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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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MULTIPLIERS ON A NEARLATTICE

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Abstract

A nearlattice is a lower semilattice in which any two elements have a supremum whenever they are bounded above. Here we generalize the concept of direct summand to nearlattices and show that the direct summands of a nearlattice S with 0 are precisely the central elements of J(S), the lattice of ideals. Then we discuss multipliers (meet translations) on nearlattices.

Subject Classifications (1980) : 06A12, 06A99, 06B10

1 Introduction

Nearlattices, or lower semilattices with the property that any two elements possessing a common upper bound have a supremum, provide an interesting generalization of lattices. Cornish and Hickman [2] referred this property as the *upper bound property*, and a semilattice of this nature as a *semilattice with the upper bound property*. We refer the reader to [2, 3] for necessary background on nearlattices.

Standard elements and ideals in lattices were first studied in depth by Grätzer and Schmidt [5]. Recently Cornish and Noor [3] has extended those concepts to nearlattices. An element 's' in a lattice 'L' is called *standard* if for any $x, y \in L$, $x \wedge (y \vee s) = (x \wedge y) \vee (x \wedge s)$. It is called *neutral* if it is standard and for any x, y $\in$ L, s $\wedge$ (x $\vee$ y) = (s $\wedge$ x) $\vee$ (s $\wedge$ y). An ideal of a lattice (nearlattice) is called *standard* if it is a standard element of the lattice of ideals.

Central elements in a lattice were studied by Kolibiar in [7]. An element 's' in a lattice 'L' is called *central* if it is neutral, and complemented in each interval containing it.

According to [8; 4.3, p-15], in a lattice 'L' with 0, a ∇ b denotes the fact that a $\wedge$ b = 0 and (a $\vee$ x) $\wedge$ b = x $\wedge$ b for all x $\in$ L. For a subset H of L, H^{∇} denotes the set of elements a $\in$ L such that a ∇ b for all b $\in$ H. Let L be a lattice with 0, and $H_1, \ldots, H_n$ be its subsets, each of which contains 0. We say that L is the *direct sum* of $H_1, \ldots, H_n$ and write $L = H_1 \oplus \ldots \oplus H_n$, if

(i) Every element a $\in$ L can be expressed (uniquely) in the form $a = a_1 \vee \ldots \vee a_n$ for some $a_i \in H_i$, and

(ii) $H_i \subset H_j^{\nabla}$ for i $\neq$ j, i = 1, ..., n; j = 1, ..., n.

The subsets $H_1, \ldots, H_n$ are called *direct summands* of L. By [8; 4.8, p-16], every direct summand is an ideal of L. Janowitz in [6] has shown that the direct summands of a lattice L with zero are precisely the central elements of the lattice of ideals.

For a lattice L, a map ϕ : L $\rightarrow$ L is called a *multiplier* if ϕ (a $\wedge$ b) = ϕ (a) $\wedge$ b for each a, b $\in$ L. The set of all multipliers of L is denoted by M (L) and is known as the *multiplier extension* of L.

Multipliers on semilattices and lattices have been previously studied by several authors. A good and accessible summary appears in [1], also c.f. [10].

In §2, we generalize the concept of direct summand to nearlattices. Then we show that the direct summands of a nearlattice S with 0 are precisely the central elements of J(S) , which is an extension of Janowitz's result in [6].

In §3, we discuss multipliers on nearlattices. We extend some results of Nieminen [9], and include some corrections of certain errors of Nieminen's work in [9].

2 Direct Summands of a Nearlattice

In a nearlattice S with 0, we define $a \nabla b$ to mean that $a \wedge b = 0$ and $((a \wedge x) \vee (x \wedge y)) \wedge b = x \wedge y \wedge b$ for $x, y \in S$.

