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INTERPOLATION SPACES $\overline{X}_{\varphi}(\overline{E})$

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Abstract: There are given necessary and sufficient conditions under some assumptions on the couples of Banach lattices $\overline{E}$ and $\overline{F}$, that for some couples of Banach lattices $\overline{X}$, the spaces $\overline{X}_{\varphi}(\overline{E})$ and $\overline{X}_{\psi}(\overline{F})$ intermediate with respect to $(\overline{X}_{\varphi_0}(\overline{E}),\overline{X}_{\varphi_1}(\overline{E}))$ and $(\overline{X}_{\psi_0}(\overline{F}),\overline{X}_{\psi_1}(\overline{F}))$, respectively are (positive) interpolation spaces with respect to $(\overline{X}_{\varphi_0}(\overline{E}),\overline{X}_{\varphi_1}(\overline{E}))$ and $(\overline{X}_{\psi_0}(\overline{F}),\overline{X}_{\psi_1}(\overline{F}))$.

Key words: Peetre´s K-functional, Calderón-Lozanovskii spaces, interpolation spaces.

Classification: 46E30, 46E35

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1. Introduction. Let A_0 and A_1 be two Banach spaces. We say that $\overline{A}=(A_0,A_1)$ is a Banach souple if both A_0 and A_1 are continuously embedded in some Háusdorff topological vector space.

A Banach space is called intermediate with respect to $\overline{A}$ if $A_0\cap A_1\subset A\subset A_0+A_1$ with continuous embeddings. Let $\overline{A}$ and $\overline{B}$ be two Banach couples and let T be a linear operator mapping A_0+A_1 into B_0+B_1. We write $T:\overline{A}\to\overline{B}$ if the restriction of T to A_i defines a bounded linear operator from A_i into B_i, i=0,1.

Let A and B be two intermediate spaces with respect to $\overline{A}$ and $\overline{B}$, respectively. We say that A and B are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$ if every linear operator T such that $T:\overline{A}\to\overline{B}$ maps A into B. If $\overline{A}=\overline{B}$ and A=B we say simply that A is an interpolation space with respect to $\overline{A}$.

The closed graph theorem implies that if A and B are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$, then there exists a positive constant C such that

(1) $$\|T\|_{A\to B}\le C\max\{\|T\|_{A_0\to B_0},\|T\|_{A_1\to B_1}\}$$

for any $\bar{T}:\bar{A} \to \bar{B}$ (see [4], p.34).

Let (Ω, Σ, μ) be a complete σ-finite measure space and let us denote by $L^0 = L^0(\Omega, \Sigma, \mu)$ the space of all equivalence classes of μ-measurable, real valued functions finite μ-a.e. on Ω equipped with the topology of convergence in measure. A Banach space $X \subset L^0$ is called a Banach lattice (on (Ω, Σ, μ)) if $|x(t)| \leq |y(t)|$ a.e. and $y \in X$ implies that $x \in X$ and $\|x\|_X \leq \|y\|_X$.

A Banach lattice $X \subset L^0$ has the Fatou property if for every a.e. pointwise increasing sequence $(x_n)_{n=1}^{\infty}$ of non-negative function in X with $\sup_{n \geq 1} \|x_n\|_X < \infty$, the function x, $x = \lim_{n \to \infty} x_n$, is in X with $\|x\|_X = \lim_{n \to \infty} \|x_n\|_X$.

For a Banach lattice X on (Ω, Σ, μ) and a weight function w (a.e. positive measurable function on Ω) by X_w we shall denote the space of all functions x such that $xw \in X$ with the norm $\|x\|_{X_w} := \|xw\|_X$.

<u>Notation</u>: The equivalence $f \sim g$ means that $c_1 f(t) \leq g(t) \leq c_2 f(t)$ for some positive constants c_1 and c_2 and all $t \in \mathbb{R}_+ := (0, \infty)$.

2. <u>The Calderón-Lozanovskii space</u> $\varphi(\bar{X})$. A real function $\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$ belongs to the class $\mathcal{U}$ if it satisfies the following conditions:

(i) $\varphi(\lambda s, \lambda t) = \lambda \varphi(s,t)$ for each $\lambda \geq 0$ and $s,t \in \mathbb{R}_+$,

(ii) $0 < \varphi(s,t) \leq \max\{\frac{s}{u}, \frac{t}{v}\} \varphi(u,v)$ for each $s,t,u,v \in \mathbb{R}_+$.

