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IDEALS OF UNIFORMLY CONTINUOUS MAPPINGS ON PSEUDOMETRIC SPACES

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Abstract: The concepts of an ideal $\mathfrak{I}$ of uniformly continuous mappings between pseudometric spaces and a uniform $\mathfrak{I}$-space are atudied. As examples various ideals of precompact mappings are investigated. Connections with point-finiteness and covering dimension are stated.

Key words: Ideals of mappings, uniformities.

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0. Introduction. The method of ideals of mappings as a tool of investigating special classes of spaces is often used in functional analysis (cf. e.g. [6]). We want to present here a parallel theory in the (nonlinear) case of uniform spaces and show in easy examples that it may bring new and interesting methods. The main difference here is that we have no canonical generating family of pseudometrics for a general uniform space (like the family of all continuous seminorms in the case of a locally convex topological vector space).

The first general part is devoted to definitions and fundamental properties of an ideal $\mathfrak{I}$ and a related class of $\mathfrak{I}$-spaces, so that the class of all $\mathfrak{I}$-spaces forms an epireflective subclass of uniform spaces. The second part applies the theory to the ideals of precompact and equi-precompact mappings and shows among others how the methods of metric precompactness may be used in very nonmetric and nonprecompact cases.

We refer to [2] for basic definitions and results pertaining to uniform spaces. If (X,d), (Y,σ) are pseudometric spaces, $\mathcal{J}$ a family of mappings defined between pseudometric spaces, $\mathcal{J}((X,d),(Y,\sigma))$ or only $\mathcal{J}(d,\sigma)$ will denote all $\mathcal{J}$-mappings from (X,d) into (Y,σ). In particular, $U(d,\sigma)$ will stand for all uniformly continuous ones.

1. General part. Let d be a pseudometric on a set X, $\varepsilon > 0$. We shall denote by $\mathcal{S}_d(\varepsilon)$ the cover of X consisting of all (open) balls with diameter ε. Obviously, all such covers form a basis for the uniformity of the space (X,d). For a uniform space X, a family $\mathfrak{D}$ of uniformly continuous pseudometrics on X will be called a generating family for X, if every uniform cover of X may be refined by $\mathcal{S}_d(\varepsilon)$ for some $d \in \mathfrak{D}$ and $\varepsilon > 0$.

1.1. Definition. Suppose for every two pseudometric spaces σ, d we have a family of uniformly continuous mappings $\mathcal{J}(\sigma,d)$ having the following two properties:

(1) If ρ, σ, d, η are pseudometrics, $f \in U(\rho,\sigma)$, $h \in \mathcal{J}(\sigma,d)$ $g \in U(d,\eta)$, then $ghf \in \mathcal{J}(\rho,\eta)$

(2) If $fh \in \mathcal{J}$, f is an isometry, then $h \in \mathcal{J}$.

A uniform space X will be called an $\mathcal{J}$-space, if X has a generating family $\mathfrak{D}$ of pseudometrics such that for every $d \in \mathfrak{D}$ there is a uniformly continuous pseudometric σ on X such that the identity $id: \sigma \longrightarrow d \in \mathcal{J}$ (hence σ is finer than d).

The following characterization of $\mathcal{J}$-spaces seems to be very instructive omitting the unpleasant point with the existence of a "good" generating family:

1.2. Proposition. Suppose $\mathcal{J}$ is as in 1.1, X is an $\mathcal{J}$-space if and only if for every (M,ρ) metric ANRU (absolute uniform neighborhood retract), $f \in U(X,M)$, $\varepsilon > 0$ we can find d_ε uniformly continuous pseudometric on X, $f_\varepsilon \in \mathcal{J}(d_\varepsilon,\rho)$ such that $\rho(f,f_\varepsilon) = \sup_{x \in X} \rho(fx,f_\varepsilon x) \leq \varepsilon$.