Suppose $a \nabla b$ holds in a lattice L with 0 in the sense of the introduction. Then for all $x, y \in L$,
$(a \vee ((a \wedge x) \vee (x \wedge y))) \wedge b = ((a \wedge x) \vee (x \wedge y)) \wedge b$ and so
$((a \wedge x) \vee (x \wedge y)) \wedge b = (a \vee ((a \wedge x) \vee (x \wedge y))) \wedge b =$
$(a \vee (x \wedge y)) \wedge b = x \wedge y \wedge b$. This and a part of the following result show that the concept of ∇ in a nearlattice and the one in "Lattice Theory" coincide in a lattice.

Proposition 2.1

Suppose $a \nabla b$ *holds in a nearlattice* S *for some* $a, b \in S$. *Then* $a \wedge b = 0$ *and* $(a \vee t) \wedge b = t \wedge b$ *for any* $t \in S$, *whenever* $a \vee t$ *exists. But these are not sufficient for* a *and* b *to satisfy the relation* $a \nabla b$.

Proof

Since $a \nabla b$ in S, $a \wedge b = 0$ and, for any $x, y \in S$,
$((a \wedge x) \vee (x \wedge y)) \wedge b = x \wedge y \wedge b$. Suppose $a \vee t$ exists for some $t \in S$. Putting $a \vee t = x$, we obtain
$(a \vee t) \wedge b = ((a \wedge x) \vee (x \wedge t)) \wedge b = x \wedge t \wedge b = t \wedge b$.

For the second assertion, consider the nearlattice S in Figure 1. There

$s \wedge a = 0$ and $(s \vee x) \wedge a = x \wedge a$ for all $x \in S$, whenever $s \vee x$ exists. But $((s \wedge c) \vee (c \wedge d)) \wedge a > c \wedge d \wedge a$ implies that $s \nabla a$ does not hold. •

For a subset H of a nearlattice S with 0, let $H^{\nabla} = \{ a \in S : a \nabla b$ for all $b \in H \}$. Suppose $a, b \in S$ are such that $a \nabla b$ and let $a_1 \leq a$. Then for any $x, y \in S$, $((a_1 \wedge x) \vee (x \wedge y)) \wedge b =$
$((a_1 \wedge x) \vee (x \wedge y)) \wedge ((a \wedge x) \vee (x \wedge y)) \wedge b =$
$((a_1 \wedge x) \vee (x \wedge y)) \wedge b \wedge x \wedge y = b \wedge x \wedge y$, which implies that H^{∇} is hereditary. It is well known in lattice theory that H^{∇} is an ideal, c.f. [8; 4.6, p-16]. Figure 2 shows that this is not necessarily true in a nearlattice. There, consider $H = \{ b \}$. It is easy to check that $a_1, a_2 \in H^{\nabla}$. But, $(((a_1 \vee a_2) \wedge x) \vee (x \wedge y)) \wedge b > x \wedge y \wedge b$ implies that $a_1 \vee a_2 \notin H^{\nabla}$.

Remark

In connection with the definition of ∇ in a nearlattice, it should be noted that one might define the relation ∇ in the following way : In a nearlattice S with 0, $a \nabla b$ means $a \wedge b = 0$ and $(a \vee x) \wedge b = x \wedge b$, whenever $a \vee x$ exists for any $x \in S$. The main disadvantage with this definition is that, for any subset H of S, H^{∇} is not necessarily hereditary. In Figure 3, notice that $a \in \{ b \}^{\nabla}$, but $(r \vee x) \wedge b > x \wedge b$ implies that $r \notin \{ b \}^{\nabla}$. •

Suppose $H_1, \ldots, H_n$ are the subsets of S, each of which contains 0. We say that S is the *direct sum* of $H_1, \ldots, H_n$ and write $S = H_1 \oplus \ldots \oplus H_n$ if

(i) every element $a \in S$ can be expressed in the form $a = a_1 \vee \ldots \vee a_n$ where $a_i \in H_i$, and

(ii) $H_i \subset H_j^{\nabla}$ whenever $i \neq j$.
The subsets $H_1, \ldots, H_n$ are called *direct summands* of S.