$\widehat{\mathcal{U}}$ denotes the class of functions $\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$ concave on $\mathbb{R}_+^2$, positive homogeneous. We observe that $\widehat{\mathcal{U}} \subset \mathcal{U}$.

Let $\bar{X}$ be a couple of Banach lattices on (Ω, Σ, μ) and let $\varphi \in \widehat{\mathcal{U}}$. We denote by $\varphi(\bar{X}) = \varphi(X_0, X_1)$ the Calderón-Lozanovskii space of all $x \in L^0$ such that for some $x_i \in X_i$, $\|x_i\|_{X_i} \leq 1$, $i=0,1$ and for some $\lambda \in \mathbb{R}_+$ holds $|x| \leq \lambda \varphi(|x_0|, |x_1|)$ μ-a.e. We put $\|x\|_{\varphi(\bar{X})} = \inf \lambda$.

Note that $\varphi(\bar{X})$ is a Banach lattice intermediate with respect to $\bar{X}$. If in particular we take $\varphi(s,t) = s^{1-\alpha} t^{\alpha}$, $0 < \alpha < 1$, we obtain the space $X_0^{1-\alpha} X_1^{\alpha}$ introduced by Calderón [2]. The space $\varphi(\bar{X})$ was investigated by Lozanovskii in [5].

Proposition 1. Let $\bar{X}$ be a couple of Banach lattices and let $\varphi_0, \varphi_1, \varphi \in \widehat{\mathcal{U}}$, then

$$(2) \qquad \psi(\bar{X}) = \varphi(\varphi_0(\bar{X}), \varphi_1(\bar{X}))$$

with equivalent norms, where $\psi(s,t) = \varphi(\varphi_0(s,t), \varphi_1(s,t))$.

Proof. We observe that $\psi \in \widehat{\mathcal{U}}$. If $x \in \psi(\bar{X})$, then $|x| \le \lambda \psi(|x_0|,|x_1|)$ a.e., for some $\lambda > 0$ and for some $x_i \in X_i$, $\|x_i\|_{X_i} \le 1$, i=0,1. Hence $|x| \le \lambda\, \varphi(y_0,y_1)$ a.e., where $y_i = \varphi_i(|x_0|,|x_1|)$, $\|y_i\|_{\varphi_i(\bar{X})} \le 1$, i=0,1. This implies that $x \in \varphi(\varphi_0(\bar{X}), \varphi_1(\bar{X}))$ and $\|x\|_{\varphi(\varphi_0(\bar{X}),\varphi_1(\bar{X}))} \le \|x\|_{\psi(\bar{X})}$, whence $\psi(\bar{X}) \subset \varphi(\varphi_0(\bar{X}), \varphi_1(\bar{X}))$ with continuous embedding.

On the other hand, let $x \in \varphi(\varphi_0(\bar{X}), \varphi_1(\bar{X}))$, then $|x| \le \lambda\, \varphi(|x_0|,|x_1|)$ a.e., for some $\lambda > 0$ and for some $x_i \in \varphi_i(\bar{X})$, $\|x_i\|_{\varphi_i(\bar{X})} \le 1$, i=0,1.

For an $\varepsilon > 0$ there exist $y_0, y_0' \in X_0$, $y_1, y_1' \in X_1$ such that

$$|x_0| \le (1+\varepsilon)\, \varphi_0(|y_0|,|y_1|), \quad \|y_0\|_{X_0} \le 1, \quad \|y_1\|_{X_1} \le 1,$$

$$|x_1| \le (1+\varepsilon)\, \varphi_1(|y_0'|,|y_1'|), \quad \|y_0'\|_{X_0} \le 1, \quad \|y_1'\|_{X_1} \le 1,$$

so we have

$$|x| \le \lambda\, \varphi(|x_0|,|x_1|) \le (1+\varepsilon)\, \lambda\, \varphi(\varphi_0(|y_0|,|y_1|), \varphi_1(|y_0'|,|y_1'|)) \le 2(1+\varepsilon)\, \lambda\, \varphi(\varphi_0(x_0',x_1'), \varphi_1(x_0',x_1'))$$

where

$$x_i' = \frac{1}{2}\max(|y_i|,|y_i'|) \in X_i, \quad \|x_i'\|_{X_i} \le 1, \ i=0,1.$$

Hence $x \in \psi(\bar{X})$ and $\|x\|_{\psi(\bar{X})} \le 2(1+\varepsilon)\|x\|_{\varphi(\varphi_0(\bar{X}),\varphi_1(\bar{X}))}$. Since is an arbitrary positive number, we obtain $\|x\|_{\psi(\bar{X})} \le 2\|x\|_{\varphi(\varphi_0(\bar{X}),\varphi_1(\bar{X}))}$, this implies $\varphi(\varphi_0(\bar{X}), \varphi_1(\bar{X})) \subset \psi(\bar{X})$ with continuous embedding and the proof is complete.