Proof: Suppose X is an $\mathcal{J}$-space, $f:X \longrightarrow M$ uniformly continuous, $\varepsilon > 0$. There is $d'_\varepsilon \in \mathfrak{D}$ such that $f^{-1}(\mathcal{S}_\rho(\varepsilon))$ is a uniform cover of (X,d'_ε). M is ANRU, hence using the Isbell's generalization of Hahn's lemma (see [3]) we get $f_\varepsilon \in U(d'_\varepsilon,\rho)$ with $\rho(f,f_\varepsilon) \leq \varepsilon$. Take d_ε uniformly continuous on X such that $id:d_\varepsilon \longrightarrow d'_\varepsilon \in \mathcal{J}$, then $f_\varepsilon : d_\varepsilon \longrightarrow \rho \in \mathcal{J}$ and $\rho(f_\varepsilon,f) \leq \varepsilon$.

Conversely, take any uniformly continuous pseudometric d on X, $f:(X,d) \longrightarrow \ell_\infty(X)$ the canonical isometric mapping. The latter space is ANRU, therefore for every n natural we may find d'_n uniformly continuous pseudometric on X and $f_n \in \mathcal{J}(d'_n,\rho)$ such that $\rho(f_n,f) \leq \frac{1}{n}$) ($\rho$ stands for the metric defined by the norm). Set $d_n = \rho f_n^2$. Then d_n is a uniformly continuous pseudometric on X. Moreover, if $n \geq \frac{3}{\varepsilon}$, we have for $d_n(x,y) \leq \frac{\varepsilon}{3}$: $d(x,y) = \rho(fx,fy) \leq \leq \rho(fx,f_nx) + \rho(f_nx,f_ny) + \rho(f_ny,fy) \leq \varepsilon$, hence d is uniformly continuous on X endowed with the uniformity generated by the family $\{d_n, n \in N\}$.

So the family $\mathfrak{D} = \{d_n; n \in N$, d a uniformly continuous pseudometric on $X\}$ forms a generating family for X. For every $d_n \in \mathfrak{D}$ take $\bar{d}_n = \max(d_n,d'_n)$ and denote i_1 the identity $\bar{d}_n \longrightarrow d_n$ and i_2 the identity $\bar{d}_n \longrightarrow d'_n$. Then $f_n i_2 \in \mathcal{J}$, f_n is an isometry, $f_n i_1 = f_n i_2$, hence $i_1 \in \mathcal{J}(\bar{d}_n,d_n)$, which proves that X is an $\mathcal{J}$-space.

1.3. Definition. The class $\mathcal{J}$ in 1.1 will be called an ideal (of uniformly continuous mappings on pseudometric spaces) if all projections onto a **oneppint** space are in $\mathcal{J}$.

$\mathcal{J}$ **will be called a projective ideal if it is an ideal and**

(P) If d_1, d_2 are pseudometrics on the same set, σ an arbitrary pseudometric, then $f \in \mathcal{J}(\sigma, \max(d_1,d_2))$ provided that $f \in \mathcal{J}(\sigma, d_1)$ and $f \in \mathcal{J}(\sigma, d_2)$.

If $\mathcal{J}$ is an ideal, we shall denote by $[\mathcal{J}]$ the class of all $\mathcal{J}$-spaces. We turn to the permanence properties of the class $[\mathcal{J}]$:

1.4. <u>Proposition</u>. (1) the class $[\mathcal{J}]$ is hereditary for every ideal $\mathcal{J}$.

(2) The class $[\mathcal{J}]$ is productive provided $\mathcal{J}$ is a projective ideal.

<u>Proof</u>: Let Y be a subspace of an $\mathcal{J}$-space X, j the corresponding embedding, $\mathcal{D}$ a generating family of pseudometrics from the definition of an $\mathcal{J}$-space. Set $\mathcal{D}' = \{dj^2; d \in \mathcal{D}\}$, this is a generating family for Y and for every $dj^2 \in \mathcal{D}'$ we may find a uniformly continuous pseudometric σ on X such that the identity $i: \sigma \longrightarrow d \in \mathcal{J}$, hence, $ij \in \mathcal{J}(\sigma j^2, d)$. ij is equal to the composition of the identity id: $\sigma j^2 \longrightarrow dj^2$ and $j: dj^2 \longrightarrow d$, the latter mapping is an isometry, hence id: $\sigma j^2 \longrightarrow dj^2 \in \mathcal{J}$.