Lemma 2.2.

If a nearlattice S *with* 0 *is a direct sum of* $H_1, \ldots, H_n$, *then for every element* $a \in S$ *the expression* $a = a_1 \vee \ldots \vee a_n$ *where* $a_i \in H_i$ *is unique,* *and $H_1, \ldots, H_n$ are ideals of* S.

Proof

Let $a = a_1 \vee \ldots \vee a_n = b_1 \vee \ldots \vee b_n$ where $a_i, b_i \in H_i$. Here, $b_2, \ldots, b_n \in H_1^\nabla$ by definition. Thus, $b_2 \nabla a_1, \ldots, b_n \nabla a_1$. Hence $a_1 = a \wedge a_1 = (b_1 \vee \ldots \vee b_n) \wedge a_1 = b_1 \wedge a_1$ by proposition 2.1, which implies that $a_1 \leq b_1$. By symmetry, $b_1 \leq a_1$ and hence $a_1 = b_1$. Similarly, $a_i = b_i$ for all i.

For the second part, we will only show that H_1 is an ideal of S. Let $a \in H_1$ and $b \leq a$ ($b \in S$). Then $b = b_1 \vee \ldots \vee b_n$ with $b_i \in H_i$. For $i \neq 1$, notice that $b_i \leq b \leq a$ and $b_i \in H_i \subset H_1^\nabla$. Thus, $b_i = b_i \wedge a = 0$, i.e., $b = b_1 \in H_1$ and so H_1 is hereditary. Finally, let $a, b \in H_1$ are such that $a \vee b$ exists. Suppose $a \vee b = c_1 \vee \ldots \vee c_n$ where $c_i \in H_i$. Now, if $i \neq 1$, $a, b \in H_1 \subset H_i^\nabla$, which implies $a \nabla c_i$ and $b \nabla c_i$ for $i \neq 1$. Then $c_i = (a \vee b) \wedge c_i = b \wedge c_i = 0$ by proposition 2.1, and $a \vee b = c_1 \in H_1$. Therefore, H_1 is an ideal of S. •

Our next theorem gives a generalization of a result of Janowitz [6] to nearlattices which says that the direct summands of a nearlattice S with 0 are precisely the central elements of J(S). To prove this, we need the following lemmas.

Lemma 2.3 [Janowitz [6]].

Let 'L' *be a bounded lattice with* 'z' ∈ 'L'. *If* z' *is the complement of* 'z' *in* 'L', *then the following conditions are equivalent.*

(i) *z is central and*

(ii) *both z and z' are standard.* •

Lemma 2.4

Suppose S *is a nearlattice with* 0 *and* $S = H_1 \oplus \ldots \oplus H_n$. *Then*

(i) *for any* x, y ∈ S, *where* $x = a_1 \vee \dots \vee a_n$ *and* $y = b_1 \vee \dots \vee b_n$ *with* $a_i, b_i \in H_i$, $x \wedge y = (a_1 \wedge b_1) \vee \dots \vee (a_n \wedge b_n)$.

(ii) *each* H_i *is a standard ideal of* S.

Proof

(i) Clearly, $(a_1 \wedge b_1) \vee \dots \vee (a_n \wedge b_n) \leq x, y$ and so $\leq x \wedge y$. Since $S = H_1 \oplus \dots \oplus H_n$, $x \wedge y = c_1 \vee \dots \vee c_n$ with $c_i \in H_i$, i = 1,2, ..., n. Now, notice that $c_1 \leq x \wedge y \leq x, y$. Thus, $c_1 = x \wedge c_1 = (a_1 \vee \dots \vee a_n) \wedge c_1 = a_1 \wedge c_1$ as $a_2 \nabla c_1, \dots, a_n \nabla c_1$ and $c_1 = y \wedge c_1 = (b_1 \vee \dots \vee b_n) \wedge c_1 = b_1 \wedge c_1$, as $b_2 \nabla c_1, \dots, b_n \nabla c_1$. Hence, $c_1 \leq a_1, b_1$ and so $c_1 \leq a_1 \wedge b_1$. Similarly, $c_i \leq a_i \wedge b_i$ for all i and thus $x \wedge y \leq (a_1 \wedge b_1) \vee \dots \vee (a_n \wedge b_n)$, which completes the proof of (i).

