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On the multiplicity points of monotone operators on separable Banach spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 3, 551--570

Persistent URL: http://dml.cz/dmlcz/106476

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Abstract: It is proved that the set of multiplicity points of monotone operator $T$ on a separable real Banach space is contained in a countable union of Lipschitz hypersurfaces with "linearly finite convexity on a subset". If $T$ is a subdifferential of a proper convex function, the hypersurfaces are $\delta$-convex. Analogous results are obtained for the sets of $n$-dimensional and $n$-codimensional multiplicities. Applications to singular points of convex sets are given. This paper improves and generalizes the results of L. Zajiček.

Key words: Multiplicity points of monotone operators, linearly finite convexity, Lipschitz surfaces in Banach spaces, convex analysis, subdifferentials of proper convex functions, singular points of convex sets, $\delta$-convex functions.

AMS Subject Classification: Primary 47H05
Secondary 52A20

1. Introduction

Let $T$ be a set-valued monotone operator on a separable real Banach space $X$ (i.e. $T:X \rightarrow \text{exp}X^*$ and $\langle x-y, x^*-y^* \rangle \geq 0$ whenever $x^* \in Tx$, $y^* \in Ty$) and let

$A_n = \{x \in X: \text{dim}(\text{co} Tx) \geq n\}$,

$A^n = \{x \in X: \text{co} Tx \text{ contains a ball of codimension } n\}$,

where $\text{co} Tx$ denotes a convex hull of the set $Tx$.

The smallness of the sets $A_n, A^n$ was investigated by E.H. Zarantonello [8], N. Aronszajn [1] and L. Zajiček [6], [7]. The theorems were applied to operators $F_M, V_M$ ("vertex-" and "face-operator") being connected with singular points of a closed convex set $M$, in [8], [7].

In this paper, the results from [6] and [7] were improved.
and generalized.

L. Zajiček has proved (see [7]) that the set $A_n$ can be covered by countably many Lipschitz surfaces of codimension $n$. If $T = \delta f$ for some continuous convex function on an open convex set $U \subset X$ then it is possible to write "$\delta$-convex surfaces" instead of "Lipschitz surfaces" (see [5]). In case $X$ is a Hilbert space or $n=1$ and $X^*$ is separable, the set $A^n$ of a general monotone operator $T$ can be covered by a countable union of Lipschitz surfaces of dimension $n$ (see [7]).

1.1 Problem: It is still an open problem whether the set $A_n$ (or $A^n$, if $X^*$ is separable, respectively) can be covered by countably many $\delta$-convex surfaces of codimension $n$ (or dimension $n$, respectively) if $T$ is a general monotone operator.

Following main results of the present article suggest that the answer could be positive:

a) The Lipschitz surfaces from [7] have an additional property - "linearly finite convexity on a subset". This result easily gives an existence of a Lipschitz surface of codimension $n$ (dimension $n$, respectively) which cannot be a subset of $A_n$ ($A^n$, respectively) for any monotone operator $T$.

b) If $X^*$ is separable then the set $A^n$ is contained in a countable union of curves with finite convexity. It gives a positive answer to 1.1 in the special case $X = \mathbb{R}^2$.

c) The result from [5] is generalized to the case $T = \delta f$, where $f$ is a proper convex function. It makes possible to improve the results from [7, 8] concerning singular points of convex sets.

d) It is shown that the Lipschitz surfaces covering the set $A^n$ are in a certain sense $\delta$-convex on a subset if $T = \delta f$.

The author is grateful to RNDr. L. Zajiček, CSc. for his advice and remarks which helped to get the final version of present paper.

2. Definitions and auxiliary propositions

All linear spaces of present paper will be real linear spaces. Let $M$ be a subset of the real line $\mathbb{R}$. We shall denote by $\mathcal{P}(M)$ the system of all sets $A \subseteq M$, which contain at least three elements.
Let $X$ be a Banach space and $f: M \to X$. For any $a, b \in M$, we define $Q_f(a, b) = \frac{f(b) - f(a)}{b - a}$. We shall write $Q(a, b)$ instead of $Q_f(a, b)$ when it is clear which mapping is concerned to.

2.1 Definition (cf. [2]): Let $X$ be a Banach space, $M \subset \mathbb{R}$ and $f: M \to X$. For $P = \{x_0 < x_1 < \cdots < x_n < x_{n+1}\} \in \mathcal{P}(M)$ we define

$$K(f, P) = \sum_{i=1}^{n} |Q_f(x_{i-1}, x_i) - Q_f(x_i, x_{i+1})|$$

and put

$$\mathcal{K}(f, M) = \begin{cases} \sup \{K(f, P) : P \in \mathcal{P}(M)\} & \text{if } \mathcal{P}(M) \neq \emptyset, \\ 0 & \text{if } \mathcal{P}(M) = \emptyset. \end{cases}$$

$

\mathcal{K}(f, M)$ is called convexity of $f$ on $M$.

2.2 Lemma: Let $X$ be a Banach space, $M \subset \mathbb{R}$ and $f: M \to X$. Then $K(f, P) \leq K(f, P \cup \{m\})$ holds for any $P \in \mathcal{P}(M)$, $m \in M$.

Proof: Let $P = \{x_0, x_1, \ldots, x_n, x_{n+1}\} \in \mathcal{P}(M)$. There are four possible positions of the point $m$.

- a/ $m \in P$;
- b/ $m < x_0$ or $x_{n+1} < m$;
- c/ $x_0 < m < x_1$ or $x_n < m < x_{n+1}$;
- d/ $x_j < m < x_{j+1}$ for some $1 \leq j \leq n-1$.

We shall perform the proof of the most complicated case d/ only, since the proof of c/ is similar and a/ and b/ are obvious.

If we shortly denote $x = x_{j-1}$, $y = x_j$, $z = x_{j+1}$, $w = x_{j+2}$, we have following situation:

$x < y < m < z < w$.

Let $k \in X$ be such that

$$\frac{k - f(y)}{m - y} = Q(y, z) = \frac{f(z) - k}{z - m}.$$ 

Then

$$\begin{align*}
&\|Q(x, y) - Q(y, z)\| + \|Q(y, z) - Q(z, w)\| = \|Q(x, y) - \frac{k - f(y)}{m - y}\| + \\
&+ \|\frac{f(z) - k}{z - m} - Q(z, w)\| + \|Q(m, z) - Q(z, w)\| = \|Q(x, y) - Q(y, m)\| + \|\frac{f(m) - k}{m - y} + \frac{f(z) - k}{z - m}\| + \\
&+ \|Q(m, z) - Q(z, w)\| + \|Q(m, z) - Q(z, w)\| = \|Q(x, y) - Q(y, m)\| + \|Q(y, m) - Q(m, z)\| + \\
&+ \|Q(m, z) - Q(z, w)\| + \|Q(m, z) - Q(z, w)\| = K(f, P) < K(f, P \cup \{m\}).
\end{align*}$$

(We have used following equalities:

$$\begin{align*}
\frac{f(z) - k}{z - m} = \frac{f(m) - k}{m - y} - \frac{k - f(y)}{m - y} = \|Q(y, m) - Q(z, w)\|.
\end{align*}$$

///
2.3 Proposition: Let $X$ be a Banach space, $M \subset \mathbb{R}$, $f: M \to X$. If $\mathcal{K}(f, M) < \infty$ then $f$ is a Lipschitz mapping on $M$.