(2) Take the family $\{X_a\}_{a \in J}$ of $\mathcal{J}$-spaces and let $\mathcal{D}_a$ be the corresponding generating family for X_a guaranteed by the definition of an $\mathcal{J}$-space. For every finite set $A \subset J$ and $\{d_a; a \in A\}$ a choice of members $d_a \in \mathcal{D}_a$ we define a pseudometric on the product of all X_a's: $d_A(\{x_a\},\{y_a\}) = \max_{a \in A} d_a(x_a,y_a)$.

All pseudometrics of this type form evidently a generating family for the space $\prod_{a \in J} X_a$. Let us denote this family $\mathcal{D}$. Take $d_A \in \mathcal{D}$, if d_a is one of the components of d_A, take σ_a a uniformly continuous pseudometric on X_a such that the identity from σ_a onto d_a is in $\mathcal{J}$. Define $\sigma_A(\{x_a\},\{y_a\}) = \max_{a \in A} \sigma_a(x_a,y_a)$. Now $d_A = \max_{a \in A} d_{\{a\}}$ and **similarly** σ_A. Using (2) from 1.1 and projectivity of $\mathcal{J}_1$ we easily obtain that the identity id: $\sigma_A \to d_A \in \mathcal{J}$.

From the foregoing proposition we may see that for every projective ideal $\mathcal{J}$ the class $[\mathcal{J}]$ forms a hereditarily epireflective subclass of uniform spaces. The natural question arises what epireflections may be represented in this form.

Let $(\mathcal{R},r)$ be a hereditary epireflection in uniform spaces, i.e. $\mathcal{R}$ is the class of spaces and r the corresponding epireflector. If σ, d are pseudometric, we define $\mathcal{J}_r(\sigma,d)$ as the family of all $f \in U(\sigma,d)$ such that there is a space $X \in \mathcal{R}$ and mappings $f_1 \in U(X,\bar{d})$, $f_2 \in U(\sigma,X)$ with $\Theta f = f_1 f_2$, where $\Theta : d \to \bar{d}$ is the natural projection onto the Hausdorffization $\bar{d}$ of d. The following proposition is very easy to verify.

1.5. Proposition. $\mathcal{J}_r$ is a projective ideal and $[\mathcal{J}_r] \subset \mathcal{R}$.

The converse inclusion is easily true for those epireflections which can be written as a hereditary epireflective hull of some family of metrizable spaces, but in general it fails to be true as the following examples.shows.

1.6. Example. There is a hereditary epireflection $(\mathcal{R},r)$ such that $[\mathcal{J}_r] \subsetneqq \mathcal{R}$, hence $\mathcal{R}$ cannot be represented by a projective ideal.

Proof: First observe that if $f \in \mathcal{J}_r(\sigma,d)$, then its Hausdorffization $\bar{f}$ is in $\mathcal{J}_r(\bar{\sigma},\bar{d})$, hence f factorizes through $r\bar{\sigma}$. From here it follows that if two epireflections r, s have the same values on all metrizable spaces, then $[\mathcal{J}_r] = [\mathcal{J}_s]$.

Recall that a uniform space is called distal if it has a basis for uniformity consisting of finite-dimensional covers only. The class of all distal spaces is epireflective, let us denote by D the corresponding reflector. (For details on distal

spaces we refer to [1].) Put $\mathcal{R}$ to be the class of all uniform spaces X such that $U(M,X) \subset U(DM,X)$ for all metrizable spaces M. The class $\mathcal{R}$ is hereditary epireflective (cf. [7]), take for r the corresponding epireflector.