(ii) Let $T = \{ h \vee r : h \vee r$ exist with $h \in H_1$ and $r \in R \}$ for an ideal R of S. Clearly T is closed under existent finite suprema. Suppose $x \in S$ and $x \leq h \vee r$ for some $h \in H_1$ and $r \in R$. Since $S = H_1 \oplus \dots \oplus H_n$, $x = a_1 \vee \dots \vee a_n$ and $r = h_1 \vee \dots \vee h_n$, where $a_i, h_i \in H_i$. Then
$x = x \wedge (h \vee r) = (a_1 \vee \dots \vee a_n) \wedge ((h \vee h_1) \vee \dots \vee h_n) = (a_1 \wedge (h \vee h_1)) \vee \dots \vee (a_n \wedge h_n)$ by the application of (i). (Here, $h \vee h_1$ exists by the upper bound property of S as h, $h_1 \leq h \vee r$). Thus $x \in T$; it follows that T is an ideal, and clearly $T = H_1 \vee R$. Hence, by [3; Th. 2.5], H_1 is standard in J(S), the lattice of ideals of S, and (ii) is obtained. •

Theorem 2.5

In a nearlattice S *with* 0, *an ideal* I *is a central element of* J(S) *if and only if it is a direct summand of* S.

Proof

Let I be central in J(S) and let K be its complement. Then $I \cap K = \{ 0 \}$ and $I \vee K = S$. Thus, by [3; Th. 2.5], for each $a \in S$ there exists $b \in I$ and $c \in K$ such that $a = b \vee c$. Moreover, since I is central, for any $i \in I$, $k \in K$ and $x, y \in S$,
$((i \wedge x] \vee (x \wedge y]) \cap (k] \subseteq (I \vee (x \wedge y]) \cap (k] =$
$(I \cap (k]) \vee (x \wedge y \wedge k] = (x \wedge y \wedge k]$ as $I \cap K = \{ 0 \}$.
Thus $[(i \wedge x) \vee (x \wedge y)] \wedge k = x \wedge y \wedge k$. But $i \wedge k = 0$ and so $i \nabla k$. Similarly, $k \nabla i$ and hence $S = I \oplus K$.

Conversely, let $S = H_1 \oplus \ldots \oplus H_n$. Then it is not hard to see that $H_1 \cap (H_2 \vee \ldots \vee H_n) = \{ 0 \}$ as each H_i is standard in J(S) by lemma 2.4. Moreover, each $a \in S$ has a representation of the form $a = a_1 \vee \ldots \vee a_n$ for suitable $a_i \in H_i$; it follows that $H_1 \vee \ldots \vee H_n = S$. Thus, H_1 is the complement of $H_2 \vee \ldots \vee H_n$ in J(S). But by lemma 2.4, both H_1 and $H_2 \vee \ldots \vee H_n$ are standard in J(S). Thus H_1 is central in J(S) by lemma 2.3. Similarly, H_i is central in J(S) for each i. •

Corollary 2.6

The direct summands of a nearlattice S *with* 0 *form a boolean sublattice of* J(S). •

3 Multiplier extension of a nearlattice

Let S be a nearlattice and ϕ a mapping of S into itself. Then ϕ is called a *multiplier* on S, if $\phi(x \wedge y) = \phi(x) \wedge y$ for each $x, y \in S$. Each multiplier ϕ on S has the following properties, $\phi(x) \leq x$, $\phi(\phi(x)) = \phi(x)$, and $x \leq y$ implies $\phi(x) \leq \phi(y)$. For a multiplier ϕ on S, $M_\phi = \{ x \in S : \phi(x) = x \}$ is clearly an ideal of S, and by [10; Th. 3], M_ϕ determines ϕ uniquely.

