Random Seminormed Spaces
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Abstract: This article deals with a very important case of statistical linear space. Properties of this special case lead to the definition of a random seminormed space.

Key words: Statistical metric space, statistical linear space, random seminorm, $\varepsilon-\eta$-topology, $t$-norm.

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Basic properties of statistical linear spaces in the sense of Menger (SLM-spaces) are given in [1], [2]. These spaces are special cases of statistical metric spaces introduced by Sklar and Schweizer in [3]. For simplicity we present the definition of a statistical linear space (SLM-space) here too. First we need the notion of a $t$-norm. A function $T: \langle 0,1 \rangle \times \langle 0,1 \rangle \rightarrow \langle 0,1 \rangle$ satisfying

(a) $T(a,b) = T(b,a)$; $T(a,1) = a$ for $a > 0$
(b) $T(a,b) \leq T(c,d)$ for $a \leq c$, $b \leq d$
(c) $T(T(a,b),c) = T(a,T(b,c))$
(d) $T(0,0) = 0$

will be called a $t$-norm.

Definition 1. Let $S$ be a real linear space, let $F$ be the set of all probability distribution functions defined on the real line $\mathbb{R}_+$. Let $G: S \rightarrow F$ be a given mapping. For every $x \in S$ let us denote $G(x) = F_x \in F$ and we demand that $G$ satisfies:

1. $x = 0 \iff F_x = H$ where $H(u) = 0$ if $u \leq 0$; $H(u) = 1$ if $u > 0$
2. $F_{\lambda x}(u) = F_x(u/|\lambda|)$ for every $x \in S$ and every $\lambda \neq 0$.
3. $F_x(u) = 0$ for every $u \leq 0$ and every $x \in S$.
4. $T(F_x(u), F_y(v)) \leq F_{x+y}(u+v)$ for every $u$, $v \in \mathbb{R}_+$ and every pair $x, y \in S$ where $T$ is a $t$-norm satisfying (a), (b), (c), (d).
Under these conditions the triple \((S,G,T)\) is called a linear statistical space in the Menger sense (SLM-space). We shall start with construction of a special SLM-space. Let \(S\) be a linear space of all real sequences \(x = \{x_i\}_{i=1}^\infty\) where linear operations of addition and scalar multiplication are defined coordinatewise. Let \(a = \{a_i\}_{i=1}^\infty\) be a sequence of positive reals such that \(\sum_1^\infty a_i = 1\). Let \(F\) be the set of all probability distribution functions on reals. Let us define a mapping \(\psi : S \rightarrow F\) in the following way:

if \(x = \{x_i\}_{i=1}^\infty\) then we put

\[
\psi(x)(u) = \begin{cases} 0 & \text{for } u \leq |x_1| \\ a_1 & \text{for } |x_1| < u \leq |x_1| + |x_2| \\ \vdots \end{cases}
\]

\[
\psi(x)(u) = \sum_{i=1}^n a_i \text{ for } \sum_{i=1}^n |x_i| < u \leq \sum_{i=1}^{n+1} |x_i|.
\]

In case \(\sum_1^n |x_i| < \epsilon\) we must distinguish two possibilities:

a) \(\sum_1^n |x_i|\) contains infinitely many non-zero elements then \(\psi(x)(u) = 1\) for \(u \geq \sum_1^n |x_i|\)

b) \(\sum_1^n |x_i|\) contains finitely many non-zero elements only then \(\psi(x)(u) = 1\) for \(u > \sum_1^n |x_i|\).

**Theorem 1.** The triple \((S,\psi,\min)\) is SLM-space with the \(t\)-norm \(T(a,b) = \min(a,b)\).

Proof. See [1].

Now, using the mapping \(\psi\) and the norm \(\min\) we can introduce a linear topology into \(S\), the \(\epsilon,\eta\)-topology, and as it is shown in [1], [2] \((S,\psi,\min)\) is a locally convex metrizable linear topological space with the base of neighborhoods of the null element

\[
0(\epsilon,\eta) = \{x \in S : \psi(x)(\eta) > 1 - \epsilon\}, \quad 0 < \epsilon \leq 1, \quad \eta > 0.
\]

**Theorem 2.** In SLM-space \((S,\psi,\min)\) the \(\epsilon,\eta\)-topology is equivalent to the coordinatewise convergence.

Proof. Let \(x_n \rightarrow 0\) in the \(\epsilon,\eta\)-topology, \(x_n = \{x_{n1}\}_{i=1}^\infty\) it means

\[
(\forall \epsilon \in (0,1) \forall \eta > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0) \Rightarrow (x_n \in 0(\epsilon,\eta)) \Leftrightarrow (\psi(x_n)(\eta) > 1 - \epsilon).
\]

As \(\sum_1^n a_i = 1\) then for every \(n, \epsilon \in \mathbb{N}\) there exists \(0 < \epsilon < 1\) such that \(1 - \epsilon > \sum_1^n a_i\) and hence \(\psi(x_n)(\eta) \geq \sum_1^n a_i\). It follows from the construction of \(\psi\) that

\[
\sum_{i=1}^n |x_{n1}| < \eta
\]

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for every $n \geq n_0$. But this inequality says that

$$x_{ni} \rightarrow 0 \text{ for every } i \in N.$$  

Conversely, let $x_n \rightarrow 0$ coordinatewise, i.e., $\lim_{n \to \infty} x_{ni} = 0$ for every $i \in N$. Let fix $i_0 \in N$ and let choose for arbitrarily chosen $\varepsilon > 0$ such a number $n_0(i_0) \in \mathbb{N}$ that for every $n \geq n_0(i_0)$ $|x_{ni}| < \varepsilon$ and hence

$$\sum_{i=i_0}^{n} |x_{ni}| < \varepsilon$$

for every $n \geq \max \{ n_0(i) \}$. According to the construction of $V$ we obtain that

$$Vx_n(i_0 \varepsilon) > \sum_{i=i_0}^{n} a_i$$

for $n \geq \max \{ n_0(i) \}$. With respect to the arbitrariness of $\varepsilon$ and $i_0$ it implies that $x_n \to 0$ in the $\varepsilon$-$\eta$-topology. Q.E.D.