$rM = DM$ for all metrizable M, hence $[\mathcal{J}_r]$ consists of distal spaces only. Take any proximally discrete (i.e. all finite partitions are uniform) non distal space. (Such spaces do exist, e.g. the proximally discrete coreflection of any non distal space, see also [2], V.5.) Obviously $Z \in \mathcal{R}$ and not in $[\mathcal{J}_r]$.

2. Precompact and equiprecompact mappings

2.1. Definition. A uniformly continuous mapping $f:(X,\sigma) \to (Y,d)$ between pseudometric spaces is called precompact, if there is $\varepsilon > 0$ such that $f[B_d(x,\varepsilon)]$ is precompact in (Y,d) for all $x \in X$. ($B(x,\varepsilon)$ stands for the ball with center x and radius ε.)

It can be easily verified that precompact mappings form a projective ideal.

The following proposition follows immediately from Chapter 1.

2.2. Proposition. The following properties of a uniform space X are equivalent:

(1) X is an $\mathcal{J}$-space, where $\mathcal{J}$ is the ideal of all precompact mappings.

(2) There exists a generating family $\mathcal{D}$ of pseudometrics for X such that for every $d \in \mathcal{D}$ there is a uniform cover $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$, U is d-precompact.

(3) For every set A, $f \in U(X, \ell_\infty(A))$, $\varepsilon > 0$ there are d_ε uniformly continuous pseudometrics on X and $f_\varepsilon:(X,d) \to \ell_\infty(A)$ precompact such that $\| f_\varepsilon - f \| \leq \varepsilon$.

(4) For every metric ANRU (M,ρ), $f \in U(X,M)$, $\varepsilon > 0$ we can find a uniformly continuous pseudometric d_ε on X and a precompact mapping $f_\varepsilon : (X,d) \longrightarrow M$ with $\rho(f,f_\varepsilon) \leq \varepsilon$.

Looking at the property (2) one can say that the property described in 2.2 is a direct generalization of uniform local precompactness. It seems to be interesting that the class of all such spaces is much larger, as the following theorem shows.

2.3. Theorem. A uniform space enjoys the property of 2.2 if and only if it has a basis of point-finite covers.

Proof: Every point-finite uniform space has a basis of uniformly locally finite covers. Let us denote $\mathcal{B}$ such a basis of X. The following lemma is a crucial point of the proof:

Lemma ([4], Lemma 8). For every point-finite uniform cover $\mathcal{U}$ of a uniform space X there is a star-refinement $\mathcal{P}$ of $\mathcal{U}$ such that if a set A intersects a finite number of $\mathcal{U}$ only, then A intersects a finite number of $\mathcal{P}$ only.

Starting with a cover $\mathcal{U}_0$ in $\mathcal{B}$ and applying the lemma infinitely many times, we obtain a normal sequence $\{\mathcal{U}_n\}$ of uniform covers with the property that for some uniform cover $\mathcal{V}$ of X, each $V \in \mathcal{V}$ intersects only finitely many members of every $\mathcal{U}_n$.

Using Urysohn metrization lemma we obtain a uniformly continuous pseudometric d on X such that $\mathcal{U}_n \succ \mathcal{S}_d(\frac{1}{2^n}) \succ \mathcal{U}_{n+1}$. All such pseudometrics form a generating family for X having the property (2) of 2.2.

Conversely suppose we have the generating family $\mathfrak{D}$ for uniformly continuous pseudometric as in 2.2. Take $d \in \mathfrak{D}$, we have a uniform cover $\mathcal{P}$ of X such that all $P \in \mathcal{P}$ are d-precompact. We shall prove that the cover $\mathcal{S}_d(1)$ consisting of all d-open balls with radius 1 has a point-finite uniform (in X) refinement. Take the cover $\mathcal{G}_d(\frac{1}{n})$, it has a refinement $\mathcal{F}$ (not necessarily uniform) such that $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$ and each $\mathcal{F}_n$ is uniformly discrete (in d). Take $\mathcal{U}$ a uniform cover of X such that $\mathcal{U} \prec \mathcal{G}_d(\frac{1}{4})$, $\mathcal{U} \overset{*}{\prec} \mathcal{P}$.