Let E and F be two Banach lattices, then we say that a linear operator $T: E \longrightarrow F$ is positive, if $0 \le Tx$ a.e. for each $0 \le x \in E$.

Let $\bar{X}$ and $\bar{Y}$ be two couples of Banach lattices and let X and Y be two Banach lattices intermediate with respect to $\bar{X}$ and $\bar{Y}$, respectively. We say that X and Y are positive interpolation spaces with respect to $\overline{X}$ and $\overline{Y}$, if every positive operator $T:\overline{X} \to \overline{Y}$ maps X into Y boundedly with

$$\|T\|_{X \to Y} \le c \max\{\|T\|_{X_0 \to Y_0}, \|T\|_{X_1 \to Y_1}\}$$

for some constant c independent of T. If $\overline{X}=\overline{Y}$ and X=Y we say that X is a positive interpolation space with respect to $\overline{X}$. We can easily show:

<u>Proposition 2</u>. Let $\overline{X}$ and $\overline{Y}$ be two couples of Banach lattices, then the spaces $\varphi(\overline{X})$ and $\varphi(\overline{Y})$ are positive interpolation spaces with respect to $\overline{X}$ and $\overline{Y}$.

By Proposition 1 and 2, we get the following

<u>Corollary 1</u>. Let $\overline{X}$, $\overline{Y}$ be two couples of Banach lattices and let $\varphi_i, \psi_i, \varphi \in \widehat{\mathcal{U}}$, i=0,1. Then the spaces $\varphi(\varphi_0, \varphi_1)(\overline{X})$ and $\varphi(\psi_0, \psi_1)(\overline{Y})$ are positive interpolation spaces with respect to $(\varphi_0(\overline{X}), \varphi_1(\overline{X}))$ and $(\psi_0(\overline{Y}), \psi_1(\overline{Y}))$.

<u>Proposition 3</u> (cf. [6]). Let $\varphi_0, \varphi_1, \varphi \in \mathcal{U}$, $\psi_0, \psi_1, \psi \in \mathcal{U}$ and let c be a positive constant, then the following inequality

$$\frac{\varphi(u,v)}{\psi(s,t)} \le c \max\left\{\frac{\varphi_0(u,v)}{\psi_0(s,t)}, \frac{\varphi_1(u,v)}{\psi_1(s,t)}\right\} \tag{3}$$

for each $s,t,u,v \in \mathbb{R}_+$

holds if and only if $\varphi(u,v) \le c_1 \Theta(\varphi_0(u,v), \varphi_1(u,v))$ and $\psi(u,v) \ge c_2 \Theta(\psi_0(u,v), \psi_1(u,v))$ for some function $\Theta \in \widehat{\mathcal{U}}$ and some constants $c_1, c_2 > 0$.

3. <u>The interpolation space $\overline{A}_E$</u>. Let $\overline{A}$ be a Banach couple and let $E \subset L^0(\mathbb{R}_+, dt/t)$ be a Banach lattice such that $\min(1,t) \in E$, then the space

$$\overline{A}_E := \{a \in A_0 + A_1 : K(\cdot, a; \overline{A}) \in E\}$$

is a Banach space with the norm

$$\|a\|_{\overline{A}_E} = \|K(\cdot, a, \overline{A})\|_E,$$

where $K(t,a;\overline{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\}$ $t \in \mathbb{R}_+$, is the K-functional of Peetre. For each $a \in A_0 + A_1$

$K(t,a;\bar{A})$ is a concave function on $\mathbb{R}_+$, so for each $s,t \in \mathbb{R}_+$

(4) $\quad \min(1, \frac{s}{t})\, K(t,a;\bar{A}) \le K(s,a;\bar{A})$.

If $a \in \bar{A}_E$ then by inequality (4) we get

(5) $\quad K(t,a;\bar{A}) \le \varphi_E(t)\, \|a\|_{\bar{A}_E}$,

where $\varphi_E(t) = \|\min(1, \frac{s}{t})\|_E^{-1}$. We observe that the function φ_E is quasi-concave ($0 < \varphi_E(t) \le \max(1, \frac{t}{s})\, \varphi_E(s)$ for each $s,t \in \mathbb{R}_+$).