Further, we can remind without proofs that SLM-space $(S, V, \min)$ is a complete topological space and the subset $S^* \subseteq S$ formed by all sequences with finite length, i.e.

$$S^* = \{ x \in S : x = \{ x_1, x_2, x_3, \ldots, x_n, 0, 0, 0, \ldots \} \},$$

can be identified with the topological dual space. The construction of the mapping $V$ is based on choice of a sequence $\{a_n\}_{n=1}^{\infty}$. The equivalence between the $\varepsilon$-$\eta$-topology and the coordinatewise convergence yields that the choice of $\{a_n\}_{n=1}^{\infty}$ is not so important because all $\varepsilon$-$\eta$-topologies generated by all possible sequences $a = \{a_n\}_{n=1}^{\infty}$ are mutually equivalent.

Let $x = \{x_n\}_{n=1}^{\infty}$ be an arbitrary point of $S$, let $n \in \mathbb{N}$, we can write

$$x = \sum_{i=1}^{n} x_i e_1 + x^{(n)}$$

where $e_1 = (0, 0, \ldots, 0, 0, 0, \ldots) \in S$. Unfortunately, we cannot write

$$x = \sum_{i=1}^{\infty} x_i e_1$$

because we do not know in which sense the convergence of this series could be understood. So, we can understand this convergence in a probabilistic sense, namely, instead of the basic vectors $\{e_i\}_{i=1}^{\infty}$ we shall consider random variables $\{\xi_i\}_{i=1}^{\infty}$ such that for every $x \in S$ the series $\sum_{i=1}^{\infty} \xi_i x_i$ will be absolutely convergent in the sense almost surely.

**Theorem 3.** Let SLM-space $(S, V, \min)$ be given. We can construct a sequence $\{\xi_i\}_{i=1}^{\infty}$ of random variables such that for every $x = \{x_i\}_{i=1}^{\infty} \in S$ the series

$$\sum_{i=1}^{\infty} \xi_i x_i$$

will be absolutely convergent in the sense almost surely.
\[ \xi_x = \sum_{i=1}^{n_i} \xi_i \]
is absolutely convergent a. s. and,
\[ Y_x(u) = P\{ \omega: \sum_{i=1}^{\infty} |x_i| \xi_i(\omega) < u \} \]
for every real u.