For every $F \in \mathcal{F}$ take $A(F) = St(F,\mathcal{U})$. Take $x \in X$, we may find $P_x \in \mathcal{P}$ such that $St(x,\mathcal{U}) \subset P_x$, hence if $x \in A(F)$, then $F \cap P_x \neq \emptyset$. $\mathcal{F}_n$ is uniformly discrete w.r.t. d, P_x is precompact in d, hence P_x may intersect only finitely many members from $\mathcal{F}_n$. Therefore $\{A(F), F \in \mathcal{F}_n\}$ is point-finite, hence $\mathcal{V}$ is σ-point-finite, so it may be refined by a point-finite uniform cover. Obviously $\mathcal{V}$ refines $\mathcal{G}_d(1)$.

As a corollary we present at least one special result which shows how far is the nonlinear case from the linear one:

2.4. Corollary. The identical operator of a separable Hilbert space is a uniform limit of a sequence of (nonlinear) precompact mappings.

Proof: The proof follows immediately from 2.2 and 2.3 because a separable Hilbert space is a point-finite ANRU.

2.5. Definition. Let d be a pseudometric on a set X, $\mathcal{U}$ a family of subsets of X, $\varepsilon > 0$. We shall denote

$$M_\varepsilon(\mathcal{U},d) = \sup_{U \in \mathcal{U}} \sup \{m \in N; \exists x_1,\dots,x_m \in U \text{ such that } d(x_i, x_j) \geq \varepsilon \text{ for all } i \neq j\}.$$

A uniformly continuous mapping $f:(X,\sigma) \longrightarrow (Y,d)$ between pseudometric spaces will be called equiprecompact, if there is $\delta > 0$ such that $M_\varepsilon(f[\mathcal{G}_\sigma(\delta)],d) < \infty$ for all $\varepsilon > 0$.

One can verify again that equi-precompact mappings form a projective ideal. Let us call the corresponding spaces (uniformly)

Schwartz. The name comes from a parallel concept in the theory of locally convex topological vector spaces. The class of all Schwartz spaces is closed under products and subspaces and we may formulate a similar characterization theorem to 2.2.

It is shown in [8] that every finite-dimensional (hence every distal) space is Schwartz, also all (linearly) Schwartz locally convex spaces with its natural uniformity are Schwartz. This together with the permanence properties illustrates that the class of Schwartz spaces is quite large. For all Schwartz spaces we may have the following complexity classification (see [8]):

$$\mathcal{E}(X) = \bigcup_{\mathcal{D}} \{\varphi : (0,1) \longrightarrow (0,\infty);\ \text{for every } d \in \mathcal{D} \text{ there is a uniform cover } \mathcal{U} \text{ of } X \text{ with } \overline{\lim_{\varepsilon \to 0}}\ \frac{1}{\varphi(\varepsilon)} M_{\varepsilon}(\mathcal{U},d) \text{ finite}\}.$$

The union is taken over all generating families for X.
This class of functions is well defined for all Schwartz spaces and $\mathcal{E}(X) \neq 0$ for many even infinite dimensional spaces.

Let us add several results showing that $\mathcal{E}(X)$ may serve as a dimension-like classification of X.

1) X is 0-dimensional if and only if $\mathcal{E}(X)$ contains a constant function.

2) If the large uniform covering dimension is at most n, then ε^{-n} belongs to $\mathcal{E}(X)$ (see [8])

3) If X is a nuclear locally convex space with its natural uniformity, then $2^{\varepsilon^{-1}}$ belongs to $\mathcal{E}(X)$ (see [5]).

Problem: Is every (topologically) fine uniform space Schwartz?

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