C + R\bar{x}$ for some $\tilde{x} \in X$. Then there is some $x' \in X' \setminus \{0\}$ such that

$$\langle \tilde{x}, x' \rangle \leq \langle c + \mu\bar{x}, x' \rangle \quad \forall c \in C, \ \mu \in R.$$

It follows that $\langle \bar{x},x' \rangle = 0$ and $\langle \tilde{x},x' \rangle \leq \langle c,x' \rangle$ for every $c \in \bar{C}$, whence $0 \leq \langle x,x' \rangle$ for any $x \in \bar{C}_{\infty}$, and so $0 \leq \langle x + \mu\bar{x},x' \rangle$ for all $x \in \bar{C}_{\infty}$ and $\mu \in R$. As $\bar{C}_{\infty} + R\bar{x} = X$, by Proposition 1, we get the contradiction $x' = 0$. Therefore $C + R\bar{x} = X$.

References

[1] J. BAIR and R. FOURNEAU: Etude Géométrique des Espaces Vectoriels. Une Introduction, Lecture Notes in Mathematics 489, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[2] C. GERSTEWITZ and E. IWANOW: Dualität für nichtkonvexe Vektoroptimierungsprobleme, Wiss. Z. TH Ilmenau 31(1985), 61-81.

[3] R. HOLMES: Geometrical Functional Analysis and its Applications, Springer-Verlag, Berlin, 1975.

Faculty of Mathematics, University of Iaşi, 6600-Iaşi, Romania

(Oblatum 3.4. 1986)