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# A NOTION OF MEASURE FOR CLASSES IN AST

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Abstract: The idea of approximating semisets by sets from within and from without is quite natural and analogous to that of the inner and outer measure of measure theory, where in the place of real numbers we now have cuts of natural numbers. However, not too a large part of the classical theory is expected to be saved under this analogy, a fact due to the rather crude structure of cuts. Finer results are obtained if we suppose that the cuts satisfy certain closure properties.

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N, FN are the classes of natural numbers and finite natural numbers respectively. We use a,b,c,... to denote elements of the first class and m,n,k, ... for elements of FN. Lower Greek letters $\alpha, \beta, \gamma, \ldots$ are reserved for ordinals. I,J,... denote cuts. For any set u, $|u|$ is the unique $a \in N$ such that $u \mathrel{\hat{\approx}} a$.

Given a class X let

$o(X) = \{a \in N; (\forall u)(X \subseteq u \rightarrow a < |u|)\}$ for X being a semiset,

$\quad = N$ for any proper class X;

$i(X) = \{a \in N; (\exists u)(u \subseteq X \,\&\, a = |u|)\}$

be the outer measure and inner measure of X respectively.

o(X), i(X) are, evidently, initial segments of N and $o(X)=i(X)=a \in N$ iff $X=u$ and $|u|=a$. In all other cases o(X), i(X) are cuts of N and, clearly, $i(X) \subseteq o(X)$.

To give some obvious examples:

a) For the universe V, o(V)=i(V)=N.

b) o(FN)=i(FN)=FN.

c) For any cut I, o(I)=i(I)=I.

d) $o(\Omega)=N$, $i(\Omega)=FN$, where $\Omega$ is the class of ordinals.

Definition 1. The class X is said to be measurable if o(X)=i(X) and, in such a case, the common cut I=o(X)=i(X) is called the measure of X and is denoted by $\mu(X)$.

Lemma 2. i) If X is not a semiset, then o(X)=N.

ii) Every proper set-definable class is measurable of measure N.

iii) Every I-class, i.e. every class f"I for some 1-1 function f, where I is a cut, is measurable of measure I.

iv) $X\subseteq Y$ implies $o(X)\leq o(Y)$ and $i(X)\leq i(Y)$.

v) If $X=\bigcup_{n} X_n$ is a $\Sigma$-class, then X is measurable and $\mu(X)=\bigcup_{n}\mu(X_n)$.

vi) If $X=\bigcap_{n} X_n$ is a $\Pi$-class, then X is measurable and $\mu(X)=\bigcap_{n}\mu(X_n)$.

Proof. i) - iv) are trivial. v) Let $X=\bigcup_{n} X_n$ with $(X_n)_n$ increasing. If some $X_n$ is a proper class then X is not a semiset and o(X)=N by i). On the other hand, $i(X)\geq i(X_n)=N$. Hence $\mu(X)=N=\bigcup_{n}\mu(X_n)$. Suppose X is a $\Sigma$-semiset, that is $X=\bigcup_{n} u_n$ with $(u_n)_n$ increasing and let $|u_n|=a_n$. Since $u\subseteq\bigcup_{n} u_n$ iff $(\exists n)(u\subseteq u_n)$ we get $i(X)=\bigcup_{n} a_n$. It suffices to show that $o(X)=\bigcup_{n} a_n$, that is,

$$a>\bigcup_{n} a_n\rightarrow(\exists u)(\bigcup_{n} u_n\subseteq u\ \&\ |u|\leq a).$$

But this is an immediate consequence of the prolongation axiom.

vi) Let $X=\bigcap_{n} u_n$ with $(u_n)_n$ decreasing and let $|u_n|=a_n$. Clearly

$$i(X)\leq o(X)\leq\bigcap_{n} a_n.$$

It suffices to show that $\bigcap_{n} a_n\leq i(X)$.

Let $a\in\bigcap_{n} a_n$. Since $|u_n|=a_n$, by the prolongation axiom we can find $v\subseteq\bigcap_{n} u_n$ such that $a=|v|$. Thus, $a\in i(X)$.

