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DIMENSION OF INDISCERNIBILITY EQUIVALENCES

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Abstract: In this paper we study the concept of the topological dimension by means of the alternative set theory (AST). In the AST various topological concepts were studied (see [V]) but the dimension theory was not worked out till now. In our work we define basic notions, prove some characterizations of the dimension and describe the connection between the classical concept and ours.

Key words: Alternative Set Theory, dimension, indiscernibility equivalence.

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1. Basic notions. Let us recall some notions from [V]:

Sd-class is a set-theoretically defined class,

π-class is a class which is an intersection of countably many Sd-classes,

σ-class is a union of countably many Sd-classes,

symmetry on A is a reflexive symmetrical relation on A,

symmetry R on A is said to be compact if for every infinite set $u \subseteq A$ there exist $x,y \in u$ such that $\langle x,y\rangle \in R$,

an indiscernibility equivalence on an Sd-class A is a compact π-equivalence on A.

For a given indiscernibility equivalence R on A we define

$Fig(X)=R''X$, $Mon(x)=R''\{x\}=Fig(\{x\})$,

$Sep(X,Y) \equiv (\exists Z \text{ Sd-class})(Fig(X) \subseteq Z \,\&\, Fig(Y) \cap Z=0)$,

$X^c=\{x; \text{not } Sep(X,\{x\})\}$,

$X^o=\{x; Mon(x) \subseteq X\}$,

X is a figure if X=Fig(X),

X is closed if $X=X^c$,

X is open if A-X is closed.

Observe that X^o=A-Fig(A-X) is a dual operation to Fig and not a topological interior in the common sense. Open and closed classes have usual topological properties (they form topology of a compact metrizable space) and moreover there holds:

Theorem 1. Let X be a figure. Then the following is equivalent:

(i) X is a figure of a set u (i.e. X=Fig(u)),

(ii) X is a π-class,

(iii) X is closed.

Similarly, open classes are exactly such figures that are σ-classes.

Definition: (1) Let R be an indiscernibility equivalence on A. A sequence $(R_n; n \in FN)$ is called a generating sequence if

(i) R_n is an Sd-symmetry on A,

(ii) $R_{n+1} \circ R_{n+1} \subseteq R_n$,

(iii) $R_0 = A^2$,

(iv) $R = \bigcap\{R_n; n \in FN\}$.

(2) Let R be an indiscernibility equivalence on a set u. A sequence r= $=\{r_\alpha; \alpha < \gamma\}$, $\gamma \in (N-FN)$, is called a prolongation of a generating sequence if

(i) r_α is a symmetry on u,

(ii) $r_{\alpha+1} \circ r_{\alpha+1} \subseteq r_\alpha$ for $\alpha < \gamma$,

(iii) $r_0 = u^2$,

(iv) $R = \bigcap\{r_n; n \in FN\}$.

It is easy to prove the following theorem (see [V]):

Theorem 2. (1) For any indiscernibility equivalence there exists a generating sequence.

(2) For any indiscernibility equivalence on a set there exists a prolongation of a generating sequence.

An indiscernibility equivalence S is called totally disconnected if there exists a generating sequence $\{S_n, n \in FN\}$ such that S_n are equivalences.

Under a prolongation of a generating sequence of a totally disconnected S we understand a prolongation $\{s_\alpha; \alpha < \gamma\}$ such that s_α are equivalences.

2. Dimension. Now we are going to define the dimension of an indiscernibility equivalence and to prove its basic properties. We define one technical notion.

Definition: Let S_1, S_2 be symmetries. We define

S_1 divides S_2 on $\leq d \equiv_{df}$

$(\forall x_0, \ldots, x_d)((\forall i,j)(\langle x_i, x_j\rangle \in S_2) \Rightarrow (\exists i,j, i \neq j)(\langle x_i, x_j\rangle \in S_1))$.

Now let us suppose that R is an indiscernibility equivalence on an Sd-class A. The following definition is due to P. Vopěnka.