We say that a Banach couple $\bar{A}$ is of type $(\mathcal{A})$ (cf. [1]) if for each $t \in \mathbb{R}_+$ there exists an element a_t, such that

(6) $\quad c_1 \min(1, \frac{s}{t}) \le K(s,a_t;\bar{A}) \le c_2 \min(1, \frac{s}{t})$

for some positive constants c_1, c_2 and all $s \in \mathbb{R}_+$.

Example. Let X_0 and X_1 be two symmetric spaces defined on $(0,\infty)$ (see [4]) with the fundamental functions $\Phi_{X_i}(t) := \|\chi_{(0,t)}\|_{X_i}$, $i=0,1$, where $\chi_{(0,t)}$ is the characteristic function of the interval $(0,t)$. If the function $\Phi_{01}(t) = \Phi_{X_0}(t)/\Phi_{X_1}(t)$ is such that $\Phi_{01}(\mathbb{R}_+) = \mathbb{R}_+$, then a couple (X_0,X_1) is of type $(\mathcal{A})$.

Really we have $K(s, \chi_{(0,t)};\bar{X}) = \min(\Phi_{X_0}(t), s\,\Phi_{X_1}(t))$. Since for each $t \in \mathbb{R}_+$ there exists t_* such that $\Phi_{01}(t_*) = t$, so for $x_t = \frac{1}{\Phi_{X_0}(t)}\, \chi_{(0,t)}$ we obtain $K(s,x_t;\bar{X}) = \min(1, \frac{s}{t})$.

Theorem 1. Let $\bar{A}$ be a Banach couple of type $(\mathcal{A})$. If the spaces $\bar{A}_E$, $\bar{A}_F$ intermediate with respect to $(\bar{A}_{E_0}, \bar{A}_{E_1})$ and $(\bar{A}_{F_0}, \bar{A}_{F_1})$, respectively are interpolation spaces with respect to $(\bar{A}_{E_0}, \bar{A}_{E_1})$ and $(\bar{A}_{F_0}, \bar{A}_{F_1})$, then there exists a constant $c > 0$ such that

(7) $$\frac{\varphi_E(s)}{\varphi_F(t)} \le c \max\left\{ \frac{\varphi_{E_0}(s)}{\varphi_{F_0}(t)}, \frac{\varphi_{E_1}(s)}{\varphi_{F_1}(t)} \right\}$$

for each $s,t \in \mathbb{R}_+$.

Proof. Let $\bar{A}$ be a couple of type $(\mathcal{A})$. Put $A_s = \{\lambda a_s : \lambda \in \mathbb{R}\}$, $f_s(\lambda a_s) = \lambda$, $s \in \mathbb{R}_+$. Then $K(s,a_s;\bar{A}) \ge c_1$ and

$|f_s(a)| \leq \frac{1}{c_1} K(s,a;\bar{A})$ for $a \in A_s$. Hence f_s is a continuous linear functional on a linear subspace A_s of a Banach space A_0+A_1 with the norm $K(s,a;\bar{A})$. By the Hahn-Banach theorem the functional f_s can be extended to the functional $\bar{f}_s$ defined on the whole space A_0+A_1 such that

$$(8) \qquad |\bar{f}_s(a)| \leq \frac{1}{c_1} K(s,a;\bar{A}) \text{ for each } a \in A_0+A_1 .$$

For each $s,t \in \mathbb{R}_+$ we define operators $T_{s,t}: A_0+A_1 \to A_0+A_1$, $T_{s,t}a = \bar{f}_s(a)a_t$. Let $a \in \bar{A}_{E_i}$, $i=0,1$, then from (5),(6) and (8) we have

$$\|T_{s,t}a\|_{\bar{A}_{F_i}} = \|K(\xi,\bar{f}_s(a)a_t;\bar{A})\|_{F_i} = |\bar{f}_s(a)|\, \|K(\xi,a_t;\bar{A})\|_{F_i} \leq$$

$$\leq c_2|\bar{f}_s(a)|\, \|\min(1,\xi/t)\|_{F_i} = c_2 \frac{|\bar{f}_s(a)|}{\varphi_{F_i}(t)} \leq \frac{c_2}{c_1} \frac{K(s,a;\bar{A})}{\varphi_{F_i}(t)} \leq$$

$$\leq \frac{c_2}{c_1} \frac{\varphi_{E_i}(s)}{\varphi_{F_i}(t)} \|a\|_{\bar{A}_{E_i}}$$

