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Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 3, 413--419

Persistent URL: http://dml.cz/dmlcz/106553

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,3 (1987)

A NON ERGODIC VERSION OF GORDIN'S CLT FOR INTEGRABLE STATIONARY PROCESSES

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Abstract: The non ergodic version of Gordin's central limit theorem for strictly stationary sequences of integrable sequences of integrable random variables is proved. The result is obtained by using the fact that the ergodic decomposition of an invariant measure preserves some important properties of the stationary process.

Key words: Central limit theorem, strictly stationary sequence of random variables, ergodic measure.

Classification: 60F05, 60G10, 60G42, 28D05

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1. Introduction and results. Let $(\Omega,\mathcal{A},\mu)$ be a probability space where $\mathcal{A}$ is a σ-algebra of subsets of Ω and μ is a probability measure on $(\Omega,\mathcal{A})$. T is a 1-1 measurable and measure preserving transformation of Ω onto itself. It is well known that for any measurable function f on Ω, $(f\circ T^i)$ is a strictly stationary process and, for any strictly stationary process (X_i), $(\Omega,\mathcal{A},\mu)$, T and f exist such that (X_i) and $(f\circ T^i)$ have the same distribution. We shall deal with the central limit problem for the process $(f\circ T^i)$ here. From now on, f will be a fixed integrable function on Ω.

Let us recall that the measure μ is ergodic iff each invariant set $A\in\mathcal{A}$ (i.e. such that A=TA) has measure 1 or 0. A σ-algebra $\mathcal{M}\subset\mathcal{A}$ will be said to be invariant if $\mathcal{M}\subset T^{-1}\mathcal{M}$. From now on, $\mathcal{M}$ will be a fixed invariant σ-algebra. We shall define $S_n(f)=\sum_{j=1}^{n} f\circ T^j)$ and $s_n(f)=S_n(f)/\sqrt{n}$. One of the strongest central limit theorems for stationary processes is the following one (see [5]).

Theorem 1. (M.I. Gordin.) If the measure μ is ergodic and

(i) $\sum_{j=0}^{\infty} E(|E(f|T^j\mathcal{M})|+|f-E(f|T^{-j}\mathcal{M})|)<\infty$,

(ii) $\limsup_{n\to\infty} E|s_n(f)|<\infty$,

then there exists $\sigma=\lim_{n\to\infty}E|s_n(f)|$ and the measures $\mu s_n^{-1}(f)$ weakly converge to a distribution with a characteristic function $\varphi(t)=\exp(-\frac{\pi}{2}\sigma^2)$.

The theorem was communicated by M.I. Gordin at the Vilnius Conference on Probability Theory in 1973. The original proof remained almost unknown and a new one appeared in the monography [5]; the proof from [5] was corrected in [2] (see also [6]).

Our main aim is to prove a non ergodic version of Theorem 1. Let $\mathcal{J}$ be the σ-algebra of all invariant sets from $\mathcal{A}$.

Theorem 2. If

(i) $\sum_{j=0}^{\infty} E(|E(f|T^{j}\mathcal{M})|+|f-E(f|T^{-j}\mathcal{M})||\mathcal{J})<\infty$ almost everywhere $[\mu]$ and

(ii) $\limsup_{n\to\infty} E(|s_n(f)||\mathcal{J})<\infty$ almost everywhere $[\mu]$,

then there exists $h=\lim_{n\to\infty} E(|s_n(f)||\mathcal{J})$ and the measures $\mu s_n^{-1}(f)$ converge weakly to the distribution with a characteristic function $\varphi(t)=E\exp(-\frac{\pi}{2}h^2t^2)$.

2. An ergodic decomposition of μ and conditional expectations. We shall reduce our problem to the case when a family $(m_\omega;\ \omega\in\Omega)$ of regular conditional probabilities with respect to μ induced by $\mathcal{J}$ exists. It will turn out that m_ω are invariant and ergodic probability measures and the assumptions of Theorem 2 are preserved in each probability space $(\Omega,\mathcal{A},m_\omega)$. The $\mathcal{J}$-measurable functions become constants in these spaces and the assumptions of Theorem 2 will thus be reduced to those of Theorem 1.

