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ON THE CARDINALITY OF LINDELÖF SUBSPACES OF FUNCTION SPACES

A. V. ARHANGEL'SKII, V. V. USPENSKII

Abstract: Let X be a compact space. If Y is a Lindelöf subspace of $C_p(X)$, the space of all continuous real-valued functions on X in the topology of pointwise convergence, then $|Y| \le \exp(c(X))$, where $c(X)$ is the Souslin number of X. If X is dyadic, then any Lindelöf subspace of $C_p(X)$ has a countable network.

Key words: Lindelöf space, Souslin number, tightness, function space.

Classification: 54A25, 54C35.

Let X be a compact space having the Souslin property. Then compact subspaces of $C_p(X)$ are metrizable. This fact can be deduced from the equality $w(X)=c(X)$ which holds for Eberlein-compact spaces. We show that Lindelöf subspaces of $C_p(X)$ also cannot be too large: if $Y \subseteq C_p(X)$ is Lindelöf, then $|Y| \le 2^{\omega}$. This is a special case of the following theorem:

Theorem 1. If X is compact and $Y \subseteq C_p(X)$, then $|\overline{Y}| \le \exp(\ell(Y) \cdot c(X))$.

We consider only Tyhonoff spaces. See [1] - [3] for the definition and notation of cardinal functions: $C_p(X)$ is the space of all continuous real-valued functions on X in the topology of pointwise convergence; $\ell(X)$ is the Lindelöf number of X, $w(X)$ is the weight of X, and $e(X)$ is the extent of X i.e. $e(X) = \sup \{|A| : A \text{ is a closed discrete subspace of } X\}$.

We start with a list of facts that we need for the proof.

Theorem B. If X is compact and $Y \subseteq C_p(X)$, then $\ell(Y)=e(Y)$.

This is a recent very beautiful and very powerful result of D. Baturov of Moscow.

Theorem S (B.E. Shapirovskii). If X is compact, then $w(X) \le t(X)^{c(X)}$.

This is a combination of two other results of Shapirovskii: (1) $w(X) \le \pi\chi(X)^{c(X)}$ if X is regular; (2) $\pi\chi(X) \le t(X)$ if X is compact, see [2],[3].

Theorem A (A.V. Arhangel'skii [2]). Let X be a T_1-space and m be a cardinal. Suppose that: (1) $\ell(X) \le m$, (2) $t(X) \le m$; (3) $\psi(X) \le 2^m$; (4) if $A \subseteq X$ and $|A| \le m$, then $|\bar{A}| \le 2^m$. Then $|X| \le 2^m$.

A space Y is monolithic if $nw(\bar{A}) \le |A|$ whenever $A \subseteq Y$. If X is compact, $C_p(X)$ is monolithic and countably tight, [2],[4], so for any $Y \subseteq C_p(X)$ the inequality $|\bar{Y}| \le |Y|^{\omega} \le 2^{|Y|}$ holds.

We turn to the proof of Theorem 1. Let $m = \ell(Y) \cdot c(X)$. It suffices to prove that $|Y| \le 2^m$, for then also $|\bar{Y}| \le |Y|^{\omega} \le 2^m$.

1. First let us consider the case when there exists a point y^* in Y such that $|Y \setminus Oy^*| \le m$ for every neighborhood Oy^* of y^*. Without any loss of generality we can assume that y^* is the constant zero.

For any $x \in X$ and $\varepsilon > 0$ the set $\{f \in Y : |f(x)| \ge \varepsilon\}$ has cardinality $\le m$; hence $|\{f \in Y : f(x) \ne 0\}| \le m$. Let $X' \subseteq R^Y$ be the image of X under the diagonal product $\Delta Y : X \to R^Y$. Then X' lies in the Σ_m-product of lines and therefore $t(X') \le m$, [2]. Theorem S implies $d(X') \le w(X') \le m^{c(X')} \le m^{c(X)} \le 2^m$. Since Y embeds in $C_p(X')$, we have $|Y| \le |C_p(X')| \le d(X') \le 2^m$.

II. Now consider the general case. By Theorem A it suffices to show that $\psi(Y) \le 2^m$. Suppose $\psi(y, Y) > 2^m$ for some $y \in Y$. Then $\ell(Y \setminus \{y\}) \ge \psi(y, Y) > 2^m$. Theorem B implies there is a closed discrete subset $A \subseteq Y \setminus \{y\}$ of cardinality $> 2^m$. Let $A' = A \cup \{y\}$. Then $\ell(A') \le m$, since A' is closed in Y, and A' has only one non-isolated point. Hence A' satisfies the condition in I. But $|A'| > 2^m$. This contradicts the first part of the proof, and we are done.

If X is dyadic, a better estimate can be obtained:

Theorem 2. If a compact space X is dyadic and $Y \subseteq C_p(X)$, then $nw(Y) = \ell(Y)$.

In particular, any Lindelöf subspace of $C_p(X)$ has a countable network. Note that $nw(\overline{Y})=nw(Y)$ since $C_p(X)$ is monolithic and $\ell(Y)=e(Y)$ by Theorem B, so we also have $nw(\overline{Y})=e(Y)$.

Proof. Let $X'=\Delta Y(X)\subseteq R^Y$. Then Y is homeomorphic to a subspace of $C_p(X')$ which separates the points of X', so $nw(Y)\le$ $\le nw(C_p(X'))=nw(X')=w(X')$. It remains to show that $w(X')\le\ell(Y)$. Since X' is dyadic, $w(X')=\sup\{m:\mathfrak{D}^m$ embeds in $X'\}$, [5], and also $w(X')=\sup\{m: m+1$ embeds in $X'\}$, where m+1 is the linearly ordered space of ordinals $\le m$. The following lemma completes the proof:

Lemma. Suppose m is a cardinal and $Y\subseteq C_p(m+1)$. If Y separates the points of m+1, then $\ell(Y)=m$.

Proof. We may assume m is regular. For every $\alpha<m$ pick a function $f_\alpha\in Y$ and two rationals s_α, t_α such that either $f_\alpha(\alpha)<s_\alpha<t_\alpha<f_\alpha(m)$ or $f_\alpha(\alpha)>s_\alpha>t_\alpha>f_\alpha(m)$. If $\alpha>0$, there is an ordinal $\beta(\alpha)<\alpha$ such that for any $\gamma\in(\beta(\alpha),\alpha]$ either $f_\alpha(\gamma)<s_\alpha<t_\alpha$ or $f_\alpha(\gamma)>s_\alpha>t_\alpha$. The pressing-down lemma [6] implies there is an unbounded subset $E\subseteq m$, an ordinal $\beta<m$ and rationals s, t such that $\beta(\alpha)=\beta$, $s_\alpha=s$ and $t_\alpha=t$ for every $\alpha\in E$. The subset $\{f_\alpha:\alpha\in E\}$ of Y has no complete accumulation point in $C_p(m+1)$. Hence $\ell(Y)\ge m$. The reverse inequality is obvious.

Recall that $\sup\{t(X^n):n\in\omega\}\le\ell(C_p(X))$ for any Tychonoff space X (M. Asanov, see [4]). Our lemma suggests the following question. Suppose X is compact, $Y\subseteq C_p(X)$ and Y separates the points of X. Is it true that $t(X)\le\ell(Y)$? Note that $t(X)\le\ell^*(Y)=$ $=\sup\{\ell(X^n):n\in\omega\}$, since X embeds in $C_p(Y)$ and $t(C_p(Y))=\ell^*(Y)$ [4]. For non-compact spaces our question can easily be answered in the negative.

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