Proof. Let us consider the sequence of vectors \( \{e_i\}_{i=1}^{\infty} \) where
\[ e_1 = (1, 0, 0, \ldots), \quad e_2 = (0, 1, 0, \ldots), \quad e_3 = (0, 0, 1, 0, \ldots), \ldots \]
The mapping \( \gamma \) assigns to \( e_n \) the probability distribution function
\( F_n \) of the form
\[
\begin{align*}
F_n(u) &= 0 \quad \text{for } u \leq 1 \\
F_n(u) &= 1 \quad \text{for } u > 1 \\
F_n(u) &= \frac{\sum_{j=1}^{n-1} a_j}{n} \quad \text{for } 0 < u \leq 1 \\
F_n(u) &= 1 \quad \text{for } u > 1.
\end{align*}
\]
Further, for every k-tuple \( \{e_{n_1}, e_{n_2}, \ldots, e_{n_k}\} \) we shall define the
common probability distribution function by
\[ F_{n_1, n_2, \ldots, n_k}(u_1, u_2, \ldots, u_k) = \min_{1 \leq j \leq k} F_{n_j}(u_j). \]
This system of probability distribution functions satisfies Kolmogorov's consistence conditions and hence we can construct a sequence
\( \{\xi_n\}_{n=1}^{\infty} \) of random variables which satisfy (for every \( n \in \mathbb{N} \))
\[ P\{ \omega: \left( \sum_{i=1}^{\infty} |x_i| \xi_i(\omega) < u \right), 1 \} = F_1, 2, \ldots, n(u_1, u_2, \ldots, u_n) = \min_{l \leq i \leq n} F_1(u_i). \]
Let \( x = \{x_i\}_{i=1}^{\infty} \in \mathcal{S} \) be quite arbitrary and let us consider random variables
\[ \gamma_{i=1}^{n}, x_1 \xi_1(\omega), \quad \gamma_{i=1}^{n} |x_i| \xi_1(\omega). \]
We shall prove that the sequence \( \{\gamma_{i=1}^{n} |x_i| \xi_1(\omega)\}_{n=1}^{\infty} \) is fundamental
in probability. Surely,
\[ P\{ \omega: |x_1| \xi_1(\omega) = |\sum_{i=1}^{n} |x_i| \xi_1(\omega) | < 1 \} = P\{ \omega: |x_1| = |\sum_{i=1}^{n+1} |x_i| \xi_1(\omega) | < 1 \} \]
\[ \geq P\{ \omega: |x_i| (\omega) = 0, \ i = 1, 2, \ldots, k \} = P\{ \omega: |x_i| (\omega) = 0, \ i = 1, 2, \ldots, k \} = \min_{1 \leq i \leq k} F_{n+i}(1) = \gamma_{j=1}^{n} a_j \]
A random variable \( \xi_1 |x_1| (\omega) = p - \lim_{n \to \infty} \gamma_{i=1}^{n} |x_i| \xi_1(\omega) \) must exist.
As for every \( n \in \mathbb{N} \)
\[ \gamma_{i=1}^{n} |x_i| \xi_1(\omega) \leq \gamma_{i=1}^{n+1} |x_i| \xi_1(\omega) \quad \text{a.s.} \]
then \( \xi_1 |x_1| (\omega) = \lim_{n \to \infty} \gamma_{i=1}^{n} |x_i| \xi_1(\omega) \) a. s. and we can write
We have proved that for every \( x \in S \) there exists a sum \( \xi_x(\omega) = \sum_{i=1}^{\infty} x_i \xi_i(\omega) \) satisfying
\[
|\xi_x(\omega)| \leq \xi_{|x|}(\omega) \quad \text{a. s.}
\]
The common probability distribution function \( F_{1,2,3,\ldots,n}(u_1,u_2,\ldots,u_n) \) gives the probability distribution function \( G_n \) for the random variable
\[
\xi_{|x|}(\omega) = \sum_{i=1}^{n} x_i \xi_i(\omega).
\]
It can be easily shown that
\[
P(\omega: \xi_1(\omega) = \rho_1, \ldots, \xi_n(\omega) = \rho_n) = 0 \quad (\rho_i = 0 \text{ or } 1)
\]
for every \( n \)-tuple \((\rho_1, \rho_2, \ldots, \rho_n)\) if \( \rho_i < \rho_{i+k} \) at least for one pair. In other words,
\[
P(\omega: \xi_1(\omega) = \rho_1, \ldots, \xi_n(\omega) = \rho_n) > 0 \quad \text{only for the combinations:}
\]
\[
\begin{align*}
\rho_1 &= 1, \rho_i = 0 & i &= 2,3,\ldots,n \\
\rho_1 &= 1, \rho_2 = 1, \rho_i = 0 & i &= 3,4,\ldots,n \\
\vdots & & \vdots \\
\rho_1 &= 1, \rho_2 = 1, \ldots, \rho_{n-1} = 1, \rho_n = 0 \\
\rho_1 &= 1, \rho_2 = 1, \ldots, \rho_{n-1} = 1, \rho_n = 1.
\end{align*}
\]
From this fact we can easily derive that (under assumption \(|x_j| > 0\) for simplicity)
\[
P(\omega: \sum_{j=1}^{n} x_j | x|_{j}(\omega) = \sum_{j=1}^{n} x_j | x|_{j}(\omega) = 0) = \alpha_1.
\]
So \( G_n(u) = \sum_{j=1}^{n} a_j \) for \( \sum_{j=1}^{n} |x_j| \leq u < \sum_{j=1}^{n+1} |x_j| \).
As \( \sum_{i=1}^{n} |x_i| \xi_i(\omega) \) a. s., \( G_n(u) \) weakly, where \( G \) is the probability distribution function belonging to \( \sum_{i=1}^{n} |x_i| \xi_i \).
We obtain that
\[
P(\omega: \sum_{i=1}^{n} |x_i| \xi_i(\omega) < u) = \nu_x(u). \quad \text{Q.E.D.}
\]
We constructed a mapping \( T: \mathbb{X} \) which to every \( x \in (\mathbb{S},\mathbb{Y},\min) \) assigns
a random variable \( \xi_x = \sum_{i=1}^{\infty} x_i \xi_i \). At the first sight we see the mapping \( T \) is a linear and one-to-one mapping because
\[
\xi_{x+y} = \sum_{i=1}^{\infty} (x_i + y_i) \xi_i = \sum_{i=1}^{\infty} x_i \xi_i + \sum_{i=1}^{\infty} y_i \xi_i = \lambda \xi_x + \nu \xi_y
\]
and the equality \( \sum_{i=1}^{\infty} x_i \xi_i(\omega) = \sum_{i=1}^{\infty} y_i \xi_i(\omega) \) a. s. implies that \( x_i = y_i \)
for every \( i \in \mathbb{N} \) thanks to that fact that
\[
P(\omega: \xi_1(\omega) = 0, \xi_{1+k}(\omega) = 1) = 0
\]
for every \( i \in \mathbb{N} \) and every \( k \in \mathbb{N} \).

All random variables \( \xi_x \) for \( x \in (S, V, \min) \) form a linear set which is closed with respect to convergence a. s. That follows from the decomposition \( P\left( \bigcup_{i=1}^{\infty} A_i \right) = 1 \) if we define

\[
A_i = \{ \omega: \xi_i(\omega) = \ldots = \xi_k(\omega) = l, \xi_{k+1}(\omega) = 0 \forall k \in \mathbb{N} \};
\]

then \( P(A_i) = a_i \). Now, we can easily prove that the mapping \( T \) is topological mapping because convergence \( x_n \to 0 \) in the \( c \)-topology implies convergence \( x_n \to 0 \) a. s. and vice versa. Further, the convergence in probability of a sequence \( \{x_n\} \) implies convergence in the sense a. s.

We proved in Theorem 3 the absolute convergence of \( \sum_{i=1}^{\infty} x_i \xi_i \) a. s., i.e. for every \( x \in (S, V, \min) \)

\[
\xi_{|X|}(\omega) = \sum_{i=1}^{\infty} |x_i| \xi_i(\omega) \text{ exists a. s.}
\]

This random variable will be called a random seminorm defined on \( S \).