First, let us give some definitions. For a collection $\mathcal{G}$ of measurable sets, $\sigma\mathcal{G}$ will denote the smallest σ-algebra containing $\mathcal{G}$; if a σ-algebra $\mathcal{C}$ contains a countable collection $\mathcal{G}$ such that $\mathcal{C}=\sigma\mathcal{G}$, we shall say that $\mathcal{C}$ is separable (compare [7]). For a measurable function g, let us denote $\sigma\{g\}=\sigma\{g^{-1}(A)$: A is an interval$\}$. Given measurable functions g' and g'', $g'=g''$ mod μ means that $\mu\{g'=g''\}=1$. For σ-algebras $\mathcal{C},\mathcal{D}$, $\mathcal{C}\subset\mathcal{D}$ mod μ means that for any $A\subset\mathcal{C}$ there exists a set $A'\in\mathcal{D}$ such that $\mu(A\triangle A')=0$ where $\triangle$ denotes the symmetrical difference.

Let $\mathcal{B}^{\mathbb{Z}}$ be the Borel σ-algebra on $\mathbb{R}^{\mathbb{Z}}$ and S be the shift transformation on $\mathbb{R}^{\mathbb{Z}}$ (i.e. $(S\omega)_i=\omega_{i+1}$). We shall show that if $\mathcal{A}$ is separable, we can restrict ourselves to the case $\Omega=\mathbb{R}^{\mathbb{Z}}$, $\mathcal{A}=\mathcal{B}^{\mathbb{Z}}$ and T=S.

Let $\mathcal{A}$ be separable. Then there exists a mapping $\psi:\Omega\longrightarrow\mathbb{R}^{\mathbb{Z}}$ such that $\psi\circ T=S\circ\psi$ and $\psi^{-1}(\mathcal{B}^{\mathbb{Z}})=\mathcal{A}$.

Proof: There exists a function g on Ω such that $\mathcal{A}=\sigma\{g\}$. The mapping $\psi:\Omega\longrightarrow\mathbb{R}^{\mathbb{Z}}$ defined by $(\psi\omega)_j=g(T^j\omega)$ is the one we need.

Lemma 1. For any $\mathcal{A}$-measurable function g there exists a $\mathcal{B}^{\mathbb{Z}}$-measurable function $\bar{g}$ such that $g=\bar{g}\circ\psi$. If $\bar{\mathcal{J}}$ is the σ-algebra of S-invariant sets from $\mathcal{B}^{\mathbb{Z}}$, we have $\psi^{-1}\bar{\mathcal{J}} = \mathcal{J}$ and $E_{\mu}(f|\mathcal{J}) = E_{\psi^{-1}\mu}(\bar{f}|\bar{\mathcal{J}}) \circ \psi$ mod μ.

Thus, our claim that (in the case of separable $\mathcal{A}$) we can restrict ourselves to $\Omega = \mathbb{R}^{\mathbb{Z}}$ is justified. Recall that in that case there exists a family $(m_{\omega};\ \omega \in \Omega)$ of regular conditional probabilities w.r.t. μ which is induced by $\mathcal{J}$ (see [7]).

Proof of Lemma 1: For any $n \in N$ (where N is the set of all positive integers) and $i \in \mathbb{Z}$ let $A(n,i) = \{\omega : i\ 2^{-n} \leq g(\omega) < (i+1)\cdot 2^{-n}\}$, $g_n = \sum_{i \in \mathbb{Z}} \chi_{A(n,i)} \cdot i/2^n$. There exist sets $\bar{A}(n,i) \in \mathcal{B}^{\mathbb{Z}}$ such that $A(n,i) = \psi^{-1}(\bar{A}(n,i))$. Let $\bar{C} = \bigcup\{\bar{A}(n,i) \cap \bar{A}(n,j) : n \in N,\ i,j \in \mathbb{Z},\ i \neq j\}$ and $\bar{D} = \bigcup\{\bar{A}(n,i) \triangle (\bar{A}(n+1,2i) \cup \bar{A}(n+1,2i+1)) : n \in N, i \in \mathbb{Z}\}$. It holds $\psi^{-1}\bar{C} = \psi^{-1}\bar{D} = \emptyset$. Let $\hat{A}(n,i) = \bar{A}(n,i) \setminus (\bar{C} \cup \bar{D})$ and $\bar{g}_n = \sum_{i \in \mathbb{Z}} \chi_{\hat{A}(n,i)} \cdot i/2^n$. We have $g_n \uparrow g$, $g_n = \bar{g}_n \circ \psi$ and $\bar{g}_n \leq \bar{g}_{n+1} \leq \dots$. For $\bar{g} = \lim_{n \to \infty} \bar{g}_n$ it is $g = \bar{g} \circ \psi$.