Each $a \in S$ induces a multiplier μ_a defined by $\mu_a(x) = a \wedge x$ for each $x \in S$. A multiplier of this form is called an *inner multiplier*. Note that the identity function on S, which will be denoted by ι, is always a multiplier. M(S) (respectively μ(S)) denotes the set of all multipliers (respectively inner multipliers) on S. It is trivial that M(S) has a zero ω (say) if and only if S has 0.

The following result is due to [9, Lemma 1].

Lemma 3.1

An ideal I *of a nearlattice* S *generates a multiplier* ϕ *on* S, *that is* $M_\phi =$ I, *if and only if for each* a $\in$ S *there is an element* b $\in$ I *such that* I $\cap$ (a] = (b], *and moreover*, b = ϕ(a). •

If ϕ and λ are multipliers on a nearlattice S, then $\phi \wedge \lambda$ and $\phi \vee \lambda$ are defined by $(\phi \wedge \lambda)(x) = \phi(x) \wedge \lambda(x)$ and $(\phi \vee \lambda)(x) = \phi(x) \vee \lambda(x)$. Notice that $\phi(x) \vee \lambda(x)$ always exists by the upper bound property of S, as $\phi(x)$, $\lambda(x) \leq x$, though $\phi \vee \lambda$ is not necessarily a multiplier. Also,
$\phi(\lambda(x)) = \phi(\lambda(x \wedge x)) = \phi(\lambda(x) \wedge x) = \phi(x) \wedge \lambda(x)$.
As shown by [11; Th. 3], M(S) is a meet semilattice.

The following result is also due to [9].

Proposition 3.2

Let ϕ *and* λ *be two multipliers on a nearlattice* S. *Then* $\phi \vee \lambda$ *is a multiplier on* S *if and only if*
$(M_\phi \vee M_\lambda) \cap (x] = (M_\phi \cap (x]) \vee (M_\lambda \cap (x])$ *for each* x $\in$ S. •

In case of lattices, the following corollary follows immediately from above proposition, and was already proved by Nieminen in [9]. But in our situation, a little more care is required, as the supremum of two ideals in a nearlattice is not as well behaved as that in a lattice.

Corollary 3.3

Let ϕ *be a multiplier on a nearlattice* S. *The mapping* $\phi \vee \lambda$ *is a multiplier on* S *for each* $\lambda \in$ M(S) *if and only if* M_ϕ *is a standard ideal of* S.

Proof

If M_ϕ is standard then $(M_\phi \vee M_\lambda) \cap (x] = (M_\phi \cap (x]) \vee (M_\lambda \cap (x])$ for each $\lambda \in M(S)$. Then $\phi \vee \lambda$ is a multiplier by proposition 3.2.

Conversely, let $\phi \vee \lambda$ be a multiplier for each $\lambda \in M(S)$. By proposition 3.2, $((a] \vee M_\phi) \cap (x] = ((a] \cap (x]) \vee (M_\phi \cap (x])$ for each μ_a, $a \in S$. Now, let I be any ideal of S and suppose $T = \{ i \vee j : i \vee j$ exists and $i \in I, j \in M_\phi \}$. Obviously, T is closed under existent finite suprema. Suppose $r \in S$ with $r \leq i \vee j$ for some $i \in I$ and $j \in M_\phi$. Then from the above observation,
$(r] = (r] \cap ((i] \vee (j]) \subseteq (r] \cap ((i] \vee M_\phi) =$
$((r] \cap (i]) \vee ((r] \cap M_\phi) \subseteq ((r] \cap I) \vee ((r] \cap M_\phi)$.