Proof: Suppose $f$ is not Lipschitz. It is evident that there exist two points $a, b \in M$ such that $a < b$ and $f$ is not Lipschitz on at least one of the sets $M_+ = M \cap (b, +\infty)$, $M_- = M \cap (-\infty, a)$. We can assume $f$ to be not Lipschitz on $M_+$ without any loss of generality. There exist $u, v \in M_+$ such that $u < v$ and $|Q(u, v)| > \mathcal{K}(f, M) + |Q(a, b)|$. Then $\mathcal{K}(f, M) < |Q(u, v)| - |Q(a, b)| - |Q(a, b) - Q(u, v)| - |Q(b, u) - Q(u, v)| = K(f, \{a, b, u, v\}) < \mathcal{K}(f, M)$ and this is a contradiction. \///

2.4 Proposition: Let $X$ be a Banach space, $M \subset \mathbb{R}$, $f: M \to X$ and $\mathcal{K}(f, M) < \infty$. If $x \in X$ is a limit point of $M$ from the right (from the left, respectively), there exist

$$
 f'_+(x, M) = \lim_{y \to x^+} Q(x, y) \quad (f'_-(x, M) = \lim_{y \to x^-} Q(x, y), \text{ resp.}).
$$

Proof: Suppose $f'_+(x, M)$ doesn't exist. Then there must exist $\epsilon > 0$ such that for any $\delta > 0$ there exist $y, z, w \in M$ satisfying $x < y < z < w < x + \delta$ and $|Q(x, y) - Q(x, z)| \geq \epsilon$.

But $|Q(x, y) - Q(x, z)| \leq |Q(x, y) - Q(y, z)| + |Q(y, z) - Q(w, z)| + |Q(x, z) - Q(w, z)| = K(f, \{x, y, z, w\}) + K(f, \{x, z, w\}) \leq 2 \mathcal{K}(f, M \cap (x, x + \delta)) = 2 \mathcal{K}(f, M \cap (x, x + \delta))$. (The last equality is an easy consequence of 2.3.)

Hence $\mathcal{K}(f, M \cap (x, x + \delta)) > 2^{-1} \epsilon$ for any $\delta > 0$. Let $N \geq \frac{2 \mathcal{K}(f, M)}{\epsilon}$ be a positive integer. Since we have for any $\delta > 0$ an existence of $P$ from $\mathcal{P}(M \cap (x, x + \delta))$ such that $K(f, P) > 2^{-1} \epsilon$, it is possible to find $P_1, P_2, \ldots, P_N \in \mathcal{P}(M)$ with following properties:

$$
\max_{k=1}^{N} P_{k+1} \leq \min_{k=1}^{N} P_k \\
K(f, P_j) > 2^{-1} \epsilon \quad \text{for } j=1, 2, \ldots, N.
$$

Then $\mathcal{K}(f, M) \leq \frac{\epsilon}{2} \leq \frac{\epsilon}{\sum_{k=1}^{N} K(f, P_k)} \leq K(f, \{P_1, P_2, \ldots, P_N\}) \leq \mathcal{K}(f, M)$ and this is a contradiction. The proof of existence of $f'_-(x, M)$ is analogous. \///

2.5 Theorem: Let $X$ be a Banach space, $M \subset \mathbb{R}$, $f: M \to X$. Then there exists a mapping $F: \mathbb{R} \to X$ such that

$$
\forall x \in M \quad F(x) = f(x) \quad (1) \\
\mathcal{K}(F, \mathbb{R}) = \mathcal{K}(f, M) . \quad (2)
$$

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Proof: If \( \mathcal{K}(f,M) = +\infty \), \( F \) can be an arbitrary extension of \( f \).

If \( M \) has two or less elements then \( F \) can be defined as affine mapping satisfying (1). Suppose \( M \) has at least three elements

and \( \mathcal{K}(f,M) < \infty \). The needed extension will be constructed in several steps.

\( g/ \) Extension on \( \overline{M} \) (closure of \( M \)).

\( X \) is complete and \( f \) is Lipschitz on \( M \) (by 2.3). Hence \( f \) has a unique continuous extension \( g \) on \( \overline{M} \). Choose \( \varepsilon > 0 \) and arbitrary \( P = \{ x_0 < x_1 < \ldots < x_{n+1} \} \in \mathcal{P}(\overline{M}) \).

The continuity of the mapping \( q(u,v) = Q_g(u,v) \) on the set \( \{ [u,v] \in \overline{M} \times \overline{M} : u \neq v \} \) gives existence of \( P_1 = \{ y_0 < y_1 < \ldots < y_{n+1} \} \in \mathcal{P}(M) \) such that

\[
\| Q_g(x_j,x_{j+1}) - Q_g(y_j,y_{j+1}) \| < \frac{\varepsilon}{2n}, \quad j = 0, 1, \ldots, n.
\]

Then

\[
K(g,P) < K(g,P_1) + 2n \frac{\varepsilon}{2n} = K(f,P_1) + \varepsilon \leq \mathcal{K}(f,M) + \varepsilon.
\]

Hence \( \mathcal{K}(g,\overline{M}) = \sup \{ K(g,P) : P \in \mathcal{P}(\overline{M}) \} \leq \mathcal{K}(f,M) + \varepsilon \).

Since \( \varepsilon \) was arbitrary and the inequality \( \mathcal{K}(f,M) \leq \mathcal{K}(g,\overline{M}) \) is evident, we have \( \mathcal{K}(g,\overline{M}) = \mathcal{K}(f,M) \).

\( \psi/ \) Extension on \( M_1 = \{ x \in \mathbb{R} : \alpha < x < \beta \} \), where \( \alpha = \inf M \), \( \beta = \sup M \).

The complement of \( \overline{M} \) can be written as a finite or countable union of disjoint open intervals:

\[
\mathbb{R} \setminus \overline{M} = J_- \cup \bigcup_{k \in A} J_k \cup J_+,
\]

where \( A \subset \{ 1, 2, 3, \ldots \} \), \( J_- = (-\infty, \alpha) \), \( J_+ = (\beta, +\infty) \), \( J_k = (a_k, b_k) \), \( a_k < b_k \), \( k \in A \). \( J_- J_+ \) can be empty and, obviously, \( a_k, b_k \in M \) for any \( k \in A \).

