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A NOTE ON COUNTABLY DETERMINED AND DISTINGUISHABLE SETS

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Abstract: We shall develop a theory of absolutely countably determined spaces (often called Lindelöf Σ-spaces, here we call them absolute Hausdorff sets) which is parallel to the theory of ω-analytic spaces (often called K-analytic). In particular, the first separation theorem for them is proved.

Key words: Hausdorff operation, Suslin operation, countably determined sets, Čech-analytic spaces, K-analytic, absolute Hausdorff sets.

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0. Introduction. Let $\omega^{<\omega}$ be the set of all finite non-void sequences ranging in ω, and let Σ be ω^{ω} with the product topology where ω is discrete. The space Σ is known to be homeomorphic with the space of irrational numbers. If M is a set-valued function from $\omega^{<\omega}$, then

$$\mathcal{S}M = \bigcup\{\bigcap\{M\sigma|n \mid n\in\omega\setminus\{0\}\}\mid \sigma\in\Sigma\}$$

is the Suslin set determined by M. Any set of the form

$$\bigcup\{\bigcap\{M\sigma|n \mid n\in\omega\setminus\{0\}\}\mid \sigma\in\Sigma'\},$$

with $\Sigma'\subset\Sigma$, is called a Hausdorff set determined by M. If $\mathcal{M}$ is a collection of sets then the set of all $\mathcal{S}M$ with M ranging in $\mathcal{M}$ is called the collection of Suslin-$\mathcal{M}$ sets, and it is denoted by $\mathcal{S}(\mathcal{M})$. Similarly we define the collection $\mathcal{H}(\mathcal{M})$ of Hausdorff-$\mathcal{M}$ sets. In what follows we develop a theory of $\mathcal{H}$(closed(X)) sets with X to be completely regular spaces, which is parallel to the theory of $\mathcal{S}$(closed(X)) sets, the latter being assumed to be known to the reader. On the other hand, if one does not insist to understand the assertions involving Suslin sets then the paper is self-contained. For the theory of absolute Suslin sets (which we call ω-analytic spaces, and which are often called analytic or K-analytic) we refer to [Fro_1] and [J-R]. The main result is Theorem 3.2 which is the first separation principle for absolutely Hausdorff sets (often called Lindelöf Σ-spaces or countably determined spaces).

1. Hausdorff and Suslin sets. A set X is said to be determined by a collection of sets $\mathcal{M}$ in Y if for each $x \in X$, and each $y \in Y \setminus X$ there exists an M in $\mathcal{M}$ with $x \in M$, $y \notin M$. If $\mathcal{N}$ is a collection of sets then by a Hausdorff set w.r.t. $\mathcal{N}$, or a Hausdorff-$\mathcal{N}$ set, we shall mean a set which is determined by a countable sub-collection of $\mathcal{N}$. Denote by $\mathcal{H}(\mathcal{N})$ the collection of all Hausdorff-$\mathcal{N}$ sets. It is obvious that

$$\mathcal{H}(\mathcal{N}) = \mathcal{H}(\mathcal{H}(\mathcal{N})) \supset \mathcal{N}_\delta \cup \mathcal{N}_\sigma$$

for each $\mathcal{N}$.

Denote by $\mathcal{S}(\mathcal{N})$ the collection of all Suslin-$\mathcal{N}$ sets, i.e. the collection of all sets which are obtained by the Suslin operation from sets in $\mathcal{N}$. Here we are interested just in the case when $\mathcal{N}$ =closed(X) for some topological space X. Recall that the Suslin sets in a space X, i.e. the elements of $\mathcal{S}$(closed(X)), are just the projections along the space Σ of the irrational numbers (conceived as the product space ω^ω) of closed subsets of $\Sigma \times X$.

For the Hausdorff sets in a topological space X, i.e. for the members of $\mathcal{H}$(closed(X)), we have the following analogous characterizations. Note that A. Archangelskij calls the Hausdorff sets in X the countably determined sets in X.