Now, $((r] \cap I) \vee ((r] \cap M_\phi) = \{ x \in S : x \leq p \vee q$ with $p \in (r] \cap I$ and $q \in (r] \cap M_\phi \}$. Because, clearly the right hand side is hereditary, and it is closed under existent finite suprema by the upper bound property of S, as each element of $(r] \cap I$ and $(r] \cap M_\phi$ is $\leq r$. Thus, $r \leq a \vee b$ for some $a \in (r] \cap I$ and $b \in (r] \cap M_\phi$. This implies $r = a \vee b$ and hence $r \in T$. That is, T is an ideal containing I and M_ϕ, and $T = I \vee M_\phi$. Hence by [3; Th. 2.5], M_ϕ is standard. •

We are now in a position to generalize an interesting result of [9].

Theorem 3.4

A nearlattice S *with* 0 *has a decomposition into a direct summand if and only if there are at least two multipliers* ϕ *and* λ *on* S *such that* $\phi \vee \lambda = \iota$ *and* $\phi \wedge \lambda = \omega$*, and both* ϕ *and* λ *have a supremum with each multiplier on* S.

Proof

Let $S = J \oplus K$. By theorem 2.5, both J and K are standard elements of J(S), $J \wedge K = (0]$ and $J \vee K = S$. Choose any $x \in S$. Since $S = J \oplus K$, $x = a_1 \vee a_2$ (unique), $a_1 \in J$ and $a_2 \in K$. Thus, $J \cap (x] = (a_1]$, $a_1 \in J$, and so by Lemma 3.1, J generates a multiplier ϕ on S. As J is standard in J(S), by 3.3, $\phi \vee \tau$ is a multiplier for each multiplier $\tau \in M(S)$. Similar facts also hold for the multiplier λ on S associated with K. Then $\phi \vee \lambda$ corresponds to the multiplier associated with the ideal $J \vee K = S$, that is, ι, while $\phi \wedge \lambda$ is the multiplier associated with $J \cap K = (0]$, i.e., ω.

Conversely, let ϕ and λ be two multipliers with the properties given in the theorem. As $\phi \vee \tau$ exists for each multiplier $\tau \in M(S)$, the ideal J associated with ϕ is a standard element of J(S). This also holds for the ideal K associated with λ. As $\phi \wedge \lambda = \omega$ and $\phi \vee \lambda = \iota$, $J \wedge K = (0]$ and $J \vee K = S$, respectively. Thus, both J and K are central by Lemma 2.3. Hence, according to Theorem 2.5, $S = J \oplus K$. •

Next theorem is due to Nieminen [9; Th. 3]. It should be mentioned that there is an error in Nieminen's proof of (iii) $\Rightarrow$ (i). There he wanted to prove that if (x] is a distributive sublattice of S for each $x \in S$ (i.e., S is distributive) then J(S) is distributive, which is well known from [2, Th. 1.1]. It is important to note that his determination of the supremum of two ideals in an arbitrary nearlattice is not correct. For two ideals I and J of a nearlattice S, he has described $I \vee J$ as $\{ x \in S : x \leq i \vee j;\ i \in I, j \in J \}$. Figure 4 shows that this is not true for a non-distributive nearlattice. There, let I = (a] and J = (b]. Observe that $c \in I \vee J$ but $c \notin \{ x \in S : x \leq i \vee j;\ i \in I, j \in J \}$. In this connection we like to mention that [4, Ex. 22, p-54] gives a formula for the supremum of two ideals in an arbitrary nearlattice.

Theorem 3.5

In a nearlattice S, *the following conditions are equivalent.*

(i) M(S) *is a lattice (in fact, distributive lattice).*

(ii) *Each multiplier on* S *is a join-partial endomorphism of* S.

