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REGULARITY FOR NONLINEAR ELLIPTIC SYSTEMS OF SECOND ORDER

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Abstract: There is shown the existence of a nonlinear elliptic system of second order whose weak solution has the gradient locally in $\mathcal{L}^{2,n}$ space.

Key words: Elliptic systems, regularity, Morrey-Campanato spaces.

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1. Notations and definitions. Let Ω be a bounded open set of $\mathbb{R}^n$, $n>2$ the points of which we denote by $x=(x_1,\dots,x_n)$, N be a positive integer, $(\,|\,)$ and $\|\ \|$ denote the scalar product and the norm in $\mathbb{R}^N$. Denote further $p=(p^1,\dots,p^n)$, $p^i\in\mathbb{R}^N$, $D_iu=\partial u/\partial x_i$, $Du=(D_1u,\dots,D_nu)$. $B(\sigma)=B(x,\sigma)$ is the open ball in $\mathbb{R}^n$ with the center x and radius σ and $\Omega(x,\sigma)=B(x,\sigma)\cap\Omega$. We set $d(\Omega)=\operatorname{diam}\Omega$, $d_x=\operatorname{dist}(x,\partial\Omega)$, where $\partial\Omega$ is the boundary of Ω. Let $H^{1,2}(\Omega,\mathbb{R}^N)$, $H_0^{1,2}(\Omega,\mathbb{R}^N)$ be usual Sobolev spaces of vector-valued functions $u:\Omega\longrightarrow\mathbb{R}^N$ with the norm

$$(1.1)\qquad \|u\|_{1,2,\Omega}=\Big[\int_\Omega \|u\|^2dx+\int_\Omega \sum_{i=1}^{n}\|D_iu\|^2\,dx\Big]^{1/2}$$

Besides the well-known spaces $C^k(\Omega,\mathbb{R}^N)$, $C^k(\bar\Omega,\mathbb{R}^N)$, $C^{k,\alpha}(\bar\Omega,\mathbb{R}^N)$ and $C_0^\infty(\Omega,\mathbb{R}^N)$ we make use of Morrey-Campanato spaces $L^{2,\lambda}(\Omega,\mathbb{R}^N)$ and $\mathcal{L}^{2,n}(\Omega,\mathbb{R}^N)$. The definition of Morrey-Campanato spaces is as follows:

Definition. Let $\lambda\in[0,n]$ and $\Omega\subset\mathbb{R}^n$ be a bounded domain. The space $L^{2,\lambda}(\Omega,\mathbb{R}^N)$ is the subspace of such functions from $L^2(\Omega,\mathbb{R}^N)$ for which

$$\|f\|_{L^{2,\lambda}}=\Big\{\sup_{\sigma>0,\,x\in\bar\Omega}\frac{1}{\sigma^\lambda}\int_{\Omega(x,\sigma)}\|f(y)\|^2\,dy\Big\}^{1/2}<\infty .$$

The space $\mathcal{L}^{2,n}(\Omega,\mathbb{R}^N)$ is the subspace of such functions from

$L^2(\Omega, \mathbb{R}^N)$ for which

$$[f]_{\mathcal{L}^{2,n}} = \left\{ \sup_{\sigma>0, x\in\bar\Omega} \frac{1}{\sigma^n} \int_{\Omega(x,\sigma)} \| f(y) - (f)_{\Omega(x,\sigma)} \|^2 dy \right\}^{1/2} < \infty,$$

where $(f)_{\Omega(x,\sigma)} = |\Omega(x,\sigma)|^{-1} \int_{\Omega(x,\sigma)} f(y)dy$, $|\Omega(x,\sigma)|$ is the n-dimensional Lebesgue measure of the set $\Omega(x,\sigma)$. Define the norm in the space $\mathcal{L}^{2,n}(\Omega, \mathbb{R}^N)$ by

$$\|f\|_{\mathcal{L}^{2,n}} = \|f\|_{L^2} + [f]_{\mathcal{L}^{2,n}} .$$

Remember that $L^{2,n}(\Omega, \mathbb{R}^N) = L^\infty(\Omega, \mathbb{R}^N) \subsetneqq \mathcal{L}^{2,n}(\Omega, \mathbb{R}^N)$ and $\mathcal{L}^{2,n} \subset L^{2,\lambda_1} \subset L^{2,\lambda_2}$ for each $0 \le \lambda_2 < \lambda_1 < n$. For the more detailed information about them see e.g. [1],[3].