Now let $X=\bigcap_{n} X_n$ and each $X_n$ is proper. Let $V_a=\{x;|x|=a\}$ for every $a\in N$. $V_a$ are set-definable and given a,

$$V_a\cap P(X_n)\neq\emptyset\quad(\text{where } P(X)=\{x;x\subseteq X\})$$

for every $n\in FN$. Then, $V_a\cap(\bigcap_{n} P(X_n))\neq\emptyset$, hence $V_a\cap P(\bigcap_{n} X_n)\neq\emptyset$, which means that $a\in i(\bigcap_{n} X_n)$. Therefore $i(\bigcap_{n} X_n)=N=\mu(X)=\bigcap_{n}\mu(X_n)$. □

Lemma 3. If $(X_n)_n$ is a decreasing sequence of fully revealed classes which are measurable, then $\bigcap_{n} X_n$ is measurable and $\mu(\bigcap_{n} X_n)=\bigcap_{n}\mu(X_n)$.

Proof. Let $\mu(X_n)=I_n$. Then $i(\bigcap_{n} X_n)\subseteq\bigcap_{n} I_n$. Let $a\in\bigcap_{n} I_n$. Then if $V_a=\{x;|x|=a\}$, $V_a\cap P(X_n)\neq\emptyset$ for all $n\in FN$. Since $\bigcap_{n} P(X_n)=P(\bigcap_{n} X_n)$, by full revealness we have $V_a\cap P(\bigcap_{n} X_n)\neq\emptyset$. Hence $a\in i(\bigcap_{n} X_n)$. Therefore $\bigcap_{n} I_n\subseteq i(\bigcap_{n} X_n)\subseteq o(\bigcap_{n} X_n)$. □

From now on we shall consider semisets only, that is, subclasses of a given fixed set w with |w|=d. This is analogous to the practice of studying measures of subsets of a given interval of the real line, say [0,1].

We sometimes write -X for the class $w\setminus X$.

If I is a cut and $I < d$, let us put

$$d-I = d \setminus \{d-a; a \in I\}$$

(d-I is not to be confused with the set theoretic difference $d \setminus I$). It is not hard to see that d-I is a cut.

Theorem 4. 1) $d-I = \{d-x; x > I\}$ and if I is closed under addition then $I < d-I$.

2) $I \subsetneq J \rightarrow d-J \subsetneq d-J$

3) $d-(d-I) = I$

4) For $X \subseteq q$, $o(-X) = d-i(X)$ and $i(-X) = d-o(X)$

5) X is measurable iff -X is measurable and $\mu(-X) = d-\mu(X)$.

Proof. 1) and 2) are straightforward.

3) Let $x \notin d-(d-I)$. Then $x = d-y$ for some $y \in d-I$, that is $y < d-z$ for all $z \in I$ or $d-y > z$ for all $z \notin I$. Thus $d-y = x \notin I$.

Conversely, let $x \notin I$. Then $x = d-(d-x)$ and since $d-x \notin \{d-y; y \in I\}$, $d-x \in d-\{d-y; y \in I\} = d-I$. Therefore $x = d-(d-x) \in \{d-z; z \in d-I\}$, consequently, $x \notin d-\{d-z; z \in d-I\} = d-(d-I)$.

4) We prove the first equality. The other follows from 1) and 2). Let $x \notin d-i(X)$. Then $x = d-a$ for some $a \in i(X)$. Take $v \subseteq X$ with $|v| = a$. Then $-v \supseteq -X$ and $|-v| = d-a = x$. Thus $x \notin o(-X)$. The converse is similar.

5) Immediate from 3). □

Given cuts I, J let us put

$I+J = \{a+b; a \in I \& b \in J\}$

$I \cdot J = \{x \subsetneq a \cdot b; a \in I \& b \in J\}$

I+J and I • J are obviously cuts, the sum and product respectively of I, J.

The semisets X, Y are called separable if there are sets $v_1$, $v_2$ such that $X \subseteq v_1$, $Y \subseteq v_2$ and $v_1 \cap v_2 = \emptyset$.

Theorem 5. If X, Y are separable, then $i(X \cup Y) = i(X) + i(Y)$ and $o(X \cup Y) = o(X) + o(Y)$. If, moreover X, Y are measurable, then $X \cup Y$ is measurable, of measure $\mu(X) + \mu(Y)$.

Proof. We show that $i(X \cup Y) \subseteq i(X) + i(Y)$ (the converse is straightforward). Let $u \subseteq X \cup Y$ with X,Y. Then, clearly $u \cap X = u \cap v_1$, $u \cap Y = u \cap v_2$. If $|u \cap v_1| = a_1$, $|u \cap v_2| = a_2$, then $a = a_1 + a_2$ hence $a \in i(X) + i(Y)$.