We are motivated by the following properties of \( \xi_{|X|} \):

1. \( \xi_{|X|}(\omega) \geq 0 \) a. s., \( \xi_{|X|}(\omega) = 0 \) a. s. if and only if \( x = 0 \) in \( S \)
2. \( \xi_{|X|}(\omega) = |x| \xi_{|X|}(\omega) \) a. s.
3. \( \xi_{|X+Y|}(\omega) \leq \xi_{|X|}(\omega) + \xi_{|Y|}(\omega) \) a. s.

Let \( \Phi \) denote all real functions defined on \( S \), i.e.

\[
\Phi = \{ f; f: S \to \mathbb{R} \}. \text{ To every measurable cylinder set}
\]

\[
C = \{ f \in \Phi; [f(x_1), f(x_2), \ldots, f(x_n)] \in B_n, x_1 \in S, B_n \in \mathcal{B}_n \}
\]

we can assign a nonnegative number

\[
\mu(C) = P[\{ \xi_{|X_1|}(\omega), \xi_{|X_2|}(\omega), \ldots, \xi_{|X_n|}(\omega) \} \in B_n].
\]

In this way we define a set function \( \mu \) on the algebra of all measurable cylinder sets in \( \Phi \) that can be under certain conditions enlarged on Kolmogorov's \( \sigma \)-algebra in \( \Phi \) into a probability measure. It is clear that in the construction of the set function \( \mu \) we are not limited by a special case of the linear space \( S \). Let \( L \) be any real linear set and let \( \Phi_L \) be the function space defined on \( L \), i.e. \( \Phi_L = \{ f; f: L \to \mathbb{R} \} \). Let \( K_\Phi \) be the smallest \( \sigma \)-algebra of subsets in \( \Phi_L \) with respect to which every \( x f \to f(x) \) becomes measurable. If \( \mu \) is a probability measure defined on \( K_\Phi \) then the triple \( (\Phi_L, K_\Phi, \mu) \) forms the underlying probability space.

**Definition 2.** Let \( L \) be any real linear set, let \( (\Phi_L, K_\Phi, \mu) \) be the probability space derived from \( L \), let \( N \) be the subset of all seminorms on \( L \). The triple \( (\Phi_L, K_\Phi, \mu) \) is said to be a random semi-
normed space if there exists a probability measure \( \nu \) on the \( \sigma \)-algebra \( \{N \cap A: A \in K_\delta\} \) such that

\[
\nu(N \cap A) = \mu(A) \quad \text{for every } A \in K_\delta.
\]

It is clear that \((\Phi_L, K_\delta, \mu)\) is a random seminormed space if and only if \( \nu^*(N) = 1 \) where \( \nu^* \) is the outer measure derived from \( \nu \).

Similarly, as it is done in [4] in case of a random metric space we can give very simple necessary and sufficient conditions for the existence of a random seminormed space.

**Theorem 4.** Necessary and sufficient conditions for \((\Phi_L, K_\delta, \mu)\) to be a random seminormed space in a linear set \( L \) are

1. \( \mu(f \in \Phi_L: f(x) \geq 0) = 1 \) for every \( x \in L \)
2. \( \mu(f \in \Phi_L: f(\lambda x) = \lambda f(x)) = 1 \) for every \( x \in L \) and every \( \lambda \in \mathbb{R} \),
3. \( \mu(f \in \Phi_L: f(x+y) \leq f(x) + f(y)) = 1 \) for every \( x, y \in L \).

Proof. The proof of Theorem 4 can be omitted because that is a simple application of Theorem 1 in [5] by aid of an obvious fact that the property of real valued functions on \( L \) to be a seminorm is extensible, hereditary and measurable with respect to the \( \sigma \)-directed covering class of all finite or countable-dimensional linear subspaces in \( L \). Q.E.D.

At this situation, it is necessary to verify that a special case of a random seminormed space considered in \((S, \gamma, \min)\) is in accordance with Definition 2. It is sufficient to verify demands (1), (2), (3) in Theorem 4. Every random variable \( \xi_{|x|} \) derived from \((S, \gamma, \min)\) in \( S \) takes a.s. at most countably many values forming the series \(|x_1|, |x_1| + |x_2|, \ldots, |x_1| + \ldots + \sum_{j=1}^n |x_j|, \ldots \) (if \( x = (x_1)_1^\infty \)). All members of this series are seminorms on \( S \) and this fact implies validity of (1), (2), (3) in this special case.

As it was remembered before, the topological dual space to \((S, \gamma, \min)\) is the subset of all vectors in \( S \) with a finite length. Now, our aim is a construction of a random seminorm in this dual space \( S^* \).

On the basis of an analogy with Banach spaces we shall consider for \( f \in S^*, f(x) = \sum_{i=1}^M f_i x_i \),

\[
\sup_{x \in S: \xi_{|x|}(\omega) = 1} \frac{|f(x)|}{\xi_{|x|}(\omega)} = \sup_{x \in S: \xi_{|x|}(\omega) = 1} \left( \sum_{i=1}^M |f_i x_i| \right).
\]
Theorem 5. For every $f \in S^*$ there exists a random variable $\eta f$ defined on the same probability space as all random variables $\{\xi_i\}_{i=1}^m$ such that

$$P(\omega; |f| (\omega) = \sup_{x \in S: \xi |x| (\omega) = 1} \{|f(x)|\}) = 1$$

and

$$P(\omega; |f| (\omega) = \max\{|f_1|, |f_2|, \ldots, |f_M|\}) = \sum_{j=1}^M a_j$$

$$P(\omega; |f| (\omega) = \infty) = \sum_{j=1}^{M-1} a_j$$

if $f(x) = \sum_{j=1}^M f_j x_j$, $f_M \neq 0$.