We consider the elliptic system of the second order in the form

$$(1.2) \qquad - \sum_{i=1}^{n} D_i a^i(x,u,Du) = a^0(x,u,Du),$$

where $a^i(x,u,p)$, $a^0(x,u,p)$ are Carathéodorian mappings from $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ into $\mathbb{R}^N$. A function $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ is called a weak solution of (1.2) in Ω if

$$(1.3) \qquad \int_\Omega \sum_{i=1}^{n} (a^i(x,u,Du) | D_i \varphi) = (a^0(x,u,Du) | \varphi), \quad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^N).$$

As it is known, in case of a general system (1.2) only partial regularity can be expected for $n > 2$ (see e.g. [1]). In this paper we study $L^{2,\lambda}(\Omega, \mathbb{R}^N)$-regularity ($\lambda \in (0,n)$) and $\mathcal{L}^{2,n}(\Omega, \mathbb{R}^N)$-regularity of the weak solutions for the system whose coefficients $a^i(x,u,Du)$ have the form

$$(1.4) \qquad a^i(x,u,Du) = \sum_{j=1}^{n} A_{ij}(x) D_j u + g^i(x,u,Du)$$

and also generalize some results (for the systems of second order) having been achieved in Campanato's work [1] pp. 104-115, where the function of g^i independent of Du only is considered.
Here $A_{ij}(x) = \{A_{ij}^{hk}(x)\}_{h,k=1}^{N}$ are matrices of continuous (Hölder-continuous) functions, the following condition of strong ellipticity

$$(1.5) \qquad \sum_{i,j=1}^{n} (A_{ij}(x) \xi^j | \xi^i) \ge \nu \sum_{i=1}^{n} \| \xi^i \|^2, \quad \nu > 0 \text{ const.}, \quad \forall \xi^i \in \mathbb{R}^N$$

holds and $g^i(x,u,p)$ are smooth functions with sublinear growth in p. In what follows, we formulate the conditions on the smoothness and the growth of the functions $A_{ij}(x)$, $g^i(x,u,p)$ and $a^0(x,u,p)$ precisely.

2. Main results. Suppose that for all $(x,u,p) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ the following conditions hold:

$$\|a^0(x,u,p)\| \le f_0(x) + L\{\|u\|^{\delta_0} + \sum_{j=1}^{n} \|p^j\|^{\beta_0}\}, \tag{2.1}$$

$$\|g^i(x,u,p)\| \le f_i(x) + L\{\|u\|^{\delta_1} + \sum_{j=1}^{n} \|p^j\|^{\beta_1}\}, \tag{2.2}$$

where L is a positive constant and

$$1 \le \delta_0 < \frac{n+2}{n-2}, \quad 1 \le \beta_0 < \frac{n+2}{n}, \tag{2.3}$$

$$1 \le \delta_1 < \frac{n}{n-2}, \quad 0 < \beta_1 < 1. \tag{2.4}$$

We set

$$q_0 = \frac{n}{n+2} \tag{2.5}$$

Theorem 2.1. Let $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.2). Suppose that the conditions (1.4),(1.5),(2.1), (2.2),(2.3) and (2.4) hold. Let further $A_{ij}^{hk}(x) \in C^0(\bar{\Omega})$ for each $i,j=1,\dots,n$, $h,k=1,\dots,N$ and

$$f_0(x) \in L^{2q_0,\lambda q_0}(\Omega), \quad f_i(x) \in L^{2,\lambda}(\Omega), \quad 0 < \lambda < n, \tag{2.6}$$

where f_0 and f_i are the functions from the estimates (2.1),(2.2) and q_0 is defined by (2.5). Then for each ball $B(\sigma) \subset \Omega$

$$\int_{B(\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dx \le c\, \sigma^{\lambda} \tag{2.7}$$

and therefore $D_i u \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^N)$ for each $i=1,\dots,n$. Here $c=c(\nu, d(\Omega), L, N_1, N_2)$.