It is easy to see that \( M_1 = \overline{M} \cup \bigcup_{k \in A} J_k \). Let us define

\[
h(x) = \begin{cases} 
g(x) & \text{if } x \notin \overline{M} 
g(a_k) + Q_g(a_k, b_k)(x - a_k) & \text{if } x \in (a_k, b_k)
\end{cases}
\]

Obviously \( Q_h(a_k, x) = Q_h(x, b_k) = Q_h(a_k, b_k) = Q_g(a_k, b_k) \) for any \( x \in (a_k, b_k) \) and \( h = f \) on \( M \).

For arbitrary \( P \in \mathcal{P}(M_1) \) we define

\[
P_1 = P \cup \bigcup_{k \in A} \{ a_k, b_k \}, \quad P_2 = P_1 \setminus \bigcup_{k \in A} J_k.
\]

Then \( P_2 \) contains at least two points and \( P_2 \subset \overline{M} \). If \( P_2 \) contains just two points then \( P \subset J_k \) for convenient \( k \in A \) and then
$K(h, P) = 0 \leq K(f, M)$. Let $P_2$ contain more than two elements. Then $P_2 \in \mathcal{P}(\mathbb{R})$ and by 2.2 and (3):

$$K(h, P_2) \leq K(h, P_1) = K(h, P_2) = K(g, P_2) \leq K(g, M) = K(f, M).$$

Since $P \in \mathcal{P}(M_1)$ was arbitrary then $K(h, M_1) = K(f, M)$.

$\phi$ Extension on $\mathbb{R}$.

Define

$$F(x) = \begin{cases} 
  h(x) & \text{if } x \in M_1 \\
  h(s) + h^s(s, M_1)(x-s) & \text{if } x \in J_+ \\
  h(s) + h^s(s, M_1)(x-s) & \text{if } x \in J_-. 
\end{cases}$$

Let us suppose $J_+ \neq \emptyset, J_- \neq \emptyset$. The other cases are more simple.

Let $P \in \mathcal{P}(\mathbb{R})$ and $\epsilon > 0$. Choose $P_1 \in \mathcal{P}(\mathbb{R})$ such that $P \subseteq P_1$, $J_- \cap P_1 = \emptyset, J_+ \cap P_1 = \emptyset, M_1 \cap P_1 = \emptyset$. Define

$$P_2 = P_1 \cup \{s, s\} = \{x_0 < x_1 < \ldots < x_i < s < x_{i+1} < \ldots < x_m < s < x_{m+1} < \ldots < x_n\}$$

and

$$P_3 = \{x_i < s < x_{i+1} < \ldots < x_m < s < x_{m+1}\}.$$ 

There exist $y \in (s, x_{i+1})$, $z \in (x_m, s)$ such that

$$\|Q_{P_2}(s, x_{i+1}) - Q_{P}(y, x_{i+1})\| < \frac{1}{6} \epsilon,$$

$$\|Q_P(x_i, s) - Q_{P}(\mathbb{S}, y)\| = \|F(\mathbb{S}) - Q_{P}(\mathbb{S}, y)\| < \frac{1}{6} \epsilon,$$

$$\|Q_{P}(x_m, z) - Q_{P}(x_m, s)\| < \frac{1}{6} \epsilon,$$

$$\|Q_{P}(z, x_{m+1}) - Q_{P}(z, s)\| = \|F_-(s) - Q_{P}(z, s)\| < \frac{1}{6} \epsilon.$$

Then 2.2, (3) and simple triangle inequalities imply

$$K(P, P) \leq K(P, P_1) \leq K(P, P_2) = K(P_2, P_2) <$$

$$< K(P_1, \mathbb{S}, y, x_{i+1}, \ldots, x_m, z, s) + \epsilon = K(h, \{s, y, x_{i+1}, \ldots, x_m, z, s\}) + \epsilon$$

$$\leq K(h, M_1) + \epsilon = K(f, M) + \epsilon.$$

$P$ and $\epsilon$ were arbitrary, hence $K(P, R) = K(f, M)$. ///

2.6 Definition: Let $X, Y$ be Banach spaces, $M \subseteq X, y : M \to Y$ and $x, h \in X$. Let $M_{x, h} = \{t \in \mathbb{R} : x + th \in M\}$ and let us define mapping

$$\gamma_{x, h} : M_{x, h} \to Y$$

by the formula

$$\gamma_{x, h}(t) = y(x + th).$$

We shall say that $y$ has linearly finite convexity on $M$, if

$$\sup \{K(y_{x, h}, M_{x, h}) : x, h \in X, \|h\| = 1\}$$

is finite.

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Thus $\psi$ has linearly finite convexity on $M$ iff its restriction on any straight line $p$ has finite convexity on $M \cap p$ and all these convexities have a common upper bound.

Let us note that a mapping $\psi$, possessing a linearly finite convexity on a neighbourhood of a point $x \in X$, has all one-sided directional derivatives at $x$ (by 2.4).

2.7 Definition: Let $X,Y$ be Banach spaces, $M \subset X$ and $\psi:M \to Y$. The mapping $\psi$ is said to be $\delta$-convex on $M$ iff there exists a convex Lipschitz function $g$ on $X$ with property: for each $\psi \in Y$; $\psi \in M$, there exists a convex Lipschitz function $h_\psi$ on $X$ such that $\psi = h_\psi - g$ on $M$.

2.8 Observation: A real function $f$ on a subset $M$ of a Banach space $X$ is $\delta$-convex on $M$ iff $f$ can be extended to a function on $X$ representable as a difference of two convex Lipschitz functions.

2.9 Remark: Let $M \subset X$, $f:M \to \mathbb{R}$. Then $f$ is $\delta$-convex on $M$ iff $\mathcal{K}(f,M)$ is finite. This yields from well-known results (cf. [2]) and 2.5.

2.10 Observation: Let $M$ be a subset of a Banach space $X$ and $\psi:M \to \mathbb{R}^n$, $\psi \equiv [\psi_1,\ldots,\psi_n]$. Then

/i/ $\psi$ is $\delta$-convex on $M$ iff $\psi_k$ is $\delta$-convex on $M$ for $k=1,\ldots,n$.

/ii/ If $\psi$ is $\delta$-convex on $M$, there exists a $\delta$-convex extension of $\psi$ defined on the whole space $X$. Both propositions are easy consequences of the definition 2.7, /ii/ yields from /i/.