Proof. Let $(\Omega, \sigma, P)$ be a probability space where all random variables $\{\xi_i\}_{i=1}^m$ are defined. We can easily derive from the properties of common probability distribution function of $\{\xi_i\}_{i=1}^m$ that $\Omega$ can be decomposed into

$$\Omega = \bigcup_{i=1}^m A_i \cup \{0\} \cup A$$

where $A_i = \{w \in \Omega: \xi_i (w) = \ldots = \xi_k (w) = 1, \xi_{k+1} (w) = 0 \text{ for every } k \leq l\}$

$\Omega_0 = \{w \in \Omega: \exists j \in N \text{ such that } \xi_j (w) = 0, \xi_{j+1} (w) = 1 \text{ for every } i \in N\}$

$\{0\} = \{w \in \Omega: \xi_i (w) = 0 \text{ for every } i \in N\}$

$A = \{w \in \Omega: \xi (w) = 1, \xi_i w \neq 0\}$

All these sets are $\sigma$-measurable and $P(A_1) = a_1$, $P(\Omega_0) = P(\{0\}) = P(A) = 0$. Now, let $\omega \in A_1$, then for every $x \in S$, $x = \{x_j\}_{j=1}^m$,

$$\xi |x| (\omega) = \xi |x| = \sup_{x \in S: \xi |x| (\omega) = 1} \{|f(x)|\} = \sup_{x \in S: \xi |x| (\omega) = 1} \{\xi_j x_j \mid j = 1\}$$

At this situation we must consider two possibilities: a) $i < M$, b) $i \geq M$. In case a) it is easy to see that

$$\sup_{x \in S: \xi |x| (\omega) = 1} \{|f(x)|\} = +\infty.$$ 

In case b) we obtain

$$\sup_{x \in S: \xi |x| (\omega) = 1} \{|f(x)|\} = \max\{|f_k|\}. \quad 1 \leq k \leq M$$

We can consider a random variable $\eta f$ defined on $\Omega$ by the following relation

$$\eta f (\omega) = \max\{|f_k|\} \quad \text{for } \omega \in \bigcup_{j=M}^\infty A_j$$

$$\eta f (\omega) = \infty \quad \text{for } \omega \in \bigcup_{j=1}^{M-1} A_j$$

$$\eta f (\omega) = 0 \quad \text{for } \omega \in \Omega_0 \cup \{0\} \cup A.$$ 

This construction yields immediately that the corresponding probability distribution function $P_f$ is equal to

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\[
\begin{align*}
F_f(u) &= 0 \quad \text{for } u \leq \max_{1 \leq k \leq M} \{ |f_k| \} \\
F_f(u) &= \ell_{i=1}^M a_j \quad \text{for } u > \max_{1 \leq k \leq M} \{ |f_k| \}
\end{align*}
\]

and, further, we have
\[
P\{\omega: \eta |f| (\omega) = \sup \{ |f(x)| \} = 1, \quad \{x: f(x) = 1, |x| = 1\} \}
\]

which completes the proof. Q.E.D.

Theorem 5 enables to define a mapping \( \gamma^*: S^* \rightarrow F^* \) \( \gamma^*_f(u) = F_f(u) \)
where \( F^* \) denotes all nondecreasing left continuous real functions defined on reals with variation less or equal to 1. We shall study properties of the mapping \( \gamma^* \). If \( f \) is the null functional on \( S \), i.e. \( f(x) = 0 \) for every \( x \in S \) then for every \( \omega \in \bigcup_{i=1}^N A_i \)
\[
\eta |0| (\omega) = \sup_{\{x: \frac{1}{j} \leq |x_j| = 1\}} 0 = 0,
\]

which means \( \eta |0| (\omega) = 0 \) for every \( \omega \in \Omega \). The corresponding probability distribution is \( \gamma^*_f = F_0 = H \) where \( H(u) = 0 \) for \( u \leq 0, H(u) = 1 \) otherwise. If \( \lambda \neq 0 \) is any real number then
\[
\lambda f(x) = \ell_{i=1}^M (\lambda f_i) x_i = \ell_{i=1}^M (\lambda f_i) x_i = (\lambda f)(x) \quad \text{and hence}
\]
\[
\eta |\lambda f| (\omega) = \eta |f| (\omega) \quad \text{for every } \omega \in \Omega. \text{ It follows that for every}
\]
\( u \in R_f \) and every \( \lambda \neq 0 \)
\[
\gamma^*_\lambda f(u) = \gamma^*_f(-\frac{u}{\lambda}.
\]