In order to obtain $\mathcal{L}^{2,n}$-regularity for the first derivatives of the weak solution we strengthen the conditions on the coefficients g^i. Namely suppose that

(2.8) $\|g^i(x,u,p)-g^i(y,v,q)\| \le L\{|f_i(x)-f_i(y)|+(\|u\|+\|v\|)^{\delta_1}+$
$$+\sum_{j=1}^{n}\|p^j-q^j\|^{\beta_1}\},$$
$$g^i(x,0,0)\in \mathcal{L}^{2,n}(\Omega,\mathbb{R}^N).$$

We can now formulate the main result of this paper.

Theorem 2.2. Let u $H^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to the system (1.2) and suppose that the conditions (1.4),(1.5),(2.1), (2.3),(2.4) and (2.8) hold. Let further $A_{ij}^{hk}(x)\in C^{0,\alpha}(\bar{\Omega})$, ($\alpha\in(0,1]$) for each i,j=1,...,n, h,k=1,...,N, and

(2.9) $$f_0(x)\in L^{2q_0,nq_0}(\Omega),\ f_i(x)\in \mathcal{L}^{2,n}(\Omega),\quad i=1,\dots,n,$$

where f_0 and f_i are the functions from the estimates (2.1),(2.8) and q_0 is defined by (2.5) Then

(2.10) $$D_iu\in \mathcal{L}^{2,n}_{loc}(\Omega,\mathbb{R}^N),\quad i=1,\dots,n.$$

3. Auxiliary lemmas. In this section we present the results needed for the proofs of Theorems 2.1, 2.2. In Lemma 3.1 we consider a linear elliptic system

(3.1) $$\sum_{i,j=1}^{n} D_i(A_{ij}(x)D_ju)=0,$$

with constant coefficients for which (1.5) holds.

Lemma 3.1. Let A_{ij} in (3.1) be constant matrices. Let $u\in H^{1,2}(B(\sigma),\mathbb{R}^N)$ be a weak solution to the system (3.1), Then for each $t\in(0,1]$

(3.2) $$\int_{B(t\sigma)}\sum_{i=1}^{n}\|D_iu\|^2dx\le ct^n\int_{B(\sigma)}\sum_{i=1}^{n}\|D_iu\|^2dx,$$

(3.3) $$\int_{B(t\sigma)}\sum_{i=1}^{n}\|D_iu-[D_iu]_{B(t\sigma)}\|^2dx\le ct^{n+2}\int_{B(\sigma)}\sum_{i=1}^{n}\|D_iu-$$
$$-[D_iu]_{B(\sigma)}\|^2dx.$$

Proof: cf.[1] pp. 54-55.

Lemma 3.2. Let $\Phi(\sigma)$ be a nonnegative function on (0,d] and let A,B,C,α,β be nonnegative constants. Suppose that for each $t \in (0,1)$ and all $\sigma \in (0,d]$ hold:

(3.4) $$\Phi(t\sigma) \le (At^{\alpha} + B)\,\Phi(\sigma) + C\sigma^{\beta},$$

(3.5) $$\Phi(d) < \infty .$$

Further let the constant $K \in (0,1)$ exist such that

(3.6) $$\varepsilon = AK^{\alpha-\beta} + BK^{-\beta} < 1 .$$

Then

(3.7) $$\Phi(\sigma) \le c\,\sigma^{\beta}, \quad \forall \sigma \in (0,d],$$

where $c = \max\left\{\frac{C}{K(1-\varepsilon)}, \sup_{\sigma \in [Kd,d]} \frac{\Phi(\sigma)}{\sigma^{\beta}}\right\}$.

Proof: cf. [2] pp. 537-538.

Lemma 3.3. Let Ω be a bounded convex domain in $\mathbb{R}^n$, $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ such that $D_i u \in L^{2,\tau}(\Omega, \mathbb{R}^N)$ with some $\tau \in (0,n)$. If $\tau < n-2$ then $u \in L^{2^*,\tau 2^*/2}(\Omega, \mathbb{R}^N)$ and for all $x_0 \in \Omega$, $\sigma \le d(\Omega)$ the estimate

(3.8) $$\int_{\Omega(x_0,\sigma)} \|u\|^{2^*} dx \le c_1 M^{2^*} \sigma^{\tau 2^*/2},$$

holds with $2^* = 2n/(n-2)$. If $\tau \ge n-2$ then $u \in L^{\infty}(\Omega, \mathbb{R}^N)$ and

(3.9) $$\|u\|_{\infty,\Omega} \le c_2 M.$$

Here

(3.10) $$M = \|u\|_{1,\Omega} + \sum_{i=1}^{n} \|D_i u\|_{L^{2,\tau}(\Omega,\mathbb{R}^N)}$$

and c_1, c_2 depend only on $d(\Omega)$.