(iii) (x] *is a distributive sublattice of* S *for each* $x \in S$. *In other words,* S *is distributive.* •

We conclude this paper with the following theorem which was also mentioned by Nieminen in] 9, Th. 4] without proof. But it is quite significant to note that there he has given an outline of a proof which is completely wrong. He has suggested to use the idea that for a nearlattice S, J(S) is modular if and only if (x] is modular for each x ∈ S. Nearlattice S of figure 2 gives a counter example to that. Notice that there (r] is modular for each r ∈ S. But in J(S), clearly { (0], $(a_1]$, $(a_1,y]$, $(a_2,b]$, S } is a pentagonal sublattice.

Still, we are able to provide an independent proof of this theorem.

Theorem 3.6

Let S *be a nearlattice. Each multiplier* ϕ *on* S *has the property that* $\phi(\phi(y) \vee z) = \phi(y) \vee \phi(z)$ *when* $\phi(y) \vee z$ *exists in* S, *if and only if* (x] *is a modular sublattice of* S *for each* x ∈ S.

Proof

Suppose (x] is modular for each x ∈ S. Let ϕ be a multiplier on S such that $\phi(y) \vee z$ exists for some y, z ∈ S. Choose any $a \in M_\phi \cap ((\phi(y) \vee z])$. Then $a = \phi(a)$ and $a \leq \phi(y) \vee z = t$ (say). Since a, $\phi(y) \leq t$, the upper bound property of S ensures that $a \vee \phi(y) = s$ (say) exists in S and $s \leq t$. Also, a, $\phi(y) \leq s$ implies that $a = \phi(a) \leq \phi(s)$ and $\phi(y) = \phi(\phi(y)) \leq \phi(s)$, i.e., $s \leq \phi(s)$, and so $s \in M_\phi$. Since (t] is a modular sublattice of S, $s = s \wedge t = s \wedge (\phi(y) \vee z) = \phi(y) \vee (s \wedge z) \in (M_\phi \cap (\phi(y)]) \vee (M_\phi \cap (z])$. Thus, $a \in (M_\phi \cap (\phi(y)]) \vee (M_\phi \cap (z])$. Since the reverse inclusion is obvious, $M_\phi \cap (\phi(y) \vee z] = (M_\phi \cap (\phi(y)]) \vee (M_\phi \cap (z])$. Hence by Lemma 3.1, $\phi(\phi(y) \vee z) = \phi(y) \vee \phi(z)$.

To prove the converse, let each multiplier ϕ on S has the property $\phi(\phi(y) \vee z) = \phi(y) \vee \phi(z)$ whenever $\phi(y) \vee z$ exists. Suppose a, b, c ∈ (x] with $c \leq a$. As the multiplier μ_a has the given property, $a \wedge (b \vee c) = \mu_a(b \vee c) = \mu_a(b \vee \mu_a(c)) = \mu_a(b) \vee \mu_a(c) = (a \wedge b) \vee (a \wedge c) = (a \wedge b) \vee c$, which implies that (x] is modular. •

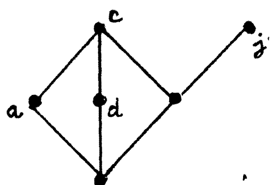

Figure 1.

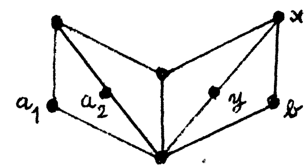

Figure 2.

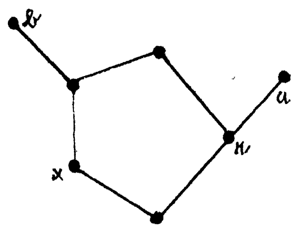

Figure 3.

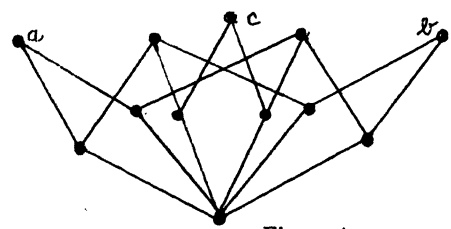

Figure 4.

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