Hence, we get

$$(9) \qquad \|T_{s,t}\|_{\bar{A}_{E_i} \to \bar{A}_{F_i}} \leq \frac{c_2}{c_1} \frac{\varphi_{E_i}(s)}{\varphi_{F_i}(t)}, \qquad i=0,1.$$

Let us see that $\varphi_E(s)a_s \in \bar{A}_{E_i}$, $i=0,1$, and

$$\|T_{s,t}(\varphi_E(s)a_s)\|_{\bar{A}_F} \geq c_1 \frac{\varphi_E(s)}{\varphi_F(t)}, \text{ whence}$$

$$(10) \qquad \|T_{s,t}\|_{\bar{A}_e \to \bar{A}_F} \geq c_1 \frac{\varphi_E(s)}{\varphi_F(t)} .$$

By inequalities (9),(10) and (1) we obtain (7). From Proposition 3 and Theorem 1, we obtain Corollary.

Corollary 2. If for a Banach couple $\bar{A}$ of type $(\mathcal{A})$ the Banach space $\bar{A}_E$ intermediate with respect to $(\bar{A}_{E_0},\bar{A}_{E_1})$ is an interpolation space with respect to $(\bar{A}_{E_0},\bar{A}_{E_1})$, then there exists a concave function Θ on $\mathbb{R}_+$ such that
$\varphi_E(t) \sim \varphi_{E_0}(t)\, \Theta(\varphi_{E_1}(t)/\varphi_{E_0}(t))$.

The following theorem can be proved in a similar way as the theorem 1.

Theorem 2. Let (X_0,X_1) be a couple of Banach lattices of type $(\mathcal{A})$. If the spaces $\bar{X}_E$, $\bar{X}_F$ intermediate with respect to $(\bar{X}_{E_0},\bar{X}_{E_1})$ and $(\bar{X}_{F_0},\bar{X}_{F_1})$, respectively are positive interpolation spaces with respect to $(\bar{X}_{E_0},\bar{X}_{E_1})$ and $(\bar{X}_{F_0},\bar{X}_{F_1})$, then there exists a constant $c>0$ such that

$$\frac{\varphi_E(s)}{\varphi_F(t)} \leq c \max\left\{\frac{\varphi_{E_0}(s)}{\varphi_{F_0}(t)}, \frac{\varphi_{E_1}(s)}{\varphi_{F_1}(t)}\right\} \tag{11}$$

for each $s,t \in \mathbb{R}_+$.

We say that a Banach lattice $E \subset L^0(R_+, \frac{dt}{t})$ is the parameter of the K-method if $L^\infty \cap L^\infty_{1/s} \subset E \subset L^1+L^1_{1/s}$ and the Calderón operator $Sx(t)= \int_0^\infty \min(1,\frac{t}{s})x(s)\,\frac{ds}{s}$ is bounded in E (see [3]).

In the sequel, let E_i, F_i, $i=0,1$ be parameters of the K-method such that $E_i=(L^\infty, L^\infty_{1/s})_{E_i}$, $F_i=(L^\infty, L^\infty_{1/s})_{F_i}$, $i=0,1$ and $(\varphi_{E_0}/\varphi_{E_1})(\mathbb{R}_+)=\mathbb{R}_+$, $(\varphi_{F_0}/\varphi_{F_1})(\mathbb{R}_+)=\mathbb{R}_+$.

Theorem 3. Let $\varphi_i, \psi_i, \varphi, \psi \in \widehat{\mathcal{U}}$ and let (X_0,X_1) be a couple of Banach lattices of type $(\mathcal{A})$. The spaces $\bar{X}_{\varphi(\bar{E})}$, $\bar{X}_{\psi(\bar{F})}$ intermediate with respect to $(\bar{X}_{\varphi_0(\bar{E})}, \bar{X}_{\varphi_1(\bar{E})})$ and $(\bar{X}_{\psi_0(\bar{F})},\bar{X}_{\psi_1(\bar{F})})$, respectively are positive interpolation spaces with respect to $(\bar{X}_{\varphi_0(\bar{E})},\bar{X}_{\varphi_1(\bar{E})})$ and $(\bar{X}_{\psi_0(\bar{F})},\bar{X}_{\psi_1(\bar{F})})$ if and only if there exists a constant $c>0$ such that the inequality (3) holds.