1.1. Lemma. For any space X, a subset Y of X is a Hausdorff set in X iff there exist a separable metric space P (which may be assumed a subspace of Σ) and a closed set C of $P \times X$ such that C projects onto Y.

Proof. If C is a closed subset of $P \times X$ which projects onto Y, and if $\{U_n\}$ is an open basis for P then the collection of all sets

$$\mathrm{c}\ell(\pi[(U_n \times X) \cap C])$$

determines Y in X, where π is the projection $P \times X \rightarrow X$ and cℓ

stands for the closure operator in X. On the other hand, if Y is determined by a sequence $\{F_n\}$ of closed sets in X, we define $F_{i_0 \ldots i_n} = \bigcap\{F_{i_k} \mid k \in n+1\}$, and for $\sigma \in \Sigma$ we put

$$F_\sigma = \bigcap\{F_s \mid s \subset \sigma\}.$$

Now if

$$\Sigma' = \{\sigma \mid F_\sigma \subset Y\}$$

then

$$C = (\bigcap\{\bigcup\{\Sigma s \times F_s \mid s \in \omega^n\} \mid n \in \omega \setminus \{0\}\}) \cap (\Sigma' \times X)$$

is a closed set in $\Sigma' \times X$ which projects onto Y; here Σs is the basic open set $\{\sigma \mid s \subset \sigma\}$ with $s \in \omega^{<\omega}$.

Example. If X is a separable metrizable space, or more generally, if X is separated and has a countable closed network, then each subset of X is a Hausdorff set in X.

All the following properties are obvious.

1.2. Permanence properties. The Hausdorff sets in any space are closed under the Hausdorff operation (since $\mathcal{H} \circ \mathcal{H} = \mathcal{H}$). If $X \hookrightarrow Y \hookrightarrow Z$, and if $Y \in \mathcal{H}(Z)$, $X \in \mathcal{H}(Y)$ then $X \in \mathcal{H}(Z)$. The preimages of Hausdorff sets under the continuous maps are Hausdorff. Finally, if X_n is Hausdorff in Y_n for each $n \in \omega$, then $\prod\{X_n\}$ is Hausdorff in $\prod\{Y_n\}$.

1.3. Existence of non-Hausdorff sets. It is obvious that the union of a discrete family of Suslin sets is Suslin. On the other hand, the discrete union of Hausdorff sets need not be Hausdorff. To show it we use the observation in Fundamental Lemma in [A-Š]. Let I be a separable metrizable space with at least $\mathbf{c}^+$ subsets. Let $X = \mathbf{c}^+ \times I$ have any topology such that each $\{\alpha\} \times I$ is a closed subspace of X; e.g. X may have the topology of the sum, and then the family $\{\{\alpha\} \times I\}$ is discrete. For any injection f of $\mathbf{c}^+$ into exp I the set

$$Y = \bigcup\{\{\alpha\} \times f\alpha \mid \alpha \in \mathbf{c}^+\}$$

is not Hausdorff. If Y were Hausdorff, then there would exist a separable metric space P and a closed set C in $P \times X$ which would project onto Y. But then each $f\alpha$ would be the projection of a closed set in $P \times I$; since $P \times I$ has at most $\mathbf{c}$ closed sets, this is a contradiction.

It should be noted that if P is a separable metric space and $\{Y_\alpha\}$ is a discrete family of P-Hausdorff sets in Y, i.e. if each Y_α is the projection of a closed set in $P \times Y$, then the union of $\{Y_\alpha\}$ is P-Hausdorff, hence Hausdorff.