Let us note that if $X,Y$ are metric spaces, $M \subset X$ and $f:M \to Y$ is a Lipschitz mapping, then there exists a Lipschitz extension $\rho:X \to Y$ of $f$ in the following cases:

/i/ $Y \equiv \mathbb{R}^n$

/ii/ $X,Y$ are Hilbert spaces

/iii/ $X=\mathbb{R}$ and $Y$ is a Banach space.

(For references see [7]).

It is not known to the author whether there exist extensions of mappings with linearly finite convexity keeping this property, if $\dim X > 1$ (even in case $X=\mathbb{R}^2,Y=\mathbb{R}$), and $\delta$-convex extensions of $\delta$-convex mappings if $\dim Y = \infty$. 

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2.11 Definition: Let $E$ be a subset of a Banach space $X$ and $n < \dim X$ be a positive integer. We shall say that $E$ is a Lipschitz fragment of dimension $n$ (of codimension $n$, respectively) and denote $E \in \kappa_n$ ($E \in \kappa^*_n$, respectively) if the following condition is satisfied: There exist a subspace $Z$ of $X$ of codimension $n$ (of dimension $n$, resp.), a topological complement $W$ of the space $Z$ in $X$, a set $M \subseteq W$ and a Lipschitz mapping $\gamma : M \to Z$ such that $E = \{w + \gamma(w) : w \in M\}$.

If $Z, M, W$ can be chosen in such way that in addition $\gamma$ is $\delta$-convex on $M$ or $\gamma$ has linearly finite convexity on $M$ then we shall say that $E$ is a $\delta$-convex fragment or $E$ is an LFC-fragment of given dimension or codimension. The notation will be following: $E \in \kappa_n$, $E \in \kappa^*_n$, $E \in \mathrm{LFC}_n$, $E \in \mathrm{LFC}^n$.

Fragments with $M = W$ are called surfaces. Surfaces of dimension $1$ (of codimension $1$, resp.) are called curves (hypersurfaces, resp.).

2.12 Notation: Let $\mathcal{Y}$ be a given system of subsets of a Banach space $X$. By $\mathcal{Y}$ we denote the system of all sets representable as a union of countably many elements from $\mathcal{Y}$. (For example: $E \in \kappa_n \mathcal{Y}$ means that $E$ can be written as a countable union of $\delta$-convex fragments of codimension $n$).

2.13 Observations: a/ Every $E \in \kappa_n$ has $\sigma$-finite $n$-dimensional Hausdorff measure. In particular, if $X = \mathbb{R}^m$, $m > n$, then $E$ is of Lebesgue measure zero.

b/ Every surface from $\kappa_n$ has infinite but $\sigma$-finite $n$-dimensional Hausdorff measure and its Hausdorff dimension is $n$.

c/ As consequences of 2.5, 2.10 and extension theorems for Lipschitz mappings we obtain the following propositions:

Every $E \in \kappa^*_n$ is a subset of a Lipschitz surface of codimension $n$.

Every $E \in \kappa_n$ is a subset of a Lipschitz curve.

Every $E \in \kappa^*_n$ is a subset of a $\delta$-convex surface of codimension $n$.

Every $E \in \mathrm{LFC}_n$ is a subset of an LFC-curve.

If $X$ is a Hilbert space then every $E \in \kappa_n$ is a subset of a Lipschitz surface of dimension $n$.
3. Multiplicity points of monotone operators

By \( \exp A \) we shall denote the system of all subsets of a set \( A \) and by \( \text{co} A \) the convex hull of \( A \).

The dimension (codimension, resp.) of a convex set is meant as the dimension (codimension, resp.) of its affine hull.

Let \( X \) be a Banach space with dual space \( X^* \) and \( T : X \to \exp X^* \) be a monotone operator. We shall use the following notation:

\[
A_n = \{ x \in X : \dim(\text{co} T x) \geq n \}
\]

\[
A_n^* = \{ x \in X : \text{co} T x \text{ contains a ball of codimension } n \}
\]

\[
\text{gph } T = \{ (x, x^*) \in X \times X^* : x^* \in T x \}.
\]

3.1 Definition: Let \( T, T' \) be monotone operators on \( X \). We shall write \( T \preceq T' \) if \( \text{gph } T \subseteq \text{gph } T' \). \( T \) is called a maximal monotone operator if \( T \preceq T' \) implies \( T = T' \).

3.2 Observation: a/ For every monotone operator \( T \) there exists a maximal monotone operator \( T_{\max} \) such that \( T \preceq T_{\max} \) by Zorn's lemma.

b/ It is easy to see that \( T x \) is always convex if \( T \) is a maximal monotone operator.

By a proper convex function (cf. [3]) it is meant a mapping \( f : X \to \mathbb{R} \cup \{+\infty\} \) satisfying following two conditions:

\[
\forall x, y \in X, \forall \lambda \in (0, 1) \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad (4)
\]

\[
\text{dom } f = \{ x \in X : f(x) < +\infty \} \neq \emptyset. \quad (5)
\]

3.3 Definition: Let \( f \) be a proper convex function on a Banach space \( X \) and \( x \in X \). If \( x \in \text{dom } f \), we define

\[
\partial f(x) = \{ x^* \in X^* : \forall z \in X \quad f(z) \geq f(x) + \langle z-x, x^* \rangle \}.
\]

We put \( \partial f(x) = \emptyset \) in case \( f(x) = +\infty \). The mapping \( \partial f : x \mapsto \partial f(x) \) is called subdifferential of \( f \).

It will fit to define \( \partial f \equiv \emptyset \) for \( f = +\infty \).

3.4 Remark: Subdifferentials of proper convex functions are monotone but not conversely. There exist monotone operators which are not subdifferentials ([3]). The characterization of subdifferentials of proper convex functions uses the notion of a cyclic-
cally monotone operator is due to R.T. Rockafellar (see [4]).

The main result of this paper is contained in the following two theorems.

3.5 Theorem: Let $T$ be a monotone operator on a separable Banach space $X$ and $n < \text{dim } X$ be a positive integer. Then $A_n \in \mathcal{L}C^n$. If in addition $T \in \mathcal{L}f$ for some proper convex function $f$ on $X$ then $A_n \in \mathcal{D}C^n$.

3.6 Theorem: Let $T$ be a monotone operator on a Banach space $X$ with separable dual space $X^*$ and $n < \text{dim } X$ be a positive integer. Then $A^n \in \mathcal{L}F_n$. If in addition $T \in \mathcal{L}f$ for some proper convex function $f$ on $X$ then $A^n \in \mathcal{D}C_n$.

These theorems say that the sets $A_n$ and $A^n$ can be written as a countable union of images of special Lipschitz mappings (defined on a subset of a Banach space of codimension $n$ or dimension $n$, respectively).

Both proofs are practically equal and we shall do it simultaneously.