The proof of the other two statements of Lemma 1 is obvious and we shall omit it.

Now, we shall give three auxiliary statements belonging to measure theory. The first one can be found in [7]. It might be easier to prove the other ones than to seek a reference.

(1) Let g be an integrable function. Then $E(g|\mathcal{J})(\omega) = \int g\, dm_{\omega}$ for almost all $[\mu]$ $\omega \in \Omega$.

(2) For any $\mathcal{J}$-measurable function g it holds $g = g(\omega)$ mod m_{ω} for almost all $[\mu]$ $\omega \in \Omega$. Hence, for an arbitrary integrable g we have $E(g|\mathcal{J}) = \int g\, dm_{\omega}$ mod m_{ω} for almost all $[\mu]$ $\omega \in \Omega$.

Proof. It is $\sigma\{g\} \subset \mathcal{J}$. Let $\mathcal{G}$ be a countable collection of sets from $\sigma\{g\}$ such that $\sigma\mathcal{G} = \sigma\{g\}$. For any $A \in \mathcal{G}$, $\mu(A|\mathcal{J}) = \chi_A$ mod μ and (following (1)) $\mu(A|\mathcal{J})(\omega) = m_{\omega}(A)$ a.s. $[\mu]$. For almost all $[\mu]$ $\omega \in \Omega$ we thus have $m_{\omega}(A) = 1$ for $\omega \in A$ and $m_{\omega}(A) = 0$ for $\omega \notin A$. This holds for any other set $A \in \sigma\{g\}$, too, so $m_{\omega}\{\omega' : g(\omega') = g(\omega)\} = 1$.

(3) Let $\mathcal{F}$ be a separable σ-algebra such that $\mathcal{J} \subset \mathcal{F}$ mod μ and g be an integrable function. Then $E(g|\mathcal{F}) = E_{m_{\omega}}(g|\mathcal{F})$ mod m_{ω} for almost all $[\mu]$ $\omega \in \Omega$.

Remark. The σ-algebra $\mathcal{A} = \mathcal{B}^{\mathbb{Z}}$ is separable so there exists a separable σ-algebra $\mathcal{J}' \subset \mathcal{J}$ such that $\mathcal{J} \subset \mathcal{J}'$ mod μ; the assumption $\mathcal{J} \subset \mathcal{F}$ mod μ can be thus fulfilled. However, the σ-algebra $\mathcal{J}$ is not separable itself. Let the contrary hold and let $\mathcal{G}$ be a countable collection such that $\sigma\mathcal{G} = \mathcal{J}$. For any $\omega \in \Omega$, let A_ω be the intersection of all $A \in \mathcal{G}$ such that $\omega \in A$. Hence, A_ω has no nonempty proper $\mathcal{J}$-measurable subset. The orbit $\{T^i\omega : i \in \mathbb{Z}\}$ is $\mathcal{J}$-measurable hence it is equal to A_ω. In the same way in which we proved (2) we can prove that $m_\omega(A_\omega)=1$ for almost all $[\mu]$ $\omega \in \Omega$. So, the sets A_ω must be finite, i.e. there exist natural numbers $n(\omega)$ such that $T^{n(\omega)}\omega = \omega$. Hence any aperiodic measure μ will make a contradiction.

On the other hand, let us put $\mathcal{J}' = \sigma\{\liminf_{n \to \infty} S_n(\chi_A) : A \in \mathcal{G}'\}$ where $\mathcal{G}'$ is a countable algebra generating $\mathcal{A}$. Following the remark after Lemma 2 we can prove that $\mathcal{J}' = \mathcal{J}$ mod ν for each invariant probability measure ν (and of course, $\mathcal{J}' \subset \mathcal{J}$ is a separable σ-algebra).