Example. The Baire space $\varkappa^\omega$ ($\varkappa$ has the discrete topology) has a subset which is not Hausdorff iff $\varkappa > \mathbf{c}$. For "if" observe that if $\varkappa \leq \mathbf{c}$ then there exists a bijection into the separable space $(2^{\omega_0})^{\omega_0}$. If $\varkappa > \mathbf{c}$ then we can find $\mathbf{c}^+$ disjoint copies of 2^{ω_0} in $\varkappa^{\omega_0}$ and then the union of these copies contains a subset which is not Hausdorff. The same argument gives: if $X = \prod\{X_n \mid n \in \omega\}$, if each X_n has at least two points and if each subset of X is Hausdorff then the cardinal of X is $\leq \mathbf{c}$.

2. Absolute Hausdorff sets. For convenience we only consider completely regular separated spaces, i.e. the spaces admitting separated compactifications.

2.1. Theorem. The following conditions on a space X are equivalent:

(1) If $X \hookrightarrow Y$ then X is a Hausdorff set in Y.

(2) X is a Hausdorff set in some compact space.

(3) There exists an usco-compact correspondence of a separable metric space (which may be assumed to be a subspace of Σ) onto X.

(4) There exists a sequence $\{\mathfrak{M}_n\}$ of countable partitions of X such that each $\mathfrak{M}_{n+1}$ refines $\mathfrak{M}_n$, and for each x in X the sequence $\{M_n\}$, where $x \in M_n \in \mathfrak{M}_n$, converges to a compact set of X.

(5) There exists a countable cover $\mathcal{F}$ of X (which may be assumed to consist of closed sets in X) such that for each $x \in X$ the filter generated by $\{F \mid x \in F \in \mathcal{F}\}$ converges to a compact set in X.

Definition. A space X is said to be absolutely Hausdorff, or an AH-space, if X satisfies the equivalent conditions in the foregoing theorem.

It should be noted that AH-spaces are often called countably determined spaces ([T_2],[V]) or spaces with countable K-network, or Lindelöf Σ-spaces.

Notice that the conditions (1)-(4) in Theorem 2.1 correspond to the characterizations of ω-analytic (often called K-analytic spaces) which are the absolute Suslin sets. The equivalence of Conditions (1),(2) and (3) is well-known, and follows from Lemma 1.1, and the fact that a correspondence into a compact space is usco-compact (i.e. upper semi-continuous and compact-valued) if and only if the graph is closed. Also, the verification of the necessity and sufficiency of the two remaining conditions is easy, and is left to the reader. Note that Condition (5) says exactly that the space is a Lindelöf Σ-space.

2.2. Permanence properties. The class of all AH-spaces is closed under countable products, and taking the images under the usco-compact correspondences. In an AH-space a subset X is Hausdorff iff X is an AH-space in the subspace topology.

Proof. Easy.

The strength of AH is illustrated by the following (it seems non-trivial) result from [Fro_2] and its easy corollary.

2.3. Theorem. A space X is ω-analytic (i.e. K-analytic) iff X is AH and Čech-analytic.

Recall that following D. Fremlin [Fre] a space X is said to be Čech-analytic iff some Čech-complete space projects onto X along a separable metric space. It is shown in [Fre] that X is Čech-analytic iff

$$X \in \mathcal{S}(\mathrm{open}(K) \cup \mathrm{closed}(K))$$

for some, and then any, compactification K of X.

Corollary. The following conditions on a subset X of a compact space K are equivalent:

(1) X is a Baire set in K.

(2) X is a Borel set in K, and both X and $K \setminus X$ are Hausforff sets in K.

(3) Both X and $K \setminus X$ are Čech-analytic and Hausdorff in K.

Proof. Clearly (1) $\Longrightarrow$(2) $\Longrightarrow$(3), and (3) $\Longrightarrow$(1) follows from Theorem 2.3 and the first separation theorem for ω-analytic spaces (proved in 1960) which says that any two disjoint ω-analytic subspaces of any space can be separated by a Baire set.