At first we state the following simple lemmas without a proof (see [7], Lemma 1, Lemma 2). An open ball with centre $c$ and radius $r > 0$ is denoted by $\Omega(c, r)$.

3.7 Lemma: Let $X$ be a separable Banach space. Then there exist a countable system $\mathcal{T}$ of $n$-codimensional subspaces of $X^*$ and a countable system $\mathcal{K}$ of $n$-codimensional affine subsets of $X$ such that: /i/ Any $n$-dimensional subspace $P \subseteq X^*$ has a topological complement $V \in \mathcal{T}$.

/ii/ If $P, V$ are as in /i/, $c \in X^*$, $\varepsilon > 0$ then there exists $t \in (c^* + P) \cap \Omega(c^*, \varepsilon)$ such that $L = t + V \in \mathcal{K}$.

3.8 Lemma: Let $Y$ be a separable Banach space. Then there exist a countable system $\mathcal{T}$ of $n$-dimensional subspaces of $Y$ and a countable system $\mathcal{K}$ of $n$-dimensional affine subsets of $Y$ such that: /i/ Any subspace $P \subseteq Y$ of codimension $n$ has a topological complement $V \in \mathcal{T}$.

/ii/ If $P, V$ are as in /i/, $c \in Y$, $\varepsilon > 0$ then there exists $t \in (c^* + P) \cap \Omega(c^*, \varepsilon)$ such that $L = t + V \in \mathcal{K}$. 

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3.9 Proof of 3.5 and 3.6: Let $X,T$ be as in 3.5 (in 3.6, resp.) and $A=\mathbb{A}_n$ ($A=A^n$, resp.). Without any loss of generality we can suppose $Tx$ to be convex for any $x$ (see 3.2).

a/ Decomposition of $A$.
Let $x$ be an arbitrary element of $A$. Then there exist a point $c_x \in Tx$, a positive rational number $r_x$ and a subspace $P_x \subset X^*$ of dimension $n$ (of codimension $n$, resp.) such that

$$(c_x + P_x) \cap \bigcap (c_x, r_x) \subset Tx.$$ 

Let $m_x$ be a rational number such that $Lc_x \leq m_x$. Lemma 3.7 (3.8, resp.) guarantees an existence of a topological complement $V_x \subset T$ of $P_x$ and a point $t_x \in (c_x + P_x) \cap \bigcap (c_x, \frac{1}{2} r_x)$ such that

$L_x = t_x + V_x \in \mathcal{E}.$

Let us find a rational number $q_x$ such that $Lt^T \geq q_x$, where

$\pi_x : X^* \to P_x$ is a projection in the direction of $V_x$.

For any $r,m,q$ positive rational, $V \in T$, let us denote

$B(r,m,V,q,L) = \{ x \in A : r_x = r, m_x = m, V_x = V, q_x = q, L_x = L \}.$

It is clear that $A = \bigcup B(r,m,V,q,L)$ and the union is countable.

Let $r,m,V,q,L$ be fixed. We shall show that the set $B(r,m,V,q,L)$ is a Lipschitz fragment of codimension $n$ (of dimension $n$, resp.).

b/ "Parametrization" of $B$.
Define $Z = V$. Let $W$ be an arbitrary topological complement of $Z$ in $X$ and $Y = W^\perp$. Then $Y$ is a topological complement of $V$ in $X^*$.

The following proposition is true:

(6) $z^* \in Z$ iff there exists $y^* \in Y$ such that $z^* = y^*$ on $Z$.

There exists a point $y_0 \in Y$ such that $L = y_0 + V$. Let us denote

$M = \{ w \in W : \exists z \in Z \ w + z \in B \},$

i.e. $M$ is a projection of $B$ on the subspace $W$ in the direction of $Z$.

c/ $B$ is a Lipschitz fragment.
Let $B \neq \emptyset$. Let $w_1, w_2 \in M$, $z_1, z_2 \in Z$ such that $x_i = w_i + z_i \in B$ for $i = 1, 2$. Let us denote $t_i = t_{x_i}, \ x_i = \pi_{x_i}$

Let $y^* \in Y$ be an arbitrary functional from a unit sphere in $Y$. Define

$t_1^+ = t_1 + \frac{2}{q} \pi_{x_1}(y^*)$

$t_1^- = t_1 - \frac{2}{q} \pi_{x_1}(y^*).$
The fact \( t_1^+, t_1^- \in T_{x_1} \) follows from inequalities
\[
\| t_1^- - c_{x_1} \| \leq \| t_1^- - c_{x_1} + \frac{r}{2q} \pi_1(y^*) \| < r, \quad \| t_1^- - c_{x_1} \| < r.
\]
The monotonicity of \( T \) implies
\[
0 < \langle x_1 - x_2, t_1 - t_2 \rangle + \frac{r}{2q} \pi_1(y^*) \leq \langle x_1 - x_2, \frac{r}{2q} y^* \rangle.
\]
(We have used the fact that the functionals \( t_1 - t_2 \), \( y^* - \pi_1(y^*) \) are elements of \( V_1 \).) Now we obtain
\[
\| x_1 - x_2, t_1 - t_2 \| \leq \| x_1 - x_2, t_1 - t_2 \| + \frac{r}{2q} \pi_1(y^*) \leq \langle x_1 - x_2, \frac{r}{2q} y^* \rangle.
\]
Then by (6)
\[
\| x_1 - x_2, y^* \| = \sup \{ \langle x_1 - x_2, y^* \rangle : y^* \in Y, \| y^* \| = 1 \} \leq \frac{\| x_1 - x_2, y^* \|}{r}.
\]
If we take \( y(w) \in Z \) (for \( w \in M \)) such that \( w + y(w) \in B \), we obtain a correctly defined mapping which is Lipschitz on \( M \) and satisfies
\[
B = \{ w + y(w) : w \in M \}.
\]
\( y \) has linearly finite convexity on \( M \).
Let \( w_0 \in W, h \in W, \| h \| = 1 \). Denote \( D = W_0, h \rangle, P = W_0, h \rangle \) (see 2.6).
If \( D \) contains less than three elements then \( \chi(D) = 0 \) by the definition. Let \( D \) have at least three elements and
\[
\{ d_0 < d_1 < \ldots < d_s < d_{s+1} \} \in \rho(D).
\]
For \( 0 \leq j \leq s+1 \) let us introduce following simplifications:
\[
x_j = w_0 + d_j h + F(d_j),
\]
\[
t_j = t_j x_j,
\]
\[
x_j = x_j x_j.
\]
x_j's are obviously points from \( B \). The monotonicity of \( T \) implies
\[
0 < 0 < t_j - t_i \Rightarrow \langle h, t_j - t_i \rangle \geq 0.
\]
Let us choose an arbitrary number \( i \in \{ 1, 2, \ldots, s \} \) and a functional \( y^* \in Y \) such that \( y^* \| = 1 \).
Denote \( t_1^+ = t_1 + \frac{r}{2q} x_1(y^*); \quad t_1^- = t_1 - \frac{r}{2q} x_1(y^*) \). We have
\[
t_1^+, t_1^- \in T_{x_1} \quad \text{similarly as in the part } g/. \text{ Using the monotonicity of } T \text{ we obtain}
\]