Proof: cf. [1] pp. 23-24.

We set

(3.11) $$v_0 = \min\left\{n\left(1 - \frac{n-2}{n+2}\delta_0\right), n\left(1 - \frac{n}{n+2}\beta_0\right)\right\},$$

(3.12) $$v_1 = n\left(1 - \frac{n-2}{n}\delta_1\right).$$

Lemma 3.4. Let the assumptions of (2.1),(2.2),(2.6) be satisfied and let $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ with $D_i u \in L^{2,\tau}(\Omega, \mathbb{R}^N)$,

($\tau \in [0, n)$) for each $i=1,\dots,n$.

(i) Then $a^0(x,u,Du) \in L^{2q_0,\lambda_0}(\Omega, \mathbb{R}^N)$ and for each ball $B(\sigma)$ we have

$$\int_{B(\sigma)} \|a^0\|^{2q_0}\, dx \le c_1 N_1 \sigma^{\lambda_0}, \tag{3.13}$$

Here $N_1 = (\|f_0\|_{L^{2q_0,\lambda q_0}} + M^{\sigma_0} + M^{\beta_0})^{2q_0}$, M is defined by (3.10), $\lambda_0 = \min\{\lambda q_0,\ v_0 + \tau q_0\}$ and $c_1 = c_1(d(\Omega), L)$.

(ii) For each $\varepsilon \in (0,1)$ and all $B(\sigma) \subset \Omega$

$$\int_{B(\sigma)} \|g^i\|^2 dx \le 4L^2 \varepsilon \int_{B(\sigma)} \sum_{j=1}^{m} \|D_j u\|^2 dx + cN_2 \sigma^{\lambda_1}. \tag{3.14}$$

Here $N_2 = (\|f_i\|_{L^{2,\lambda}(\Omega)} + M^{\sigma_1})^2$, $\lambda_1 = \min\{\lambda, v_1 + \tau\sigma_1\}$ and $c_2 = c_2(\varepsilon, d(\Omega), L)$.

<u>Proof</u>: (i) cf. [1] pp. 106-107.

(ii) Let $B(\sigma) \subset \Omega$. According to (2.2) it follows that

$$\int_{B(\sigma)} \|g^i(x,u,Du)\|^2 dx \le 2 \int_{B(\sigma)} |f_i|^2 dx + 4L^2 \{ \int_{B(\sigma)} \|u\|^{2\sigma_1} dx + \int_{B(\sigma)} \sum_{j=1}^{m} \|D_j u\|^{2\beta_1} dx \}. \tag{3.15}$$

From (2.6) we have

$$\int_{B(\sigma)} |f_i|^2 dx \le c\, \sigma^{\lambda} \|f_i\|_{L^{2,\lambda}} \tag{3.16}$$

and from Hölder inequality and by means of (2.4) we get

$$\int_{B(\sigma)} \|u\|^{2\sigma_1} dx \le c\, \sigma^{n(1-\frac{n-2}{n}\sigma_1)} \left[\int_{B(\sigma)} \|u\|^{2n/(n-2)} dx \right]^{\frac{n-2}{n}\sigma_1}.$$

Now by Lemma 3.3 (in case $\tau=0$ by Sobolev imbedding theorem) we have

$$\int_{B(\sigma)} \|u\|^{2\sigma_1} dx \le cM^{2\sigma_1} \sigma^{v_1 + \tau\sigma_1} \tag{3.17}$$

where M is defined by (3.10). By Young inequality

$$\int_{B(\sigma)} \|D_j u\|^{2\beta_1} dx \le \varepsilon \int_{B(\sigma)} \|D_j u\|^2 dx + \varepsilon^{1/(\beta_1 - 1)} \sigma^n \tag{3.18}$$

for each $\varepsilon \in (0,1)$. From (3.16), (3.17) and (3.18) we obtain (3.14).