Remark. Let w be the weak topology of a Banach space X. It is well-known that $\langle X,w\rangle$ is universally measurable. On the other hand, $\langle X,w\rangle$ need not be Čech-analytic; indeed, if $\langle X,w\rangle$ is AH but not ω-analytic (see [T_1] for such an example), then $\langle X,w\rangle$ is not Čech-analytic by the preceding theorem. It may be interesting to describe those Banach spaces which are Čech-analytic in the weak topology. The Banach spaces with the Čech-complete closed unit ball have been simply characterized by Edgar and Wheeler.

3. The first separation principle. A set X in a space Y is called distinguishable [Fro_2] if there exists a continuous mapping f of Y into a separable metric space such that $F^{-1}[f[X]] = X$. The collection Dstg(Y) of all distinguishable subsets of Y is complemented and closed under the Hausdorff operation, and if $f: Y \longrightarrow Z$ is continuous then

$$f^{-1}[\mathrm{Dstg}(Z)] \subset \mathrm{Dstg}(Y).$$

Countable products of distinguishable sets are distinguishable.

3.1. Proposition. If X is distinguishable in Y, then both X and $Y \setminus X$ are Hausdorff.

A set X in Y is called bi-Hausdorff if both X and $Y \setminus X$ are Hausdorff. Thus distinguishable sets are bi-Hausdorff. Now we are going to prove the first separation principle for AH-spaces. As a consequence we obtain the converse to 3.1 for AH-spaces X in 3.5 below.

Remark. The spaces X such that each subset of X is distinguishable, are quite interesting. A number of results about them is announced in [A-Š].

3.2. Theorem. If A is an AH-subspace of X, C is a Hausdorff set in X, and $A \cap C = \emptyset$ then there exists a distinguishable set B in X such that $A \subset B \subset X \setminus C$.

The proofs of Theorem 3.2 as well as of the next one are given in 3.4.

3.3. Theorem. If X is a normal countably compact space then any two disjoint Hausdorff sets in X can be separated by a distinguishable set (like in 3.2).

Example. If X is a countably compact space which is not normal then there are two closed disjoint sets F_1 and F_2 which cannot be separated by a distinguishable set. Indeed, it is easy to see that if two pseudocompact sets in a space are separated by a distinguishable set, then they are 0-1 separated, i.e. there exists a continuous function f on the space which is 0 on F_1 and 1 on F_2 (note that if F_1 and F_2 are distinguished by a continuous g, then $g[F_1]$ and $g[F_2]$ are disjoint compact sets).

3.4. Proofs of Theorems 3.2 and 3.3. Assume that A is determined in X by a countable collection $\mathcal{F}$ of closed sets, and C is determined by a countable collection $\mathcal{G}$ of closed sets in X. We may assume that both $\mathcal{F}$ and $\mathcal{G}$ are closed under formation of finite intersections. Denote by D the set of all pairs $\langle F,G\rangle$, $F\in\mathcal{F}$, $G\in\mathcal{G}$, such that there exists a continuous function $f_{F,G}$ on X which is 0 on F and 1 on G. Let f be the diagonal product of all f_{FG}, $\langle F,G\rangle\in D$, i.e.

$$fx=\{f_{FG}x \mid \langle F,G\rangle\in D\}.$$

We shall try to prove that $f[A]\cap f[C]=\emptyset$.

First assume that X is normal and countably compact, $x\in A$, $y\in C$. Let $\mathcal{F}_x=\{F \mid x\in F\in\mathcal{F}\}$, $\mathcal{G}_y=\{G \mid y\in G\in\mathcal{G}\}$. Since $A\cap C=\emptyset$, we have that

$$\bigcap\mathcal{F}_x\cap\bigcap G_y=\emptyset.$$

Since X is countably compact and $\mathcal{F}_x$ and $\mathcal{G}_y$ are centred, there exist an F in $\mathcal{F}_x$ and a G in $\mathcal{G}_x$ with $F\cap G=\emptyset$. Now by normality $\langle F,G\rangle\in D$ and hence $fx\neq fy$.