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Analogous calculations with 

$$0 \leq <x_{i-1} - x_{i-1}, t_{i-1}^+ - t_{i-1}^+> = (d_{i-1} - d_{i-1})<h, t_{i-1}^+ - t_{i-1}^-> - \frac{2a}{r} \left< f(d_{i-1}) - f(d_{i-1}), y^* \right>$$

and hence

$$<Q_p(d_{i-1}, d_i), y^*> \leq \frac{2a}{r} <h, t_{i-1}^+ - t_{i-1}^-> - <h, \pi_i(y^*)>.$$

will afford the following inequalities:

$$- <Q_p(d_{i-1}, d_i), y^*> \leq \frac{2a}{r} <h, t_{i-1}^+ - t_{i-1}^-> + <h, \pi_i(y^*)>$$

$$- <Q_p(d_{i}, d_{i+1}), y^*> \leq \frac{2a}{r} <h, t_{i+1}^+ - t_{i}^-> + <h, \pi_i(y^*)>$$

$$<Q_p(d_{i}, d_{i+1}), y^*> \leq \frac{2a}{r} <h, t_{i+1}^+ - t_{i}^-> - <h, \pi_i(y^*)>.$$

Then for any $y^* \in Y, \|y^*\| = 1$ 

$$|<Q_p(d_{i-1}, d_i), Q_p(d_{i}, d_{i+1}), y^*>| \leq \frac{2a}{r} <h, t_{i+1}^+ - t_{i-1}^->$$

and hence by (6)

$$|Q_p(d_{i-1}, d_i) - Q_p(d_{i}, d_{i+1})| \leq \frac{2a}{r} <h, t_{i+1}^+ - t_{i-1}^->.$$

Then 

$$\frac{1}{t_i} |Q_p(d_{i-1}, d_i) - Q_p(d_{i}, d_{i+1})| \leq \frac{2a}{r} <h, t_{s+1} - t_{s-1} - t_o> \leq \frac{2a}{r} (|t_s| + |t_t| + |l_t| + |l_t|) < \frac{2a}{r} (r + m)$$

because $|t_j| \leq |t_j - c_{x_j}| + |c_{x_j}| < \frac{r}{2} + m$.

So we managed to estimate $X_t(F,w)\) from above independently on the choice of $w$ and $h$, and that is why $y^*$ has linearly finite convexity on $M$.

$d)$ $y^*$ is $\delta$-convex on $M$ if $T \subset \partial f$.

Let $T \subset \partial f$ for some proper convex function on $X$. Without any loss of generality we can suppose $T = \partial f$. Now $T$ is always convex

Let $y^* \in Y, \|y^*\| = 1$. For any $x \in \mathcal{B}$ we shall denote $w_x$ the projection of $x$ on $W$ in the direction of $Z$. Then $w_x \in M$ and $x = w_x + y(w_x)$. A functional $t_x^+ t_x^-$ is an element of $T$ because $\|\frac{1}{2q} x_x(y^*)\| < \frac{r}{2}$.

Let us denote

$$s_1(w_x) = f(x) - <y(w_x), t_x^->$$

and

$$s_1(w_x) = f(x) - <y(w_x), t_x^->.$$

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Let $x_0 \in B$ be fixed. We shall define two continuous affine functions on $W$: 

$$a_{x_0}(w) = f(x_0) + \langle w-x_0, t_{x_0} \rangle$$

$$b_{x_0}(w) = f(x_0) + \langle w-x_0, t_{x_0}^- \rangle .$$

For any $x \in B$ the functionals $t_x^- t_{x_0}, t_x^- t_{x_0}^-$ are in $V$ and $t_{x_0}, t_{x_0}^-$ are in $\mathcal{D}(x_0)$, hence 

$$a_{x_0}(w_x) = f(x_0) + \langle x-x_0, t_{x_0} \rangle - \langle \psi(w_x), t_{x_0} \rangle \leq$$

$$\leq f(x) - \langle \psi(w_x), t_{x_0} \rangle = f(x) - \langle \psi(w_x), t_x \rangle = g_1(w_x) ,$$

$$a_{x_0}(w_x) = f(x_0) - \langle \psi(w_x), t_x \rangle = g_1(w_x) .$$

Similarly 

$$b_{x_0}(w_x) \leq h_1(w_x) , \quad b_{x_0}(w_x) = h_1(w_x) .$$

The functions $a_{x_0}, b_{x_0}$ are Lipschitz with the constant $m+r$ (since $|t_x^-| m+r, |t_{x_0}| m+ \frac{r}{2} < m+r$). The former properties enable us to say that the functions 

$$g(w) = \sup \{a_{x_0}(w): x_0 \in B \}$$

$$h(w) = \sup \{b_{x_0}(w): x_0 \in B \}$$

are Lipschitz convex functions on $W$ satisfying $g = g_1, h = h_1$ on $M$ and the function $g$ does not depend on the choice of $y^*$. For any $x \in B$

$$h_1(w_x) - g_1(w_x) = \frac{r}{2q} \langle \psi(w_x), \pi_x(y^*) \rangle = \frac{r}{2q} \langle \psi(w_x), y^* \rangle .$$

Put $G(w) = \frac{2q}{r} g(w), H_y(w) = \frac{2q}{r} h(w) .$

We have proved that for any $y^* \in Y$: $y^*, \psi = H_y, G$ on $M$ where $H_y, G$ are convex Lipschitz functions on $W$ and $G$ is independent on $y^*$. Hence $\psi$ is $\delta$-convex on $M$ regarding (6).

The theorems 3.5, 3.6 are proved. ///

The following proposition is a direct consequence of 3.5, 3.6 and 2.13.

3.10 Corollary: Let $T$ be a monotone operator on a separable Banach space $X$ and $n < \dim X$ be a positive integer. Then the set $A_n$ can be covered by countably many Lipschitz surfaces of codi-
mension \( n \). If \( T < \partial f \) for some proper convex function \( f \) then the set \( A^n \) can be covered by countably many DC-surfaces of codimension \( n \).

If \( X^* \) is separable then the set \( A^1 \) for a general monotone operator \( T \) can be covered by countably many LFC-curves.

If \( X \) is a separable Hilbert space then \( A^n \) can be covered by countably many Lipschitz surfaces of codimension \( n \).