4. Proofs of the theorems. If $A_{ij}^{hk}(x) \in C^0(\bar{\Omega})$ for each $i,j=1,\dots,n$, $h,k=1,\dots,N$ we set

$$(4.1) \qquad \omega(\sigma) = \sup_{\substack{x,y\in\Omega \\ \|x-y\|\le\sigma}} \left\{ \sum_{i,j=1}^{n} \|A_{ij}(x)-A_{ij}(y)\|^2 \right\}^{1/2} ,$$

where $\|A_{ij}(x)\| = \left\{ \sum_{h,k=1}^{N} \| A_{ij}^{hk}(x)\|^2 \right\}^{1/2}$.

Proof of Theorem 2.1. Let $B(\sigma)=B(x_0,\sigma) \subset \Omega$ be an arbitrary ball with the center x_0 and let $w \in H_0^{1,2}(B(x_0,\sigma),\mathbb{R}^N)$ be a solution of the following system:

$$(4.2) \qquad \int_{B(\sigma)} \sum_{i,j=1}^{n} (A_{ij}(x_0)\, D_j w | D_i \varphi)dx =$$

$$= \int_{B(\sigma)} \sum_{i,j=1}^{n} ([A_{ij}(x_0)-A_{ij}(x)]D_j u - g_i(x,u,Du) | D_i \varphi)dx +$$

$$+ \int_{B(\sigma)} (a^0(x,u,Du)|\varphi)dx, \quad \forall\varphi \in H_0^{1,2}(B(\sigma),\mathbb{R}^N).$$

It is known that under the assumption of Theorem 2.1 such a solution exists and it is unique. From Lemma 3.4 (with $\tau=0$) follows that for each $\varepsilon \in (0,1)$ we have

$$(4.3) \qquad \int_{B(\sigma)} \sum_{j=1}^{n} \|D_j w\|^2 dx \le c_1 [\omega^2(\sigma)+\varepsilon] \int_{B(\sigma)} \sum_{j=1}^{n} \|D_j u\|^2 dx + c_2(\varepsilon)\sigma^{\lambda'},$$

where $\lambda' = \min\{\lambda_1, \lambda_0/q_0\}$, $c_1=c_1(d(\Omega),L)$ and c_2 depend on the constants from Lemma 3.4.

The function $v=u-w \in H^{1,2}(B(\sigma),\mathbb{R}^N)$ is the solution of the system

$$(4.4) \qquad \int_{B(\sigma)} \sum_{i,j=1}^{n} (A_{ij}(x_0)D_j v | D_i \varphi)dx = 0, \quad \forall\varphi \in H_0^{1,2}(B(\sigma),\mathbb{R}^N).$$

From Lemma 3.1 we have for $t \in (0,1]$

$$(4.5) \qquad \int_{B(t\sigma)} \sum_{i=1}^{n} \|D_i v\|^2 dx \le c_3 t^n \int_{B(\sigma)} \sum_{i=1}^{n} \|D_i v\|^2 dx.$$

By means of (4.3) and (4.5) we obtain for $t \in (0,1]$, $\sigma \le d_{x_0}$ and $\varepsilon > 0$

$$(4.6)\quad \int_{B(t\sigma)} \sum_{i=1}^{m} \|D_i u\|^2 dx = \int_{B(t\sigma)} \sum_{i=1}^{m} \|D_i v + D_i w\|^2 dx \le$$

$$\le \{c_4 t^n + c_5(\omega^2(\sigma) + \varepsilon)\} \int_{B(\sigma)} \sum_{i=1}^{m} \|D_i u\|^2 dx + c_6 \sigma^{\lambda'}.$$

Set $\Phi(\sigma) = \int_{B(\sigma)} \sum \|D_i u\|^2 dx$, $A = c_4$, $B = c_5(\omega^2(\sigma) + \varepsilon)$ and $C = c_6$.