Let $a \in o(X)$, $b \in o(Y)$. Then $a < |v|$ $\forall v \supseteq X$, and $b < |s|$ $\forall s \supseteq Y$. Let $r \supseteq X \cup Y$. By separability there are disjoint sets $r_1 \supseteq X$, $r_2 \supseteq Y$ such that $r_1 \cup r_2 \subseteq r$. Thus $a+b < |r_1| + |r_2| \subsetneq |r|$:

Therefore, $a+b < |r|$ for all $r \supseteq X \cup Y$. It means that $a+b \in o(X \cup Y)$ and one inclusion is proved.

Now suppose $a > o(X)+o(Y)$. Then

$$(\forall b \in o(X))(\forall c \in o(Y))(b+c < a).$$

By an overspill argument we can show that there are $a_1 > o(X)$, $b_1 > o(Y)$ such that $a_1+b_1 < a$. Choose $u_1 \supseteq X$, $u_2 \supseteq Y$ with $|u_1|=a_1$, $|u_2|=b_1$. If $v_1$, $v_2$ separate X,Y and $w_1=u_1 \cap v_1$, $w_2=u_2 \cap v_2$, then $X \cup Y \subseteq w_1 \cup w_2$ and $|w_1 \cap w_2| \leq a_1+b_1 < a$. Thus $a \notin o(X \cup Y)$.

The other claim follows immediately. □

<u>Theorem 6.</u> For any cuts X, Y, $i(X \times Y)=i(X) \cdot i(Y)$ and $o(X \times Y) \subseteq o(X) \cdot o(Y)$. If X, Y are measurable, then $X \times Y$ is measurable and $\mu(X \times Y)=\mu(X) \cdot \mu(X)$.

<u>Proof.</u> a) $i(X) \cdot i(Y) \subseteq i(X \times Y)$ is straightforward. Conversely, suppose $u \subseteq X \times Y$ and $|u|=a$. If $u_1=dom(u)$, $u_2=rng(u)$, then $u \subseteq u_1 \times u_2$ and $|u| \leq |u_1| \cdot |u_2|$. Since $|u_1| \in i(X)$, $|u_2| \in i(Y)$, it follows $|u| \in i(X) \cdot i(Y)$.

b) Let $a > o(X) \cdot o(Y)$. Then

$$(\forall b \in o(X))(\forall c \in o(Y))(b \cdot c < a).$$

By the overspill argument used in Theorem 5, there are $b_1 > o(X)$, $c_1 > o(Y)$ such that $b_1 \cdot c_1 < a$. Thus, there are $u_1 \supseteq X$, $v_1 \supseteq Y$ with $|u_1|=b_1$, $|v_1|=c_1$. Hence $u_1 \times v_1 \supseteq X \times Y$ and $|u_1 \times v_1|=b_1 \cdot c_1 < a$. This shows that $a \notin o(X \times Y)$. □

<u>Theorem 7.</u> If $(X_n)_n$ is a sequence of measurable classes and the cut $\bigcup_{\mathfrak{n}} \mu(X_n)$ is closed with respect to addition, then $\bigcup_{\mathfrak{n}} X_n$ is measurable and $\mu(\bigcup_{\mathfrak{n}} X_n)= \bigcup_{\mathfrak{n}} \mu(X_n)$.

<u>Proof.</u> Let $\mu(X_n)=I_n$. Since clearly $\bigcup_{\mathfrak{n}} I_n \subseteq i(\bigcup_{\mathfrak{n}} X_n) \subseteq o(\bigcup_{\mathfrak{n}} X_n)$ it suffices to show that $o(\bigcup_{\mathfrak{n}} X_n) \subseteq \bigcup_{\mathfrak{n}} I_n$.