**3.11 Observation:** Let us observe that in case \( X = \mathbb{R}^2 \), 3.10 ensures a countable covering of the set \( A_1 \) of a general monotone operator \( T \) on \( \mathbb{R}^2 \) by LFC-curves which are simultaneously DC-hypersurfaces in this case. (Compare the problem 1.1.)

There are sometimes considered monotone operators on \( X^* \) with values in \( X \), e.g., an operator \( T_{-1} \) "inverse" to a monotone operator \( T \) on \( X \):

\[
T_{-1}: X^* \to \exp X
T_{-1}(x^*) = \{ x \in X : x^* \in T(x) \}.
\]

In this cases the following version of 3.6 is useful. (The proof is similar; instead of \( \| x^* \| = \sup\{ \langle x^*, x^{**} \rangle : \| x \| = 1 \} \) use \( \| x^* \| = \sup\{ \langle x, x^* \rangle : \| x \| = 1 \} \) and change the roles of \( X \) and \( X^* \).)

**3.12 Theorem:** Let \( X \) be a separable Banach space, \( T: X^* \to \exp X \) be a monotone operator and \( n < \dim X \) be a positive integer. Then \( A^n \in \mathcal{LFC}_n \). If \( T < \partial f \) for some proper convex function \( f \) on \( X^* \) then \( A^n \in \mathcal{S} \mathcal{D}_n \).

4. Operators \( V_M, F_M \)

Let \( M \) be a nonvoid convex subset of a Banach space \( X \). We shall state the definition of a vertex-operator \( V_M: X \to \exp X^* \) and a face-operator \( F_M: X^* \to \exp X \) which are in close connection with singular points of \( M \) (cf. [58]).

**4.1 Definition:** Let

\[
\delta_M(x) = \begin{cases} 
0 & \text{if } x \in M, \\
+\infty & \text{if } x \notin M;
\end{cases}
\]

\[
s_M(x^*) = \sup\{ \langle m, x^* \rangle : m \in M \}, \quad x^* \in X^*.
\]
\( \delta_M \) is called indicator-function of \( M \) and is a proper convex function on \( X \). The function \( s_M \) satisfies \( s_M(tx^*) = t \cdot s_M(x^*) \), \( s_M(x^*+y^*) \leq s_M(x^*)+s_M(y^*) \) for any \( t > 0, x^*, y^* \in X^* \). Hence if \( \text{dom } s_M \) is not empty then \( s_M \) is a proper convex function on \( X^* \).

### 4.2 Definition:

\[
V_M(x^*) = \begin{cases} 
\{ y^* \in X^* : \langle x^*, y^* \rangle = s_M(y^*) \} & \text{if } x^* \in M, \\
\emptyset & \text{if } x^* \notin M;
\end{cases}
\]

\[
P_M(x^*) = \{ y \in M : \langle y^*, x^* \rangle = s_M(x^*) \}, \quad x^* \in X^*.
\]

### 4.3 Note:

a/ If \( X = \mathbb{R}^m \) then \( V_M(x) \) is the set of all normals of \( M \) at \( x \) and is called vertex of \( M \) at \( x \). The set \( P_M(x^*) \) forms a face of \( M \) perpendicular to \( x^* \).

b/ It is obvious that the operators \( V_M, P_M \) are monotone and their images \( V_M(x), P_M(x^*) \) of each point are convex. Following simple lemma says a little more.

### 4.4 Lemma:

\( V_M = \partial \delta_M, \quad P_M \subset \partial \delta_M \).

**Proof:** a/ If \( x \notin M \) then \( V_M(x) = \emptyset \). Let \( x^* \in M \). Then the following equivalences hold:

\[
x^* \in V_M(x) \iff V_M(x) = \emptyset \iff \forall y \in X, \exists \epsilon > 0 \forall z \in X, d_M(z) > d_M(x) + \epsilon \iff z \cong x^*.
\]

b/ If \( F_M(x^*) = \emptyset \) then \( F_M(x^*) \subset \partial \delta_M(x^*) \) is evident. Let \( x^* \in F_M(x^*) \). Then any \( z^* \in X^* \) satisfies \( s_M(z^*) \leq \langle x^*, z^* \rangle = s_M(x^*) \) and hence \( x \in \partial \delta_M(x^*) \).

### 4.5 Theorem:

If \( X \) is separable then \( A_n(V_M) \in \mathcal{SDC}^n(X) \), \( A_n(P_M) \in \mathcal{SDC}^n(X^*) \).

If \( X^* \) is separable then \( A_n(V_M) \in \mathcal{SDC}^n(X), \quad A_n(P_M) \in \mathcal{SDC}^n(X^*) \).

**Proof:**

The propositions of the theorem yield from 3.5, 3.6, 4.4.

Using known extension theorems it is possible to obtain following new result.

### 4.6 Theorem:

Let \( M \) be a nonempty convex subset of a separable Banach space \( X \). Then:

1/ The set of points \( x \in M \) for which \( V_M(x) \) is at least \( n \)-dimensional can be covered by countably many DC-surfaces of codimen-

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If in addition $X^*$ is separable then the set of all normals $x^*$ to $M$ at faces $F_M(x^*)$ being at least $n$-dimensional can be covered by countably many DC-surfaces of codimension $n$, and the set of all points $x \in M$ with a vertex $V_M(x)$ containing a ball of codimension 1 can be covered by countably many LFC-curves.

5. Existence of "bad" Lipschitz surfaces

We shall show that there exist Lipschitz surfaces of codimension $n$ (dimension $n$, respectively) which cannot be a subset of $A_n$ ($A^n$, resp.) for any monotone operator $T$ satisfying assumptions of the theorems 3.5, 3.6.

We shall use the local geometric term of a contingent of a set at a point (cf. [5]).

5.1 Definition: Let $X$ be a Banach space, $x \in X$, $M \subset X$. Then we define $\text{cont}(M,x)$ as the set of all nonzero vectors $v \in X$ which satisfy the following condition:

There exist sequences $\{x_n\} \subset X$, $\{\lambda_n\} \subset \mathbb{R}$ such that

/i/ $x_n \in M$, 
/ii/ $\lambda_n > 0$, 
/iii/ $\lambda_n \to 0$, 
/iv/ $\lim_{\lambda_n \to 0} \frac{x_n}{\lambda_n} - v \to 0$.

5.2 Construction: Let $X$ be a Banach space, $W, Z$ closed subspaces of $X$ such that $X = W \oplus Z$ (i.e. $X$ is a topological sum of $W, Z$).