Further we can choose $K_0 \in (0,1)$ such that $AK_0^{n-\lambda'} < 1/2$ $(n - \lambda' > 0)$. It is obvious (the coefficients $A_{ij}^{hk}(x)$ are uniformly continuous) that the constants $\sigma_0 > 0$, $\varepsilon_0 > 0$ exist such that $BK_0^{-\lambda'} < 1/2$ $(B = c_5(\omega^2(\sigma_0) + \varepsilon_0))$ and then $AK_0^{n-\lambda'} + BK_0^{-\lambda'} < 1$. For all $t \in (0,1)$, $\sigma \le \min\{\sigma_0, d_{x_0}\}$ the assumptions of Lemma 3.2 are satisfied and therefore

$$(4.7)\qquad \int_{B(\sigma)} \sum_{i=1}^{m} \|D_i u\|^2 dx \le c_7 \sigma^{\lambda'}, \qquad \forall \sigma \le \min\{\sigma_0, d_{x_0}\}.$$

Now let Ω_0 be an arbitrary domain such that $\Omega_0 \subset\subset \Omega$ (by the symbol $G \subset\subset \Omega$ we mean $\overline{G} \subset \Omega$ and $\overline{G}$ is a compact subset of $\mathbb{R}^N$), and let $d_0 = \mathrm{dist}(\overline{\Omega}_0, \partial\Omega)$. Since (4.7) holds for every $x_0 \in \Omega_0$ and $\sigma < \min\{\sigma_0, d_0\}$ we get

$$(4.8)\qquad \int_{\Omega_0(x_0,\sigma)} \sum_{i=1}^{m} \|D_i u\|^2 dx \le c_7 \sigma^{\lambda'}.$$

If $\min\{\sigma_0, d_0\} < d(\Omega_0)$ it is easy to check that for $\sigma : \min\{\sigma_0, d_0\} \le \sigma \le d(\Omega_0)$ we have

$$(4.9)\qquad \int_{\Omega_0(x_0,\sigma)} \sum_{i=1}^{m} \|D_i u\|^2 dx \le c_8 \sigma^{\lambda'} [\min\{\sigma_0, d_0\}]^{-\lambda'} \int_{\Omega} \sum_{i=1}^{m} \|D_i u\|^2 dx.$$

Thus we have

$$(4.10)\qquad \sum_{i=1}^{m} \|D_i u\|_{L^{2,\lambda'}(\Omega_0, \mathbb{R}^N)} \le c_9 \sum_{i=1}^{m} \|D_i u\|_{L^2(\Omega, \mathbb{R}^N)}.$$

If $\lambda' = \lambda$, the theorem is proved. If $\lambda' < \lambda$, the previous procedure can be repeated with $\tau = \lambda'$ in Lemma 3.4. It is clear that after a finite number of steps (since λ' increases in each step as it follows from Lemma 3.4) we obtain $\lambda' = \lambda$.

<u>Proof of Theorem 2.2</u>. According to Theorem 2.1 we have for each $\lambda \in (0,n)$

$$(4.11)\qquad D_j u \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^N), \qquad j = 1, \dots, n.$$

From (4.11) it follows $u \in C^{0,\gamma}(\Omega, \mathbb{R}^N)$ for each $\gamma \in (0,1)$. Let

$B(2\sigma)=B(x_0, 2\sigma)\subset\Omega$ be an arbitrary ball and let the function $w\in H_0^{1,2}(B(x_0,\sigma), \mathbb{R}^N)$ be a solution of the following system:

$$(4.12)\quad \int_{B(\sigma)}\sum_{i,j=1}^{m}(A_{ij}(x_0)D_jw|D_i\varphi)dx=\int_{B(\sigma)}\sum_{i,j=1}^{m}([A_{ij}(x_0)-A_{ij}(x)]D_ju|D_i\varphi)dx+\int_{B(\sigma)}\sum_{i=1}^{m}(g^i(x,u,Du)-(g^i(x,u,Du))_{B(\sigma)}|D_i\varphi)dx+\int_{B(\sigma)}(a^0(x,u,Du)|\varphi)dx,$$

for each $\varphi\in H_0^{1,2}(B(\sigma),\mathbb{R}^N)$. From the assumption of the theorem it follows that there exists the only solution of the system (4.12). Using Lemma 3.4, we get

$$(4.13)\quad \int_{B(\sigma)}\sum_{i=1}^{m}\|D_iw\|^2dx\le c_1\{\sigma^{2\alpha}\int_{B(\sigma)}\sum_{i=1}^{m}\|D_iu\|^2dx+\int_{B(\sigma)}\sum_{i=1}^{m}\|g^i(x,u,Du)-(g^i(x,u,Du))_{B(\sigma)}\|^2dx+N_1^{1/q_0}\sigma^n\}.$$