Without loss of generality we may assume that the sequence $(I_n)_n$ is increasing. Then we can find sequences $(u_n)_n$, $(a_n)_n$ such that $u_n \subseteq u_{n+1}$, $X_n \subseteq u_n$, $|u_n|=a_n$ and $\bigcup_{\mathfrak{n}} a_n= \bigcup_{\mathfrak{n}} I_n$. Suppose $u_n$, $a_n$ are defined such that $I_n < a_n \in \bigcup_{\mathfrak{n}} I_n$, $X_n \subseteq u_n$ and $|u_n|=a_n$. Then take some $u \supseteq X_{n+1}$ with $|u|=a > I_{n+1}$ and put $u_{n+1}= =u_n \cup u$, $a_{n+1}=|u_{n+1}|$. Then $X_{n+1} \subseteq u_{n+1}$, $I_{n+1} < a_{n+1}$ and $a_{n+1} \in \bigcup_{\mathfrak{n}} I_n$ by the closure condition.

Let $a \notin \bigcup_{\mathfrak{n}} I_n$. By the prolongation axiom we can find u such that $|u| < a$, and $\bigcup_{\mathfrak{n}} u_n \subseteq u$. Then $\bigcup_{\mathfrak{n}} X_n \subseteq u$, thus $a \notin o(\bigcup_{\mathfrak{n}} X_n)$. This proves the inclusion. □

<u>Corollary 8.</u> If $(X_n)_n$ is a sequence of classes such that $\mu(X_n) \leq FN$ (that is, $\mu(X_n)=FN$ or $\mu(X_n)=m \in FN$) then $\mu(\bigcup_{\mathfrak{n}} X_n) \leq FN$. □

Classes of measure $\leq FN$ are the analogues of sets of measure zero. Corollary 8 as well as Theorem 10 below remind us of the well known facts of measure theory.

The following is an easy consequence of the prolongation axiom.

Lemma 9. Let $(u_n)_n$ be a descending sequence of sets and let Y be countable such that $Y \subseteq \bigcap_n u_n$. Then, for any infinite natural number a such that $a < \dots |u_n| < \dots < |u_1| < |u_0|$, there is a set u such that $Y \subseteq u \subseteq \bigcap_n u_n$ and $|u| = a$. □

Theorem 10. Any infinite set includes an uncountable class of measure FN.

Proof. Let w be a set with $|w| = d > FN$ and let $(a_\alpha)_{\alpha \in \Omega}$ be a decreasing $\Omega$-sequence of natural numbers with $a_0 = d$ and coinitial to $N \setminus FN$. We shall define by transfinite induction a class $X = \{x_\alpha ; \alpha \in \Omega\}$ and a descending sequence of sets $(u_\alpha)_{\alpha \in \Omega}$ such that $u_0 = w$, $|u_\alpha| = a_\alpha$ and for every $\alpha \in \Omega$, $\{x_\beta ; \beta \in \alpha \cap \Omega\} \subseteq u_\alpha$. Then, clearly, $X \subseteq u_\alpha$ for every $\alpha \in \Omega$ and since $(|u_\alpha|)_{\alpha \in \Omega}$ is coinitial to $N \setminus FN$ we have $o(X) = FN = \mu(X)$.

Construction. Suppose $u_\alpha$ and $x_\beta$ for $\beta \in \alpha \cap \Omega$ have been defined. Then, $\{x_\beta ; \beta \in \alpha \cap \Omega\} \subseteq u_\alpha$. By prolongation we can find a set $u_{\alpha+1}$ such that $\{x_\beta ; \beta \in \alpha \cap \Omega\} \subseteq u_{\alpha+1} \subseteq u_\alpha$ and $|u_{\alpha+1}| = a_{\alpha+1}$. Choose some $x \in u_{\alpha+1} \setminus \{x_\beta ; \beta \in \alpha \cap \Omega\}$ and put $x_\alpha = x$.

Suppose now that $\alpha$ is a limit ordinal and $u_\beta$, $x_\beta$ have been defined for $\beta \in \alpha \cap \Omega$. Then, for each $\beta \in \alpha \cap \Omega$, $\{x_\gamma ; \gamma \in \beta \cap \Omega\} \subseteq u_\beta$, $u_\beta$ descend and $|u_\beta| = a_\beta$. Then $\{x_\gamma ; \gamma \in \alpha \cap \Omega\} \subset \bigcap \{u_\beta ; \beta \in \alpha \cap \Omega\}$. Indeed, if $\beta, \gamma \in \alpha \cap \Omega$, take some $\delta$, such that $\beta, \gamma < \delta < \alpha$. Then $\{x_\xi ; \xi \in \delta \cap \Omega\} \subseteq u_\delta \subseteq u_\beta$, hence $x_\gamma \in u_\beta$. By Lemma 9 we can find u such that $|u| = a_\alpha$, $\{x_\gamma ; \gamma \in \alpha \cap \Omega\} \subseteq u \subseteq$ $\subseteq \{u_\beta ; \beta \in \alpha \cap \Omega\}$. Put $u_\alpha = u$. The proof is complete. □

The following shows that there are no limits in the possible divergence between inner and outer measures.