When \( \lambda = 0 \) it is reasonable to put \( \gamma^*_f(\frac{u}{0}) = 1 \) for every \( u > 0 \) and every \( f \in S^* \). Further, let \( f, g \in S^* \), \( f(x) = \ell_{i=1}^M f_i x_i, g(x) = \ell_{j=1}^N g_j x_j \), then
\[
(f + g)(x) = \ell_{i=1}^{\max(M,N)} (f_i + g_i) x_i
\]

and \( \gamma^*_f(u + v) = 0 \) for \( u + v \leq \max_{1 \leq k \leq \max(M,N)} \{ |f_k| + |g_k| \} \). As for every \( k \mid f_k + g_k \mid \leq |f_k| + |g_k| \) we have \( u + v \leq \max_{1 \leq k \leq \max(M,N)} \{ |f_k| + |g_k| \} \) and hence at least one member of \( u, v \) must satisfy
\[
u \leq \max_{1 \leq k \leq \max(M,N)} \{ |f_k| \} \text{ or } v \leq \max_{1 \leq k \leq \max(M,N)} \{ |g_k| \}
\]
and therefore \( \gamma^*_f(u) = 0 \) or \( \gamma^*_g(v) = 0 \). If \( \gamma^*_f(u + v) = \ell_{k=\max(M,N)} a_k \), i.e. \( u + v > \max_{1 \leq k \leq \max(M,N)} \{ |f_k| + |g_k| \} \), then \( \ell_{k=\max(M,N)} a_k \leq \min_{1 \leq k \leq \max(M,N)} (l_k = \max(M,N) a_k) \) and this inequality proves the generalized triangular inequality
\[
\gamma^*_f(u + v) \geq \min(\gamma^*_f(u), \gamma^*_g(v)). \text{ This result leads us to the following}
\]

Definition 3.
Definition 3. Let $F^*$ be the set of all real-valued left continuous nondecreasing nonnegative functions defined on reals with values less or equal to 1. Let $L$ be a linear space and $Y^*$ be a mapping $Y^*: L \rightarrow F^*$ satisfying the following conditions:

1. $Y^*_x(u) = H(u)$ if and only if $x = 0$ in $L$
2. $Y^*_{\lambda x}(u) = Y^*_x(\frac{u}{\lambda})$ for every $u \in R_y$, $\lambda \neq 0$
3. $Y^*_{x+y}(u+v) \geq T(Y^*_x(u), Y^*_y(v))$ for every $x, y \in L$ and every $u, v \in R_y$ where $T$ is a t-norm.

Then the triple $(L, Y^*, T)$ is called a generalized statistical linear space in the sense of Menger (GSLM-space).

We shall introduce a topology into $(L, Y^*, T)$ under assumption that the t-norm $T$ is continuous. Let $U = \{0(\varepsilon, n) = \{y \in L: Y^*_y(n) > 1 - \varepsilon\}, 0 < \varepsilon \leq 1, n > 0\}$. As it is proved in [1] this system $U$ of neighbourhoods forms a base for topology in $(L, Y^*, T)$. We shall call this topology the $\varepsilon$-$n$-topology, too.

Theorem 6. The $\varepsilon$-$n$-topology in GSLM-space $(S^*, Y^*, \min)$ is stronger than the $\beta$-topology in $S^*$.

Proof. Let $(f_n)$ be a sequence in $S^*$ convergent to 0 in the $\varepsilon$-$n$-topology, i. e.

$(\forall \varepsilon > 0 \forall u > 0 \exists n_0 \forall n \geq n_0) \Rightarrow (Y^*_x(u) > 1 - \varepsilon) \Rightarrow (\sum_{j=1}^{Mn} a_j > 1 - \varepsilon)$, if $f_n(x) = \sum_{i=1}^{Mn} f_{n,i}^x$. Since $\{a_j\}$ are positive numbers then for every $\varepsilon \in (0, 1)$ a natural number $M_\varepsilon$ must exist such that $\sum_{j=1}^{M_\varepsilon} a_j > 1 - \varepsilon \geq \sum_{j=M_\varepsilon+1}^{M} a_j$ and hence for every $n \geq n_0$ $M_n \leq M$. It means that all functionals $f_n \in 0(\varepsilon, n)$ have a uniformly bounded length. In other words, using the mapping $Y^*$ we can state that for every $n \geq n_0$

$max(|f_n^x|, \ldots, |f_{Mn}^x|) < u$.

We have proved that $\lim_{n \rightarrow \infty} f_n^x = 0$ for every $i \in N$ and $M_n \leq M < \infty$ for every $n \in N$, too. This convergence in the dual space $S^*$ is the so-called $\beta$-topology. Now, let us consider a sequence $(f_n)$ of functionals in $S$ defined by

$f_n(x) = \sum_{i=1}^{M} \frac{x_i}{n}$

where $M$ is fixed and $M > 1$. It is clear that $f_n \nrightarrow 0$ in the $\varepsilon$-$n$-topology but $f_n \neq 0$ in the $\beta$-topology. Surely,
\[ y_n^* \left( \frac{1}{n} + \delta \right) = \sum_{j=1}^{m} a_j < 1 \] for every \( n \in \mathbb{N} \) and every \( \delta > 0 \).

If we choose \( \varepsilon^* \) in such a way that \( \sum_{i=1}^{m} a_i \leq 1 - \varepsilon^* \) then

\[ f_n \not\in 0(\varepsilon^*, \frac{1}{n} + \delta_n) \quad \text{where} \quad \delta_n + 0. \]

This example proves that the \( \varepsilon-\eta \)-topology is stronger than the \( \beta \)-topology in the dual space \( S^* \).  Q.E.D.

Theorem 7. A sequence \( \{f_n\} \subset (S^*, Y^*, \min) \) tends to the null element in the \( \varepsilon-\eta \)-topology if and only if

1. \( \lim_{n \to \infty} f_n = 0 \) for every \( i \in \mathbb{N} \),
2. \( \lim_{n \to \infty} M_n = 1 \) where \( M_n \) denotes the length of \( f_n \),

\[ f_n(x) = \sum_{i=1}^{M_n} f_{n_i}^1 x_i. \]

Proof. The proof can be omitted because it is an easy conclusion of Theorem 6 and the construction of \( Y^* \).