Definition: $\dim(R) \leq d \equiv_{df}$
$(\exists S$ totally disconnected indiscernibility equivalence on $A)\cdot$
$(S \subseteq R \,\&\, S$ divides R on $\leq d+1)$.

We call this dimension the inner dimension (to differ from the covering dimension). We need also a notion of the local dimension in a point.

Definition: $\dim(R,x) \leq d \equiv_{df}$
$(\exists B$ Sd-class$)(\mathrm{Mon}(x) \subseteq B \,\&\, \dim(R \cap B^2) \leq d)$.

This definition can be expressed in the following form:

Theorem 1. Let $\mathcal{R} = \{R_n;\ n \in FN\}$ be a generating sequence of R. Then $\dim(R,x) \leq d \equiv (\exists n)(\dim(R \cap (R_n''\{x\})^2) \leq d)$.

Proof: The implication $\Leftarrow$ is trivial.

$\Rightarrow$: If B is the Sd-class from the definition, we have
$$\bigcap\{R_n''\{x\};n \in FN\} = \mathrm{Mon}(x) \subseteq B,$$
and by the axiom of prolongation we have $R_n''\{x\} \subseteq B$ for some $n \in FN$. □

It is trivial that $\dim(R)=0$ iff R is totally disconnected. For an illustration of the definition we show an elementary example.

Example: Let $R = \bigcap\{R_n;n \in FN\}$ be the usual equivalence of the real numbers,
$$R_n = \{\langle x,y\rangle \in RN^2;\ |x-y| < 1/n \text{ or } (|x| \geq n \,\&\, |y| \geq n)\}.$$
We are going to demonstrate that $\dim(R) \leq 1$ and thus $\dim(R)=1$ because R is not totally disconnected. We take

$$S_n = \{\langle x,y\rangle \in RN^2;(|x| < 1/n \,\&\, |y| < 1/n) \text{ or } (|x| \geq n \,\&\, |y| \geq n) \text{ or}$$
$$(x.y > 0 \,\&\, \mathrm{not}(\exists \alpha)(\exists k \leq n)(|x| < \alpha/k \leq |y| \text{ or } |y| < \alpha/k \leq |x|)\} .$$

Then S_n are equivalences, $S = \bigcap\{S_n;n \in FN\}$ is totally disconnected and $S \subseteq R$. S divides only monads of rational numbers and these ones only into two parts, consequently we can conclude that S divides R on ≤ 2 and $\dim(R)=1$.

3. A characterization of the dimension. In this paragraph we are going to characterize the dimension by means of generating sequences (Theorem 3). Let us suppose that $\mathcal{R} = \{R_n;n \in FN\}$ is a generating sequence of an indiscernibility equivalence R.

$\mathcal{S} = \{S_n;n \in FN\}$ is a generating sequence of a totally disconnected indiscernibility equivalence S (i.e. S_n are equivalences).

Theorem 1. Let $\mathcal{R}$ and $\mathcal{S}$ be given such that for a $d \in FN$ the following holds:

$(\forall n \in FN)(S_{n+1} \subseteq R_n \& S_n$ divides R_n on $\leq d+1)$.

Then $\dim(R) \leq d$.

Proof: Obviously $S = \bigcap\{S_n; n \in FN\} \subseteq \bigcap\{R_n; n \in FN\} = R$. It remains to prove that S divides R on $\leq d+1$. Suppose that $x_0, \dots, x_{d+1}$ are in one monad of R, but each two in distinct monads of S. Then for $i,j \leq d+1$, $i \neq j$ there exists $a_{i,j} \in FN$ such that $\langle x_i, x_j \rangle \notin S_{a_{i,j}}$. We can take $a \in FN$ such that a is greater than all $a_{i,j}$'s. Then S_a does not divide R on $\leq d+1$ - a contradiction. Hence $\dim(R) \leq d$. □

Lemma: Suppose that $\varphi(x)$ is a set-theoretical formula, X_n $(n \in FN)$ are Sd-classes, $X_{n+1} \subseteq X_n$, $X = \bigcap\{X_n; n \in FN\}$. Then it holds

$(\forall x \subseteq X)\varphi(x) \Rightarrow (\exists n)(\forall x \subseteq X_n)\varphi(x)$.