Proof of (3). Let $\mathcal{G}$ be a countable algebra of sets from $\mathcal{F}$ such that $\sigma\mathcal{G} = \mathcal{F}$. For any $A \in \mathcal{G}$ we have $E(\chi_A \cdot E(g|\mathcal{F})|\mathcal{J}) = E(\chi_A \cdot g|\mathcal{J})$ mod μ, so (following (1)) $\int_A E(g|\mathcal{J})\, dm_\omega = \int_A g\, dm_\omega$ for almost all $[\mu]$ $\omega \in \Omega$. The system of sets for which the last equality holds, forms a monotone class containing $\mathcal{G}$.

Remark. The condition $E(\chi_A \cdot E(g|\mathcal{F})|\mathcal{J}) = E(\chi_A \cdot g|\mathcal{J})$ mod μ is for $E(g|\mathcal{F}) = E_{m_\omega}(g|\mathcal{F})$ mod m_ω for almost all $[\mu]$ $\omega \in \Omega$ necessary and sufficient (as R. Yokoyama pointed out, see [12]).

Following the Kryloff-Bogoliouboff theory (see [8]) there exist ergodic probability measures with disjoint supports which form a decomposition of any invariant probability measure. Here we shall use the fact that these measures can be obtained as regular conditional probabilities induced by the σ-algebra $\mathcal{J}$.

Lemma 2. For almost all $[\mu]$ $\omega \in \Omega$, m_ω is an invariant and ergodic probability measure.

Proof. Let $\mathcal{G}$ be a countable algebra such that $\mathcal{A} = \sigma\mathcal{G}$. For $A \in \mathcal{G}$ we have $\mu(T^{-1}A|\mathcal{J}) = \mu(A|\mathcal{J})$ mod μ, hence $m_\omega(A) = m_\omega(T^{-1}A)$ for almost all $[\mu]$ $\omega \in \Omega$. The collection of sets $A \in \mathcal{A}$ for which the previous equality holds, forms a monotone class containing $\mathcal{G}$, so m_ω are invariant measures.

For any $A \in \mathcal{G}$ we have $S_n(\chi_A) \longrightarrow \mu(A|\mathcal{J})$ pointwise (by Birkhoff's ergodic theorem). Following (1) and (2) we thus have $S_n(\chi_A) \longrightarrow m_\omega(A)$ almost surely $[m_\omega]$. According to [1], Theorem 1.4, p. 17, the measure m_ω is ergodic.

Remark. For each $\omega \in \Omega$ let $B_\omega = \{\omega' : S_n(\chi_A)(\omega') \longrightarrow m_\omega(A)\}$ for each $A \in \mathcal{G}$. Then $B_\omega \in \mathcal{J}$ and it is a support of the ergodic probability measure m_ω. For different measures m_ω, the sets B_ω are disjoint. Using regular conditional probabilities we can thus partially mimick the theory given in [8].

The following lemma is basic for the proof of Theorem 2.

Lemma 3. Let $\mathcal{M}$ be an invariant and separable σ-algebra and let the integrable function g be measurable with respect to $\bigvee_{i\in\mathbb{Z}} T^i\mathcal{M}$ $(=\sigma(\bigcup_{i\in\mathbb{Z}} T^i\mathcal{M}))$. Then $E(g|T^i\mathcal{M})=E_{m_\omega}(g|T^i\mathcal{M}) \bmod m_\omega$ for almost all $[\mu]$ $\omega\in\Omega$.

Corollary. If $(g\circ T^i)$ is a martingale in $L^p(\mu)$, $1\le p\le\infty$, then it is also a martingale in $L^p(m_\omega)$ for almost all $[\mu]$ $\omega\in\Omega$.

Proof of Lemma 3. Following [10] and [11] we have

(4) $E(g|T^i\mathcal{M})=E(g|\mathcal{J}\vee T^i\mathcal{M}) \bmod \mu$.