It is remarkable that the \( \varepsilon-\eta \)-topology in the dual space \( (S^*, Y^*, \min) \) is not a linear topology because for some \( f \in S^* \), \( f \not= 0 \) and \( \{\lambda_n\}_{n=1}^{\infty} \) such that \( \lambda_n = 0 \) and \( \lambda_n f \not= 0 \) in the \( \varepsilon-\eta \)-topology. One can simply prove that the \( \varepsilon-\eta \)-topology in a GSLM-space \( (S^*, Y^*, T) \) is linear if and only if for every \( x \not= 0 \) \( \lim_{u \to 0} x^*(u) = 1 \).

Now, we shall return to the SLM-space \( (S, Y, \min) \) again. We proved that for every \( x = \{x_i\}_{i=1}^{\infty} \) the sum \( \sum_{i=1}^{\infty} x_i \xi_i(\omega) \) is a.e. absolutely convergent and hence for every pair \( x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \) from \( S \) the sum

\[ \sum_{i=1}^{\infty} x_i y_i \xi_i(\omega) \]

is absolutely convergent a.e. too. As \( P(\omega: \xi_i(\omega) = 0) + P(\omega: \xi_i(\omega) = 1) = 1 \) for every \( i \in \mathbb{N} \) then \( \xi_i(\omega) = \xi_i(\omega) \) a.e. and we can write

\[ \sum_{i=1}^{\infty} x_i y_i \xi_i(\omega) = \sum_{i=1}^{\infty} x_i y_i \xi_i^2(\omega) \] a.e.

In this way we constructed a mapping which is defined on \( S \times S \) and takes its values among random variables defined on the underlying probability space \( (\Omega, \sigma, P) \) where all \( \xi_i(.) \) are defined. This mapping satisfies the following properties. If we shall denote by \( \xi_{<x,y>}(\omega) = \sum_{i=1}^{\infty} x_i y_i \xi_i^2(\omega) \) then
1. $\xi_{<x,y>}^\omega + \xi_{<z,y>}^\omega = \xi_{<x+z,y>}^\omega \text{ a. s.}$

2. $\xi_{<ax,y>}^\omega = a\xi_{<x,y>}^\omega \text{ a. s., } a \in \mathbb{R}$

3. $\xi_{<x,x>}^\omega \geq 0 \text{ a. s.}$

4. $\xi_{<x,y>}^\omega = \xi_{<y,x>}^\omega \text{ a. s.}$

These properties lead us to call this mapping a random scalar product defined on $S$. This random scalar product defines a random seminorm on $S$ by the relation

$$\xi_{<x,x>}^\omega = \xi_{|x|^2}^\omega.$$  

The inequality $|\sum x_j y_j| \leq \left(\sum x_j^2\right)^{\frac{1}{2}} \left(\sum y_j^2\right)^{\frac{1}{2}}$ holding for every $n \in \mathbb{N}$ yields the inequality

$$|\xi_{<x,y>}^\omega| \leq \xi_{|x|^2}^\omega \xi_{|y|^2}^\omega \text{ a. s.}$$

We can introduce a notion of orthogonality by aid of the random scalar product in $S$. We shall say that $x, y \in S$ are orthogonal if $\xi_{<x,y>}^\omega = 0 \text{ a. s.}$. We immediately see that two vectors $x, y \in S$ are orthogonal if and only if $x_i \cdot y_i = 0$ for every $i \in \mathbb{N}$.

Under orthogonality the Pythagorean theorem holds in the usual form

$$\xi_{|x+y|^2}^\omega = \xi_{|x|^2}^\omega + \xi_{|y|^2}^\omega \text{ a. s.}$$

Now, we need the probability distribution function of a random variable $\xi_{|x|^2}^\omega$, $x \in S$. According to the definition of the random scalar product on $S$ we see that ($\lambda \geq 0$)

$$P\{w: \xi_{|x|^2}^\omega < \lambda\} = P\{w: \left(\sum_{i=1}^{n} x_i^2 T_i^2(\omega)\right)^{\frac{1}{2}} < \lambda\} =$$

$$= P\{w: \sum_{i=1}^{n} x_i^2 T_i^2(\omega) < \lambda^2\} = \max\{\lambda \in \mathbb{N}: \sum_{i=1}^{n} x_i^2 < \lambda^2\} = \max\{\lambda \in \mathbb{N}: \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} < \lambda\}.$$

We obtain a mapping $\psi_\lambda: S \rightarrow \mathbb{R}$

$$\psi_\lambda(x)(\lambda) = P\{w: \xi_{|x|^2}^\omega < \lambda\}.$$  

**Theorem 8.** The triple $(S,\psi_\lambda,\min)$ is a SLM-space and the corresponding $\epsilon$-$\eta$-topology in $S$ is equivalent to the coordinatewise convergence in $S$.  

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Proof. When \( x \approx 0 \) in \( S \) then \( \xi |x|^2(\omega) = 0 \) a. s. and hence \( \nu_2(x)(u) = H(u) \) for every \( u \in \mathbb{R} \). On the contrary, if \( \xi |x|^2(\omega) = 0 \) a. s. then \( P(\omega: \sum_{i=1}^{n} x_{i}^2(\omega) < u) = 1 \) for every \( u > 0 \) and it implies \( x = 0 \) in \( S \). Let \( \lambda \neq 0 \) then \( \nu_2(\lambda x)(u) = \frac{u}{|\lambda|} \).