Proof: Let us suppose that the assertion does not hold. Consequently there exists a sequence $x_n \subseteq X_n$ such that not $\varphi(x_n)$ for all n. We prolong this sequence and take $\alpha_n \notin FN$ such that

$(\forall \beta < \alpha_n)(x_\beta \subseteq X_n \&$ not $\varphi(x_\beta))$ (such α_n exists because X_n is an Sd-class).

We take a $\beta \in \bigcap\{\alpha_n; n \in FN\}$, $\beta \notin FN$. It holds $x_\beta \subseteq X$, not $\varphi(x_\beta)$ - a contradiction. □

Theorem 2. Let $\mathcal{R}$ and $\mathcal{S}$ be given such that $S \subseteq R$ and for a $d \in FN$ S divides R on $\leq d+1$.

Then there exists a selected sequence $\overline{\mathcal{R}}$ from $\mathcal{R}$ and a selected sequence $\overline{\mathcal{S}}$ from $\mathcal{S}$ such that

$(\forall n)(\overline{S}_{n+1} \subseteq \overline{R}_n \& \overline{S}_n$ divides $\overline{R}_n$ on $\leq d+1)$.

Proof: We take $\overline{R}_0 = \overline{S}_0 = R_0 = S_0 = (\mathrm{dom}(R))^2$ and then we select step by step $\overline{S}_{i+1}$ such that $\overline{S}_{i+1} \subseteq \overline{R}_i$ and $\overline{R}_{i+1}$ such that $\overline{S}_{i+1}$ divides $\overline{R}_{i+1}$ on $\leq d+1$. It suffices to prove the following two statements:

(1) $(\forall n_0)(\forall m)(\exists n \geq n_0)(S_n \subseteq R_m)$.

We use the lemma for $X_n = S_{n_0+n}$, $X = S$ and $\varphi(x) \equiv (x \subseteq R_m)$.

(2) $(\forall n_0)(\forall m)(\exists n \geq n_0)(S_m$ divides R_n on $\leq d+1)$.

We use the lemma for $X_n = R_{n_0+n}$, $X = R$,

$\varphi(x) \equiv (\forall x_0, \dots, x_{d+1})(x = \{x_0, \dots, x_{d+1}\}^2 \Rightarrow (\exists i,j,\ i \neq j)(\langle x_i, x_j \rangle \in S_m))$

(i.e. $(\forall x \subseteq R)\varphi(x) \equiv S_m$ divides R on $\leq d+1$).

Consequently we have constructed the desired $\overline{\mathcal{R}}, \overline{\mathcal{S}}$. □

Theorem 3. Let $\mathcal{R}$, $d \in FN$ be given. Then $\dot{\dim}(R) \leq d$ iff

$(\exists \overline{\mathcal{R}}$ selected sequence from $\mathcal{R})(\exists \mathcal{S})$

$(\forall n \in FN)(S_{n+1} \subseteq \overline{R}_n \,\&\, S_n$ divides $\overline{R}_n$ on $\leq d+1)$.

Proof: It follows immediately from Theorems 1 and 2. $\square$

4. Coverings. Let R be a given indiscernibility equivalence on an Sd-class A; in the sequel all classes will be considered as parts of A.

The following two propositions are wellknown (see [V]).

Proposition 1. Let X be an Sd-class then X^{o} is open (w.r.t. R).

Proposition 2. Let $X \subseteq Y$, X closed, Y open. Then there exists an Sd-class Z such that $X \subseteq Z \subseteq Y$.

Definition. $\{X_1,\dots,X_m\}$ is a covering (R-covering) $\equiv_{df}$ $(\forall x \in A)(\exists i)(\mathrm{Mon}(x) \subseteq X_i)$.

It is called to be an open (closed, Sd) covering if each class X_i is open (closed, Sd).