Let $\mathcal{J}'\subset\mathcal{J}$ be a separable σ-algebra such that $\mathcal{J}\subset\mathcal{J}' \bmod \mu$ and $\overline{\mathcal{M}}=\mathcal{M}\vee\mathcal{J}$. Following (4),(3) and (4) it holds $E(g|T^i\mathcal{M})=E(g|T^i\overline{\mathcal{M}})=E_{m_\omega}(g|T^i\overline{\mathcal{M}})=$ $=E_{m_\omega}(g|T^i\mathcal{M}) \bmod m_\omega$ for almost all $[\mu]$ $\omega\in\Omega$.

Remarks.

1) Without the assumption that g is measurable w.r.t. $\bigvee_{i\in\mathbb{Z}} T^i\mathcal{M}$ (4) need not hold.

2) R. Yokoyama (see [12]) proved Lemma 3 using the equation $E(g|\mathcal{J}\vee\mathcal{M}_{-\infty})=$ $=E(g|\mathcal{M}_{-\infty}) \bmod \mu$, $\mathcal{M}_{-\infty}=\bigcap_{i\in\mathbb{Z}} T^i\mathcal{M}$, instead of (4).

Our aims are almost fulfilled now: Using the results given up to now it can be shown that the assumptions of Theorem 2 are preserved in $(\Omega,\mathcal{A},m_\omega)$ and m_ω's are invariant and ergodic probability measures. The last lemma will eventually assure us that we can pick the limit theorems which hold in probability spaces $(\Omega,\mathcal{A},m_\omega)$ together and we obtain a result in $(\Omega,\mathcal{A},\mu)$.

Lemma 4. Let g be a measurable function and let for almost each $[\mu]$ $\omega\in\Omega$ the measures $m_\omega s_n^{-1}(g)$ weakly converge to a probability measure with a characteristic function φ_ω. Then the measures $\mu s_n^{-1}(g)$ weakly converge to a measure with a characteristic function $\varphi(t)=\int\varphi_\omega(t)\,d\mu(\omega)$.

The proof is an easy application of the Lebesgue dominated convergence theorem.

3. Proof of Theorem 2. Let $\overline{\mathcal{C}}=\sigma\{f,E(f|T^i\mathcal{M}):i\in\mathbb{Z}\}$ and $\mathcal{C}=\bigvee_{i\in\mathbb{Z}} T^i\overline{\mathcal{C}}$. $\mathcal{C}$ is a separable σ-algebra (we consider $E(f|T^i\mathcal{M})$ as a function here) and

$T^{-1}\mathcal{C} = \mathcal{C} = T\mathcal{C}$. The σ-algebra $\mathcal{M} \cap \mathcal{C}$ is invariant and for each $i \in \mathbb{Z}$, $E(f|T^{i}\mathcal{M}) = E(f|\mathcal{C} \cap T^{i}\mathcal{M}) \bmod \mu$. Following the Birkhoff ergodic theorem we have $E(g|\mathcal{J}) = E(g|\mathcal{C} \cap \mathcal{J}) \bmod \mu$ for each integrable and $\mathcal{C}$-measurable g. We can thus suppose that $\mathcal{A} = \mathcal{C}$ and that $\mathcal{M}$ is separable. Following Lemma 1 we can suppose that $\Omega = \mathbb{R}^{\mathbb{Z}}$ and $\mathcal{A} = \mathcal{B}^{\mathbb{Z}}$. Let us put $g = \sum_{j=0}^{\infty} E(|E(f|T^{j}\mathcal{M})| + |f - E(f|T^{-j}\mathcal{M})| \,|\, \mathcal{J})$.