Further, we estimate the second integral on the right hand side of (4.13), From the assumption (2.8) we obtain

$$(4.14)\quad \int_{B(\sigma)}\|g^i(x,u,Du)-(g^i(x,u,Du))_{B(\sigma)}\|^2dx\le$$

$$\le\sigma^{-n}\int_{B(\sigma)}dx\int_{B(\sigma)}\|g^i(x,u(x),Du(x))-g^i(y,u(y),Du(y))\|^2dy\le$$

$$\le c_2\{\int_{B(\sigma)}|f_i(x)-(f)_{B(\sigma)}|^2dx+\sigma^n\|u\|^{2\delta_1}_{C^0(B(\sigma),\mathbb{R}^N)}+\int_{B(\sigma)}\sum_{j=1}^{m}\|D_ju(x)-(D_ju)_{B(\sigma)}\|^{2\beta_1}dx\}\le$$

$$\le c_3\{\varepsilon\int_{B(\sigma)}\sum_{j=1}^{m}\|D_ju(x)-(D_ju)_{B(\sigma)}\|^2dx+c_4\sigma^n\},$$

where $\varepsilon\in(0,1)$ is arbitrary, $c_3=c_3(d(\Omega),L)$ and

$$c_4=\|f_i\|_{\mathcal{L}^{2,n}(\Omega)}+\|u\|^{2\delta_1}_{C^0(B(\sigma),\mathbb{R}^N)}+c_5(\varepsilon).$$

Using the relations (4.11) and (4.14) for the estimation of the remaining terms on the right hand side of (4.13), we get the estimate

$$(4.15)\quad \int_{B(\sigma)}\sum_{i=1}^{m}\|D_iw\|^2dx\le c_6\,\varepsilon\int_{B(\sigma)}\sum_{i=1}^{m}\|D_iu-(D_iu)_{B(\sigma)}\|^2dx+c_7(\varepsilon)\sigma^n,$$

which is valid for each $\varepsilon \in (0,1)$.

The function $v=u-w \in H^{1,2}(B(\sigma),\mathbb{R}^N)$ is the solution of the system

(4.16) $$\int_{B(\sigma)} \sum_{i,j=1}^{n} (A_{ij}(x_0)D_j v | D_i \varphi)dx=0, \qquad \forall \varphi \in H_0^{1,2}(B(\sigma),\mathbb{R}^N)$$

and by means of Lemma 3.1 we have for each $t \in (0,1]$

(4.17) $$\int_{B(t\sigma)} \sum_{i=1}^{m} \| D_i v-(D_i v)_{B(t\sigma)}\|^2 dx \le$$
$$\le c_8 t^{n+2} \int_{B(\sigma)} \sum_{i=1}^{m} \| D_i v-(D_i v)_{B(\sigma)}\|^2 dx.$$

From (4.15) and (4.17) we obtain by the standard manner

(4.18) $$\int_{B(t\sigma)} \sum_{i=1}^{m} \| D_i u-(D_i u)_{B(t\sigma)}\|^2 dx \le$$
$$\le \{c_9 t^{n+2}+c_{10}\varepsilon\} \int_{B(\sigma)} \sum_{i=1}^{m} \| D_i u-(D_i u)_{B(\sigma)}\|^2 dx+c_{11}\sigma^n.$$

Since the inequality (4.18) holds for all $t \in (0,1]$, $\sigma \le d_{x_0}/2$ and $\varepsilon \in (0,1)$ we may use Lemma 3.2 from which we obtain

(4.19) $$\int_{B(\sigma)} \sum_{i=1}^{m} \| D_i u-[D_i u]_{B(\sigma)}\|^2 dx \le c_{12}\sigma^n, \quad \forall \sigma \le d_{x_0}/2,$$

where $c_{12}=c_{12}(\nu, d(\Omega), L, \|f_0\|_{L^{2q_0,nq_0}}, \|f_i\|_{\mathcal{L}^{2,n}}, d_{x_0})$.

The remaining part of the proof is analogous to the corresponding part of the proof of Theorem 2.1.

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