Proof. We easily obtain that $\varphi_{\varphi(E)}(t) \sim \varphi(\varphi_{E_0}(t),\varphi_{E_1}(t))$, so the necessity follows from Theorem 2. Now, let the inequality (3) hold, then by Proposition 3, there exists the function $\Theta \in \widehat{\mathcal{U}}$ and constants $c_1,c_2>0$ such that $\varphi(u,v) \leq c_1\,\Theta(\varphi_0(u,v), \varphi_1(u,v))$ and $\psi(u,v) \geq c_2\,\Theta(\psi_0(u,v), \psi_1(u,v))$ for all $u,v \in \mathbb{R}_+$. From Proposition 1 we have

(12) $\varphi(\bar{E}) \subset \Theta(\varphi_0, \varphi_1)(\bar{E}) = \Theta(\varphi_0(\bar{E}), \varphi_1(\bar{E}))$,

$\psi(\bar{F}) \supset \Theta(\psi_0, \psi_1)(\bar{F}) = \Theta(\psi_0(\bar{F}), \psi_1(\bar{F}))$

with continuous inclusions. Since the operator S is positive, by Proposition 2 the spaces $\Theta(\varphi_0, \varphi_1)(\bar{E})$ and $\Theta(\psi_0, \psi_1)(\bar{F})$ are the parameters of the K-method. By Corollary 2 in [8] and (12) we get $\bar{X}_{\varphi(\bar{E})} \subset \Theta(\bar{X}_{\varphi_0(\bar{E})}, \bar{X}_{\varphi_1(\bar{E})}) = \bar{X}_{\Theta(\varphi_0(\bar{E}), \varphi_1(\bar{E}))} = \bar{X}_{\Theta(\varphi_0, \varphi_1)(\bar{E})}$,

$\bar{X}_{\Theta(\psi_0, \psi_1)(\bar{F})} = \bar{X}_{\Theta(\psi_0(\bar{F}), \psi_1(\bar{F}))} = \Theta(\bar{X}_{\psi_0(\bar{F})}, \bar{X}_{\psi_1(\bar{F})}) \subset \bar{X}_{\psi(\bar{F})}$ with continuous inclusions. Now, if the operator

$$T : (\bar{X}_{\varphi_0(\bar{E})}, \bar{X}_{\varphi_1(\bar{E})}) \longrightarrow (\bar{X}_{\psi_0(\bar{F})}, \bar{X}_{\psi_1(\bar{F})})$$

is positive and $x \in \bar{X}_{\varphi(\bar{E})}$, then

$$\|Tx\|_{\psi(\bar{F})} \le c_1 \|Tx\|_{\Theta(\bar{X}_{\psi_0(\bar{F})}, \bar{X}_{\psi_1(\bar{F})})} \le c_2 \|x\|_{\Theta(\bar{X}_{\varphi_0(\bar{E})}, \bar{X}_{\varphi_1(\bar{E})})} \le$$

$$\le c_3 \max\{\|T\|_{\varphi_0(\bar{E}) \to \psi_0(\bar{F})}, \|T\|_{\varphi_1(\bar{E}) \to \psi_1(\bar{F})}\} \|x\|_{\bar{X}_{\varphi(\bar{E})}},$$

by Proposition 2, where c_1, c_2 and c_3 are some positive constants. The proof is complete.

From Proposition 3 and Theorem 3 we obtain

Corollary 3. Let $\varphi_0, \varphi_1, \varphi \in \widehat{\mathcal{U}}$ and let $\bar{X}$ be a couple of Banach lattices of type $(\mathcal{A})$. The spaces $\bar{X}_{\varphi(\bar{E})}, \bar{X}_{\varphi(\bar{F})}$ are positive interpolation spaces with respect to $(\bar{X}_{\varphi_0(\bar{E})}, \bar{X}_{\varphi_1(\bar{E})})$ and $(\bar{X}_{\varphi_0(\bar{F})}, \bar{X}_{\varphi_1(\bar{F})})$ if and only if $\varphi(u,v) \sim \Theta(\varphi_0(u,v), \varphi_1(u,v))$ with some function $\Theta \in \widehat{\mathcal{U}}$.

If the spaces X_i, F_i, i=0,1 have the Fatou property, then by a result of Ovčinnikov [7] we obtain an analogous interpolation theorem if we take "interpolation" instead of "positive interpolation" in Theorem 3.

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