Theorem 11. For any cuts $I < J$ there is a class X such that $i(X) = I$ and $o(X) = J$.

Proof. We assume for simplicity that I is not a $\Sigma$-class and J is not a $\Pi$-class, so there is an increasing $\Omega$-sequence $(a_\alpha)_{\alpha \in \Omega}$ of natural numbers, cofinal in I and a decreasing $\Omega$-sequence $(b_\alpha)_{\alpha \in \Omega}$ coinitial in $N \setminus J$. (Else consider $\omega$-sequences and make minor modifications in the construction).

Let $(w_\alpha)_\alpha$ be an enumeration of all the sets w such that $I < |w| < J$. We shall write $\alpha < \beta$ instead of $\alpha \in \beta \cap \Omega$.

We define sequences $(u_\alpha)_\alpha$, $(v_\alpha)_\alpha$, $(r_\alpha)_\alpha$, $(s_\alpha)_\alpha$ such that:

$u_\alpha$ is increasing and $v_\alpha$ decreasing in inclusion,

i) $|u_\alpha|=a_\alpha$ and $|v_\alpha|=b_\alpha$ $\forall\alpha\in\Omega$,

ii) $\bigcup_{\beta<\alpha} u_\beta \subseteq \bigcap_{\beta<\alpha} v_\beta$ $\forall\alpha\in\Omega$,

iii) $\{r_\beta;\beta<\alpha\}\subseteq\bigcup_{\beta<\alpha} u_\beta$ and $r_\beta\notin w_\beta$,

iv) $\{s_\beta;\beta<\alpha\}\cap(\bigcup_{\beta<\alpha} u_\beta)=\emptyset$ and $s_\beta\in w_\beta$.

If this is done and if we put $X=\bigcup\{u_\alpha;\alpha\in\Omega\}$ then $I\subseteq i(X)$, $o(X)\subseteq J$, $\{r_\alpha;\alpha\in\Omega\}\subseteq X$, $r_\alpha\notin w_\alpha$, $\{s_\alpha;\alpha\in\Omega\}\cap X=\emptyset$, $s_\alpha\in w_\alpha$, that is, $w\nsubseteq X\nsubseteq w$ for every $w$ with $I<|w|<J$, hence $I=i(X)$ and $o(X)=J$.

Construction. Suppose $u_\beta$, $v_\beta$, $r_\beta$, $s_\beta$ have already been defined for $\beta<\alpha$.

Then $\bigcup_{\beta<\alpha} u_\beta \subseteq \bigcap_{\beta<\alpha} v_\beta$.

Clearly $\bigcap_{\beta<\alpha} v_\beta \nsubseteq w_\alpha \nsubseteq \bigcup_{\beta<\alpha} u_\beta$ since $|v_\beta|>J$, $I<|w_\alpha|<J$ and $|u_\beta|<I$.

Therefore we can choose $r_\alpha\in(\bigcap_{\beta<\alpha} v_\beta)\setminus w_\alpha$ and $s_\alpha\in w_\alpha\setminus\bigcup_{\beta<\alpha} u_\beta$.

Then take a set $u_\alpha\subseteq\bigcap_{\beta<\alpha} v_\beta$ such that $r_\alpha\in u_\alpha$, $|u_\alpha|=a_\alpha$ and $\{s_\beta;\beta\leq\alpha\}\cap u_\alpha=\emptyset$. This is clearly possible since $\{s_\beta;\beta\leq\alpha\}$ is countable. Then, find by prolongation a set $v_\alpha\subseteq\bigcap_{\beta<\alpha} v_\beta$ such that $|v_\alpha|=b_\alpha$ and $\bigcup_{\beta\leq\alpha} u_\beta\subseteq\bigcap_{\beta\leq\alpha} v_\beta$. Obviously $u_\alpha$, $v_\alpha$, $r_\alpha$, $s_\alpha$ are as required and the construction is complete. □

Next, we show that there is hardly any connection between measurability and revealness (even in its strongest form).

