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ON THE MULTIPLICITY POINTS OF MONOTONE OPERATORS ON SEPARABLE BANACH SPACES II

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Abstract: The results from [1] are sharpened, e.g. it is proved that the set of multiplicity points of a monotone operator on a separable real Banach space can be written as a union of countably many subsets of Lipschitz hypersurfaces, having "finite convexity on curves with finite convexity".

Key words: Multiplicity points of monotone operators, finite convexity, Lipschitz surfaces in Banach spaces.

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Let T be a monotone operator on a real Banach space X,i.e. $T:X \longrightarrow \exp X^*$ and $\langle x-y,x^*-y^*\rangle \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$. Denote by coTx a convex hull of the set Tx and put

$A_n=\{x \in X:\dim(\mathrm{co}Tx) \geq n\}$,

$A^n=\{x \in X:\mathrm{co}Tx$ contains a ball of codimension $n\}$.

In [1] there was proved that if X (or X^*, respectively) is separable then A_n (or A^n, resp.) is representable as a countable union of Lipschitz fragments of codimension n (of dimension n, resp.), where F (see Definition 2) has "linearly finite convexity" (i.e. uniformly bounded convexity on lines).

By finer calculations with the Lipschitz fragments constructed in [1], the stronger result is obtained: they are in fact CFC-fragments (see Definition 3).

X will always be a real Banach space; by $\Omega(x,r)$ we shall denote an open ball in X with centre x and radius $r>0$.

Definition 1. Let $S \subset \mathbf{R}$ and $c:S \longrightarrow X$. If card $S \geq 3$ we define

$$\mathcal{K}(c,S)=\sup \sum_{j=1}^{k} \left\| \frac{c(s_{j+1})-c(s_j)}{s_{j+1}-s_j} - \frac{c(s_j)-c(s_{j-1})}{s_j-s_{j-1}} \right\|,$$

where "sup" is taken over all finite sequences $s_0 < s_1 < \ldots < s_{k+1}$ in M. We put $\mathcal{K}(c,S)=0$ if card $S \leq 2$.
$\mathcal{K}(c,S)$ is called convexity of c on S.

Basic properties of mappings with finite convexity can be found in [1], part 2.

Definition 2. Let $B \subset X$, $n \in \mathbf{N}$, and $n < \dim X$. We shall say that B is a Lipschitz fragment of dimension n (of codimension n, resp.) iff the following is satisfied:

There exist subspaces W and Z of X and a set $M \subset W$ such that

(i) $X = W \oplus Z$

(ii) $\dim W = n$ ($\operatorname{codim} W = n$, resp.)

(iii) $B = \{ w + F(w) : w \in M \}$ where $F: M \longrightarrow Z$ is a Lipschitz mapping.

($\oplus$ denotes a topological sum.)
Fragments with M=W are called surfaces.

Definition 3. Let $B \subset X$ be a Lipschitz fragment. We shall say that B is CFC-fragment (of the same dimension or codimension) iff W,Z,M,F from Definition 2 can be chosen in such way that for any mapping $c: S \longrightarrow M$ with $S \subset \mathbf{R}$ the following inequality holds:

$$\mathcal{K}(F \circ c, S) \leq a \cdot \mathcal{K}(c,S) + b \cdot \operatorname{Lip}(c),$$

where a and b are nonnegative constants independent on c and

$$\operatorname{Lip}(c) = \sup \left\{ \left\| \frac{c(s)-c(s')}{s-s'} \right\| : s,s' \in S,\ s \neq s' \right\}.$$

Theorem. Let T be a monotone operator on a separable Banach space X and $n < \dim X$ be a positive integer. Then A_n is representable as a union of countably many CFC-fragments of codimension n. If the dual space X^* is separable then A^n is representable as a countable union of CFC-fragments of dimension n.

Proof. We shall prove both the propositions of the theorem simultaneously. Without any loss of generality we can assume that T is maximal monotone, hence Tx is always convex.

There was proved in [1] that if X (or X^*, resp.) is separable then A_n (or A^n, resp.) can be written as a countable union of Lipschitz fragments B of codimension n (of dimension n, resp.), each of them having the following properties:

(I) $B=\{w+F(w): w\in M\}$, $M\subset W$, $F:M\longrightarrow Z$
where W,Z,M,F are as in Definition 2.

(II) There exist subspaces V, Y of X^* such that
$X^*=V\oplus Y$, $V=Z^{\perp}$, $Y=W^{\perp}$.

(III) For any $x\in B$ there exist $t_x\in Tx$ and a topological complement P_x of V in X^* such that
$\|t_x\|<m$, $\|\pi_x\|<q$, $(t_x+P_x)\cap\Omega(t_x,r)\subset Tx$
where $\pi_x:X^*\longrightarrow P_x$ is a projection in the direction of V and m,q,r are positive constants independent on $x\in B$.
(Our constants m, r correspond to constants $m+\frac{r}{2}$, $\frac{r}{2}$ from [1], 3.9.)

(IV) $t_x-t_y\in V$ for any $x,y\in B$.

Now it is sufficient to prove that B is in fact CFC-fragment.

Let $B\subset \mathbf{R}$ and $c:S\longrightarrow M$ be arbitrary. If card $S\leq 2$ then $\mathcal{K}(F\circ c,S)=0$ by Definition 1. So let card $S\geq 3$ and $s_0<s_1<\dots$ $\dots<s_{k+1}$, $s_j\in S$ $(j=0,1,\dots,k+1)$. Denote

$$w_j=c(s_j),\quad x_j=w_j+F(w_j)$$
$$t_j=t_{x_j},\quad \pi_j=\pi_{x_j}.$$

Let y^* be an arbitrary functional from a unit sphere in Y. Put

$$t_j^+=t_j+\frac{r}{q}\,\pi_j(y^*).$$

The fact $t_j^+\in Tx_j$ follows from (III).