It lasts to prove the generalized triangular inequality with the \( t \)-norm-min. Let \( x, y \in S \); \( u, v \in \mathbb{R} \), (we can consider \( u > 0, v > 0 \) only because other cases are quite trivial). We know that

\[
\nu_2(x+y)(u+v) = \frac{u}{|\lambda|} \leq \frac{u}{|\lambda|} \leq \frac{u}{|\lambda|} = \nu_2(\lambda x)(u) = \frac{u}{|\lambda|},
\]

for every \( x \in S \). That implies that the \( \epsilon \)-\( \eta \)-topology induced by the random seminorm \( \xi |x| \) is stronger than the \( \epsilon \)-\( \eta \)-topology derived from \( \xi |x|^2 \). But if \( x_0 = 0 \) in \( S \) in the \( \epsilon \)-\( \eta \)-topology induced by \( \xi |x|^2 \) then \( \epsilon > 0, \eta > 0 \) \( \exists n_0, \exists n_2 \eta, n_0 \) \( \Rightarrow (x_0 \in (\epsilon, \eta)) \Rightarrow (\nu_2(x_0)(\eta) > 1-\epsilon) \Rightarrow (\sum_{j=1}^{n_0} a_j > 1-\epsilon) \). As \( (\sum_{j=1}^{n_0} a_j > 1-\epsilon) \) is increasing then for every \( n \geq n_0 \) \( m_n \geq m_\epsilon + 1 \) where \( \sum_{j=1}^{m_n} a_j \geq 1 - \epsilon \); at the same moment \( \sum_{j=1}^{m_n} a_j > 1 \) must hold for every \( n \geq n_0 \). With respect to arbitrariness of \( \epsilon, \eta \) we can state that \( \lim_{n \to \infty} \eta = 0 \) for every \( j \in \mathbb{N} \). We proved the equivalence between the topology generated by the coordinatewise convergence and the \( \epsilon \)-\( \eta \)-topology induced by the mapping \( \nu_2 \). Q.E.D.

The system \( \{\xi_{x}, \xi_{y}\}; x, y \in S \) of random variables enables to introduce a probability measure into the measurable space \((\phi, K)\) where \( \phi \) is the set of all real-valued functions defined on \( S \times S \) and
$\mathcal{K}$ is the smallest $\sigma$-algebra generated by all measurable cylinder sets

$$\{ f \in \Phi : [f(x_1,y_1), f(x_2,y_2), \ldots, f(x_n,y_n)] \in \mathcal{B}_n \}, \quad x_i, y_i \in S,$$

$\mathcal{B}_n$ is a Borel subset in n-dimensional Euclidean space. In the $\sigma$-algebra $\mathcal{K}$ can be defined a probability measure $\mu$ by

$$\mu(C) = P\{\omega : [\xi_{<x_1,y_1>(\omega)}, \xi_{<x_2,y_2>(\omega)}, \ldots, \xi_{<x_n,y_n>(\omega)}] \in \mathcal{B}_n\}.$$

If we denote by $\pi \in \Phi$ the subset of all semiscalar products defined on $S$ then one can affirm that

$$\mu^*(\pi) = 1$$

($\mu^*$ is the outer measure corresponding to the measure $\mu$). In this way we constructed a probability space $(\pi, \mathcal{K}, \nu)$ where $\pi$ is the set of all semiscalar products on $S$, $\mathcal{K} = \mathcal{K} \cap \pi$ and

$$\nu(A \cap \pi) = \mu(A), \quad A \in \mathcal{K}.$$ 

This example enables a generalization. Let $L$ be any vector space, let $\Phi_L \times L$ be the set of all real-valued functions defined on $L \times L$ and let $\mathcal{K}$ be the smallest $\sigma$-algebra generated by all cylinder sets of the form

$$\{ f \in \Phi_L \times L : [f(x_1,y_1), \ldots, f(x_n,y_n)] \in \mathcal{B}_n \},$$

where $x_i, y_i \in L$ and $\mathcal{B}_n$ is an n-dimensional Borel subset.

**Definition 4.** A triple $(\Phi_L \times L, K, \nu)$ will be called a random semiunitary (unitary resp.) space if there exists a probability measure $\nu$ on the $\sigma$-algebra $\pi_L \cap \mathcal{K}$ such that $\nu(A \cap \pi_L) = \mu(A)$ for every $A \in \mathcal{K}$ where $\pi_L$ is the subset of all semiscalar (scalar resp.) products on $L$.

Without any proof we assert that the property "to be a semiscalar (scalar resp.) product on $L" is extensible, hereditary and $\mathcal{K}$-measurable with respect to the measurable space $(\Phi_L \times L, \mathcal{K})$. Using Theorem 1 in [5] again we can formulate Theorem 9.

**Theorem 9.** Necessary and sufficient conditions for the existence of a random semiunitary (unitary resp.) space $(\Phi_L \times L, K, \nu)$ in a vector space $L$ are

1. $\nu(f \in \Phi_L \times L : f(x,x) \geq 0) = 1$ for every $x \in L$

2. $\nu(f \in \Phi_L \times L : f(ax+ay,z) = af(x,z)+bf(y,z)) = 1$ for every $x, y, z \in L$ and every $a, b \in \mathbb{R}$.
3. \( \mu(f \in \Phi_{L \times L} : f(x,y) = f(y,x)) = 1 \) for every \( x, y \in L \)

Proof. The proof of Theorem 9 can be omitted.

References


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