Let $h \in W$, $z_0 \in Z$ be nonzero vectors and $U$ be a topological complement of $\text{lin}\{h\}$ in the space $W$. We shall define a Lipschitz mapping $F: W \to Z$ by the formula

$$F(th + u) = f(t)z_0, \quad t \in \mathbb{R}, u \in U,$$

where $f$ is a real Lipschitz function on $\mathbb{R}$ which has right derivative $f'_+(t)$ at no rational point $t$. (Existence of $f$ is guaranteed by a standard category argument.)
Denote \( E = \{w + f(w) : w \in W\} \). Let \( q \in \mathbb{R} \), \( u_0 \in U \), \( x = qh + u_0 + f(q)z_o \in E \). It is easy to prove that \( \text{cont}(E, x) \) contains the set \( C = \{dh + f(q) \in \mathbb{R} : u \in U, \alpha D_x f(q) \leq \beta = \alpha D_x f(q), \gamma \in \gamma \} \), where \( D_x f \) denote the lower and upper Dini derivatives of \( f \) and \( Y \) is a topological complement of \( \text{lin}\{z_o\} \) in \( Z \).

Hence \( \text{int}(\text{cont}(E, x)) \not= \emptyset \) if \( x = qh + u_0 + f(q)z_o \) with \( q \) rational. (7)

2.3 Lemma: Let \( X \) be a Banach space, \( W, Z \) be closed subspaces of \( X \) such that \( X = W \oplus Z \). Let \( w_o \in W \) and \( G : W \rightarrow Z \) be a Lipschitz mapping having all one-sided directional derivatives at \( w_o \). Denote

\[
M = \{w + G(w) : w \in W\},
\]

\[
x = w_o + G(w_0),
\]

\[
\pi : W \rightarrow W \text{ a projection in the direction of } Z.
\]

Then, if \( v_1, v_2 \in \text{cont}(M, x) \), \( \pi_w(v_1) = \pi_w(v_2) \) then \( v_1 = v_2 \).

Proof: Let \( v_1, v_2 \in \text{cont}(M, x) \), \( \pi_w(v_1) = \pi_w(v_2) = y \). The vector \( y \) is nonzero because \( G \) is Lipschitz. Let \( z_1, z_2 \in Z \) be such that \( v_i = y + z_i \) \((i = 1, 2)\). Let \( U_y \) be a topological complement of \( \text{lin}\{y\} \) in \( W \), \( \pi \rightarrow \pi \) a projection in the direction of \( U_y \), \( \pi : W \rightarrow W \) a projection in the direction of \( y \). By 5.1 we have

\[
x_{n,i} = w_{n,i} + G(w_{n,i}), \quad \lambda_{n,i} > 0, \quad \lambda_{n,i} \rightarrow 0,
\]

\[
\lambda_{n,i} = (\frac{x_{n,i}}{\lambda_{n,i}} - v_i) \rightarrow 0 \quad (i = 1, 2).
\]

Let \( a_{n,i} \in \mathbb{R} \) be such that \( a_{n,i} y = \pi_y(w_{n,i} - w_o) \). Then

\[
\lim_{n \rightarrow \infty} \frac{a_{n,i}}{\lambda_{n,i}} = 1
\]

because

\[
\| (\frac{a_{n,i}}{\lambda_{n,i}} - 1) y \| = \| \pi_y(\lambda_{n,i}) \| \rightarrow 0.
\]

Without any loss of generality we can suppose \( a_{n,i} > 0 \) \((i = 1, 2, \ldots, n = 1, 2, \ldots)\). Then

\[
\| \frac{G(w_o + a_{n,i}y) - G(w_o)}{\lambda_{n,i}} - v_i \| = \| \frac{w_{n,i} + G(w_{n,i}) - w_o - G(w_o)}{\lambda_{n,i}} - v_i + \frac{\lambda_{n,i}y - a_{n,i}y}{\lambda_{n,i}} + \frac{a_{n,i}y - (w_{n,i} - w_o)}{\lambda_{n,i}} + \frac{G(w_o + a_{n,i}y) - G(w_{n,i})}{\lambda_{n,i}} \| \leq \varepsilon
\]

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\[ |\frac{d}{dx} f(x)| + 1 - \frac{d}{dx} f(x) |1 + L| |1 + (1 + L)| \to 0, \]

where \( L \) is the constant from the Lipschitz property of \( g \).

Then (8) and the existence of a directional derivative \( g(x, y) \)

imply \( z_1 = g(x_0, y) = z_2 \).

5.4 Theorem: Let \( X \) be a separable Banach space (\( X \) has separable
dual \( X^* \), resp.), \( n \leq \text{dim} \ X \) be a positive integer. Then the set \( E \)
from 5.2 with \( \text{dim} Z = n \) (\( \text{codim} \ Z = n \), resp.) is a Lipschitz surface
of codimension \( n \) (of dimension \( n \), resp.) which cannot satisfy
\( E \subset A_n \) (\( E \subset A^n \), resp.) for any monotone operator \( T \) on \( X \).

Proof: Let us assume the existence of \( T \) such that \( E \subset A_n \) (\( E \subset A^n \),
resp.). Then (in the notation of 3.9) \( E \subset \cup B(r, m, v, q, L) \).

There exist \( r_0, m_0, v_0, q_0, L_0 \), a positive number \( \delta \) and a point
\( x_0 \in E \) such that the set \( B_0 = B(r_0, m_0, v_0, q_0, L_0) \) is dense in
\( E \cap \Lambda(x_0, \delta) \), by the Baire Category Theorem.

Let \( Z_0 = V_0, \ W_0 \) be a topological complement of \( Z_0 \) in \( X \) and \( x_0 \); \( X \to \ W_0 \)
be a projection in the direction of \( Z_0 \). The set \( M_0 = \Lambda(x_0) \)
is dense in \( S = \pi_0(E \cap \Lambda(x_0, \delta)) \), which is an open set
containing the point \( \pi_0(x_0) \). By the part d/ of 3.9, there exists
a Lipschitz mapping \( \psi_0: M_0 \to Z_0 \) with a linearly finite convexity
on \( M_0 \) such that \( B_0 = \{ \psi_0(w): w \in M_0 \} \).

\( \psi_0 \) has unique continuous extension \( \overline{\psi}_0 \) on \( \overline{M}_0 \). This extension is
Lipschitz, has linearly finite convexity on \( \overline{M}_0 \) and has by 2.4
all one-sided directional derivatives at each point \( \pi_0(x) \in S \).

\( \text{int}(\text{cont}(E, x)) = \emptyset \) for every \( x \in E \cap \Lambda(x_0, \delta) \) by 5.3.

But the construction of \( E \) implies that there exists a point
\( x = qh + u + f(q)z_0 \in E \cap \Lambda(x_0, \delta) \) with \( q \) rational. Then \( \text{cont}(E, x) \) has
nonempty interior by (7) and this is the needed contradiction.

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References

[1] N. Aronszajn: Differentiability of Lipschitzian mappings be­


Russian translation.


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(Oblatum 20.3. 1986)