Let us fix some endomorphism F such that the universe A=F"V has a standard extension and let us put for every class X, $X^*=Ex(F"X)$. Then the following holds:

Theorem 12. For any class X, $i(X^*)=i(X)^*$ and $o(X^*)=o(X)^*$. Thus $X^*$ is measurable iff X is measurable and $\mu(X^*)=\mu(X)^*$.

Proof.

$i(X)=I \leftrightarrow (\forall a)(a\in I \leftrightarrow (\exists u\subseteq X)(|u|=a)) \leftrightarrow (\forall a\in A)(a\in F"I \leftrightarrow$
$\leftrightarrow (\exists u\in A)(u\subseteq F"X \,\&\, |u|=a)) \leftrightarrow (\forall a)(a\in Ex(F"I) \leftrightarrow (\exists u)(u\subseteq Ex(F"X)\,\&$
$\&\,|u|=a)) \leftrightarrow (\forall a)(a\in I^* \leftrightarrow (\exists u)(u\subseteq X^* \,\&\, |u|=a)) \leftrightarrow i(X^*)=I^*$.

Similarly we see that $o(X^*)=o(X)^*$. □

We shall close this paper by showing that no non-trivial ultrafilter (restricted on a set) is measurable.

We shall work again on w with $|w|=d$.

For any $X\subseteq P(w)$ let us put

$$\overline{X}=\{w\setminus x; x\in X\}.$$

The following is trivial:

Lemma 13.

1) $X \subseteq Y \rightarrow \overline{X} \subseteq \overline{Y}$

2) $\overline{\overline{X}}=X$

3) $|u|=|\overline{u}|$ for any $u \subseteq P(w)$.

4) If $\mathfrak{M}$ is an ultrafilter on w then $\overline{\mathfrak{M}} = -\mathfrak{M}$ . □

Theorem 14. Let $\mathfrak{M}$ be non-trivial on w. Then

1) $i(\mathfrak{M})=i(-\mathfrak{M})$ and $o(\mathfrak{M})=o(-\mathfrak{M})$.

2) $i(\mathfrak{M})=2^d-o(\mathfrak{M})$.

3) $i(\mathfrak{M})< 2^{d-1}< o(\mathfrak{M})$. Thus $\mathfrak{M}$ is not measurable.

Proof. 1) By the previous lemma $u \subseteq \mathfrak{M} \subseteq v \leftrightarrow \overline{u} \subseteq -\mathfrak{M} \subseteq \overline{v}$ and $|\overline{u}|=|u|$, $|\overline{v}|=|v|$, which shows the claim.

2) By Lemma 4, $i(\mathfrak{M})=i(-\mathfrak{M})=2^d-o(\mathfrak{M})$.

3) Suppose $u \subsetneq \mathfrak{M}$ such that $|u|=2^{d-1}$. Then $|-u|=2^d-2^{d-1}=2^{d-1}$ and $\overline{u} \subseteq -\mathfrak{M} \subseteq -u$. Since $|u|=|-u|$, it follows that $-u=\overline{u}$, hence $-u \subseteq -\mathfrak{M} \subseteq -u$, or $\mathfrak{M}=u$, a contradiction. Similarly if $\mathfrak{M} \subseteq u$ and $|u|=2^{d-1}$, then $-u \subseteq -\mathfrak{M}$. But $|-u|=2^{d-1}$ and $i(-\mathfrak{M})=i(\mathfrak{M})$ which contradicts the previous result. □

Recall that given ultrafilter $\mathfrak{M}$,
$\nu(\mathfrak{M})=\{a \in N;(\forall x \in \mathfrak{M})(a<|x|)\}$ (see [S-V]). Let
$2^{d-\nu}(\mathfrak{M})=\{a;(\exists \gamma > \nu(\mathfrak{M}))(a \leq 2^{d-\gamma})\}$.

It is easy to see that

$$2^{d-\nu(\mathfrak{M})} \leq i(\mathfrak{M})< o(\mathfrak{M}) \leq 2^d-2^{d-\nu(\mathfrak{M})}.$$

However, the following is open to me:

Problem: Is it true that $2^{d-\nu(\mathfrak{M})}=i(\mathfrak{M})$? If not, find $i(\mathfrak{M})$.

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