a) Let us suppose that g is integrable. Then we have $E(f|T^{-i}\mathcal{M}) \to f$ in $L^{1}(\mu)$ so we can suppose that f is $\bigvee_{i \in \mathbb{Z}} T^{-i}\mathcal{M}$ -measurable. According to [7] there exists a family $(m_{\omega};\ \omega \in \Omega)$ of regular conditional probabilities induced by $\mathcal{J}$. By Lemma 2, m_{ω} are invariant and ergodic probability measures, by Lemma 3, $\sum_{j=0}^{\infty} E_{m_{\omega}}(|E_{m_{\omega}}(f|T^{j}\mathcal{M})| + |f - E_{m_{\omega}}(f|T^{-j}\mathcal{M})|) = E_{m_{\omega}} g$, and $\limsup_{n\to\infty} E_{m_{\omega}}|s_n(f)| < \infty$ for almost all $[\mu]$ $\omega \in \Omega$. From Theorem 1 we get that there exists $h(\omega) = \lim_{n\to\infty} E_{m_{\omega}}|s_n(f)|$ and the measures $m_{\omega}s_n^{-1}(f)$ weakly converge to a probability measure with a characteristic function $\varphi_{\omega}(t) = \exp(-\frac{\pi}{2} h^{2}(\omega)t^{2})$. From this and from Lemma 4 we obtain the statement of Theorem 2.

b) Let g be not integrable. The sets $A_k = \{\omega : k-1 \le g(\omega) < k\}$, $k=1,2,\dots$ form an $\mathcal{J}$-measurable partition of Ω and if $\mu(A_k) > 0$ then the measure $\mu_k(\cdot) = \mu(\cdot|A_k)$ is invariant. From a) it follows that there exist $h_k = E_{\mu_k}(|S_n(f)| \,|\, \mathcal{J}) \bmod \mu_k$ and the measures $\mu_k s_n^{-1}(f)$ weakly converge to a measure with a characteristic function $\varphi_k(t) = E_{\mu_k}\exp(-\frac{\pi}{2} h_k^{2}t^{2})$. There exists a function h such that $h = h_k \bmod \mu_k$ for each k such that $\mu(A_k) > 0$ and the statement of Theorem 2 follows.

4. Concluding remarks. The method used for the proof of Theorem 2 was developed in the Ph.D. Thesis [10]. In [10] it was used for a proof of a nonergodic version of Gordin's CLT for square integrable stationary processes from [4]. In [12], this approach was used for deriving a log log law for strictly stationary matingales starting from Stout's solution [9] of the ergodic case (R. Yokoyama made some technical changes; the result can, however, be obtained by the technique developed in Section 2 as well). On the other hand, from the weak convergence of measures $\mu s_n^{-1}(f)$, the convergence of measures $m_{\omega}s_n^{-1}(f)$ for almost all $[\mu]$ $\omega \in \Omega$ does not follow.

References

[1] P. BILLINGSLEY: Ergodic Theory and Information, Wiley, New York, 1965.

[2] C.G. ESSEEN and S. JANSON: On moment conditions for normed sums of independent variables and martingale differences, Stoch. proc. and their appl. 19(1985), 173-182.

[3] M.I. GORDIN: Abstracts of communications, T.1: A-K, International conference on probability theory, Vilnius, 1973.

[4] M.I. GORDIN: The central limit theorem for stationary processes, Soviet Math. Dokl. 10(1969), 1174-1176.

[5] P. HALL and C.C. HEYDE: Martingale Limit Theory and its Applications, Academic Press, New York, 1980.

[6] N. HERRNDORF: Stationary strongly mixing sequences not satisfying the central limit theorem, Ann. Probability 11(1983), 809-813.

[7] M. LOÈVE: Probability Theory, Van Nostrand, New York, 1955.

[8] J.C. OXTOBY: Ergodic sets, Bulletin of the Amer. Math. Soc. 58(1952), 116-136.

[9] W.F. STOUT: The Hartman-Wintner law of the iterated logarithm for martingales, Ann. Math. Statist. 41(1970), 2158-2160.

[10] D. VOLNÝ: The central limit problem for strictly stationary sequences, Ph.D. thesis 1984, Mathematical Institute, Charles University, Prague (in Czech).

[11] D. VOLNÝ: Approximation of stationary processes and the central limit problem, Proceedings of the Japan-USSR Symposium on probability theory, Kyoto (1986).

[12] D. VOLNÝ, R. YOKOYAMA: On the law of iterated logarithm for martingales, submitted for publication.

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This work was carried out when the author was on leave from Charles University at Osaka Kyioiku University (Japan).

The author would like to express his thanks to Professor Ryozo Yokoyama for helpful discussions.

(Oblatum 8.4. 1987)