Now for any $i,j\in\{0,1,\dots,k+1\}$, the monotonicity of T and properties (II),(III) imply:

$$0\leq\langle x_i-x_j,t_i-t_j^+\rangle=$$
$$=\langle w_i-w_j+F(w_i)-F(w_j),t_i-t_j+\frac{r}{q}(y^*-\pi_j(y^*))-\frac{r}{q}\cdot y^*\rangle=$$
$$=\langle w_i-w_j,t_i-t_j+\frac{r}{q}(y^*-\pi_j(y^*))\rangle-\frac{r}{q}\langle F(w_i)-F(w_j),y^*\rangle.$$

Hence

$$(1)\quad \langle F(w_i)-F(w_j),y^*\rangle\leq\langle w_i-w_j,\ \frac{q}{r}(t_i-t_j)+y^*-\pi_j(y^*)\rangle.$$

By the same way it is possible to obtain

$$(2)\quad -\langle F(w_i)-F(w_j),y^*\rangle\leq\langle w_i-w_j,\ \frac{q}{r}(t_i-t_j)-y^*+\pi_i(y^*)\rangle$$

using $0\leq\langle x_i-x_j,\ t_i^+-t_j\rangle$.

For simplicity let us denote $Q(j,i)=\frac{w_j-w_i}{s_j-s_i}$ if $i\neq j$. The inequalities (1),(2) give for any $j\in\{1,2,\dots,k\}$:

$$\left\langle \frac{F(w_{j+1})-F(w_j)}{s_{j+1}-s_j}-\frac{F(w_j)-F(w_{j-1})}{s_j-s_{j-1}},y^*\right\rangle \le$$

$$\le \langle Q(j+1,j)-Q(j,j-1),y^*-\sigma_j(y^*)-\frac{q}{r}\cdot t_j\rangle+\langle Q(j+1,j),\frac{q}{r}\cdot t_{j+1}\rangle -$$

$$-\langle Q(j,j-1),\frac{q}{r}\cdot t_{j-1}\rangle \le \|Q(j+1,j)-Q(j,j-1)\|\cdot(1+q+\frac{qm}{r})+$$

$$+\langle Q(j+1,j),\frac{q}{r}\cdot t_{j+1}\rangle-\langle Q(j,j-1),\frac{q}{r}\cdot t_{j-1}\rangle.$$

It is easy to see that $z^*\in Z^*$ iff there exists $\tilde{y}^*\in Y$ such that $\tilde{y}^*=z^*$ on Z.

Since y^* was an arbitrary functional with $\|y^*\|=1$ then

$$\left\|\frac{F(w_{j+1})-F(w_j)}{s_{j+1}-s_j}-\frac{F(w_j)-F(w_{j-1})}{s_j-s_{j-1}}\right\| \le$$

$$\le(1+q+\frac{qm}{r})\|Q(j+1,j)-Q(j,j-1)\|+\langle Q(j+1,j),\frac{q}{r}\cdot t_{j+1}\rangle-$$

$$-\langle Q(j,j-1),\frac{q}{r}\cdot t_{j-1}\rangle.$$

Taking the sum over $j=1,2,\dots,k$ we get

$$\sum_{j=1}^{k}\left\|\frac{F(w_{j+1})-F(w_j)}{s_{j+1}-s_j}-\frac{F(w_j)-F(w_{j-1})}{s_j-s_{j-1}}\right\| \le$$

$$\le(1+q+\frac{mq}{r})\cdot\sum_{j=1}^{k}\|Q(j+1,j)-Q(j,j-1)\|+\sum_{j=2}^{k+1}\langle Q(j,j-1),\frac{q}{r}\cdot t_j\rangle -$$

$$-\sum_{j=0}^{k-1}\langle Q(j+1,j),\frac{q}{r}\cdot t_j\rangle=(1+q+\frac{mq}{r})\sum_{j=1}^{k}\|Q(j+1,j)-Q(j,j-1)\| -$$

$$-\sum_{j=2}^{k-1}\langle Q(j+1,j)-Q(j,j-1),\frac{q}{r}\cdot t_j\rangle+\langle Q(k+1,k),\frac{q}{r}\cdot t_{k+1}\rangle +$$

$$+\langle Q(k,k-1),\frac{q}{r}\cdot t_k\rangle-\langle Q(2,1),\frac{q}{r}\cdot t_1\rangle-\langle Q(1,0),\frac{q}{r}\cdot t_0\rangle\le$$

$$\le(1+q+\frac{2mq}{r})\sum_{j=1}^{k}\|Q(j+1,j)-Q(j,j-1)\|+\frac{4mq}{r}\,\mathrm{Lip}(c)\le$$

$$\le(1+q+\frac{2mq}{r})\cdot\mathcal{K}(c,S)+\frac{4mq}{r}\,\mathrm{Lip}(c).$$

Then by Definition 1 we have

$$\mathcal{K}(F\circ c,S)\le a\cdot\mathcal{K}(c,S)+b\cdot\mathrm{Lip}(c)$$

where $a=1+q+\frac{2mq}{r}$ and $b=\frac{4mq}{r}$. The theorem is proved.

Remark. Problem 1.1 from [1], whether it is possible to write "δ-convex fragments" instead of "CFC-fragments" in Theorem, is still open.

Reference

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