Now without any assumption on X, assume that A is AH. By Theorem 2.1 we may assume that $\mathcal{F}\cap A$ satisfies Condition (5) and $F=c\ell(F\cap A)$. But then, if $x\in A$, $y\in C$, $\mathcal{F}_x$ converges to a compact set K in A which is disjoint to the closed set $\bigcap\mathcal{G}_y$. Hence there exists $\langle F,G\rangle\in D$ with $F\in\mathcal{F}_x$, $G\in\mathcal{G}_y$.

As a corollary to Theorems 3.2 and 3.3 (see Proposition 3.1) we obtain immediately:

3.5. Theorem. In AH-spaces X the distinguishable and bi-Hausdorff sets coincide. The same is true for normal countably compact spaces X.

Denote by zero(X) the collection of all zero-sets in X, i.e. the null sets of continuous functions on X. It is easy to show that

$$\mathrm{Dstg}(X)=\mathcal{H}(\mathrm{zero}(X))$$

for any X. It follows that if closed(X) $\subset \mathcal{H}$(zero(X)) (in particular, if X is perfect), then self-evidently (remember that $\mathcal{H} \cup \mathcal{H} = \mathcal{H}$)

$$\mathcal{H}(\mathrm{zero}(X))=\mathcal{H}(\mathrm{closed}(X)),$$

and hence bi-Hausdorff sets are just the distinguishable sets.

It follows now from Theorem 3.5 that under the assumptions on X in Theorem 3.5 a subset of X is in $\mathcal{H}$(zero(X)) iff it is bi-Hausdorff in X.

Clearly each distinguishable ω-analytic subspace of any X is in $\mathcal{S}$(zero(X)). Hence, it follows from Theorem 3.5:

3.6. Theorem. If X is ω-analytic then a subset Y of X is in $\mathcal{S}$(zero(X)) if (and only if) Y is a Suslin set in X and $X \setminus Y$ is a Hausdorff set in X.

In particular, a space X is in $\mathcal{S}$(zero(K)) for some, and then any, compactification K of X, iff X is ω-analytic and $K \setminus Y$ is a Hausdorff set in some, and then any, compactification K of X.

3.7. Theorem. If X is ω-Luzin, then $Y \subset X$ is a Baire set in X iff Y is a ω-Luzin subspace and $X \setminus Y$ is Hausdorff in X iff Y is obtained by the disjoint Suslin operation from the closed sets of X, and $X \setminus Y$ is a Hausdorff set in X.

Proof. Theorem 3.6 says that in the second and the third conditions the set Y is distinguishable. The rest then follows from the old results of the author.

Problem: Assume that $\mathcal{P}$ is a partition of an AH-space X such that the union of each sub-collection of $\mathcal{P}$ is a Hausdorff set. Is it true that the cardinal of $\mathcal{P}$ is at most 2^{ω}?

It should be recalled that each completely Suslin-additive partition of an ω-analytic space is countable; this follows from the first separation theorem for ω-analytic spaces (see [F-H]).

Concluding remark. In the theory of ω-analytic spaces there are further concepts like ω-Luzin, point-ω-analytic and point-ω-Luzin. In terms of characterizations by means of usco-compact correspondences from Σ it means that the correspondence is, in addition, disjoint, or single-valued, or disjoint and single-valued, respectively. Clearly, the point-AH would be just a continuous image of a separable metric space, or equivalently, a space with a countable network. It should be noted that the old problem of Christensen, whether or not every point-AH-space is a subspace of a point-ω-analytic space, is still open (Calbrix and Bešlavić have given a"positive solution" in the class of l_2 spaces). I have found nothing interesting in the routine development of the concepts analogous to ω-Luzin and point-ω-Luzin. On the other hand, it seems to be of interest to develop some "non-separable" theories of $\mathcal{H}$. For the theory analogous to analytic in the sense of [F-H] it goes smoothly. The more general theories of "analytic" are not understood well as yet.

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