A covering $P=\{X_1,\dots,X_m\}$ is inscribed into a covering $Q=\{y_1,\dots,y_k\}$ (we write $P<Q$) if $(\forall i)(\exists j)(X_i \subseteq Y_j)$.

Let $P=\{X_1,\dots,X_n\}$ be a covering; we say that the order of the covering is just d if

(i) Every d+2 classes from P have an empty intersection.

(ii) Some d+1 classes have not an empty intersection.

Proposition 3. Let $\{X_1,\dots,X_m\}$ be an Sd-covering of order $\leq d$. Then there exists an open covering of order $\leq d$ inscribed into this.

Proof: $\{X_1^{o},\dots,X_m^{o}\}$ is the desired open covering. $\square$

Proposition 4. Let $\{X_1,\dots,X_m\}$ be an open covering of order $\leq d$. Then there exists an Sd-covering $\{Z_1,\dots,Z_m\}$ inscribed into this such that $\{\mathrm{Fig}(Z_1),\dots,\mathrm{Fig}(Z_m)\}$ has order $\leq d$.

Proof: By Proposition 2 there exists an Sd-class Z_1 such that

$A-(X_2 \cup \dots \cup X_m) \subseteq Z_1 \subseteq X_1$.

$\{Z_1^{o},X_2,\dots,X_m\}$ is an open covering inscribed into $\{X_1,\dots,X_m\}$. Now we take this covering and similarly substitute X_2 by Z_2^{o}, then X_3 by Z_3^{o} and so on. Then $\{Z_1,\dots,Z_m\}$ is the desired covering. $\square$

The following definition is an analogy of the classical covering dimension.

Definition: We say that $\mathrm{Dim}(R) \leq d$ if an open covering of order $\leq d$ can be inscribed into every open covering.

Proposition 5. $\mathrm{Dim}(R) \leq d$ iff an Sd-covering of order $\leq d$ can be inscribed into every Sd-covering.

Proof: Let $\mathrm{Dim}(R) \leq d$ and $\{X_1,\dots,X_m\}$ be an Sd-covering. From the proposition 3 it follows that an open covering can be inscribed into it and into it an open covering of order $\leq d$ by the definition of covering dimension. By the proposition 4 there exists an Sd-covering inscribed into the open covering, and consequently inscribed into $\{X_1,\dots,X_m\}$, the order of which has to be also $\leq d$.

The converse implication can be proved analogously. $\square$

Proposition 6. Let $\{Y_1,\dots,Y_m\}$ be a closed covering of order $\leq d$ which is inscribed into an Sd-covering $\{X_1,\dots,X_\ell\}$. Then an Sd-covering of order $\leq d$ can be inscribed into $\{X_1,\dots,X_\ell\}$.

Proof: Let $Y_i = \bigcap\{Y_i^k; k \in FN\}$ where Y_i^k are Sd-classes, $Y_i^{k+1} \subseteq Y_i^k$ and $Y_i^0 \subseteq k_j$ for each X_j such that $Y_i \subseteq X_j$. Obviously for every $k \in FN$ the system $\{Y_1^k,\dots,Y_m^k\}$ is an Sd-covering inscribed into $\{X_1,\dots,X_\ell\}$. It suffices to prove that there exists a k such that the order of $\{Y_1^k,\dots,Y_\ell^k\}$ is $\leq d$. If the order of $\{Y_1^k,\dots,Y_\ell^k\}$ was $> d$ for every $k \in FN$ then we could choose $d+2$ indices $i_1,\dots,i_{d+2}$ such that $Y_{i_1}^k \cap \dots \cap Y_{i_{d+2}}^k = 0$ for cofinally many $k \in FN$. Hence $Y_{i_1} \cap \dots \cap Y_{i_{d+2}} = 0$ - a contradiction. $\square$

Lemma: Let S be a totally disconnected indiscernibility equivalence on A. Then an Sd-covering of order 0 can be inscribed into every Sd-covering $\{X_1,\dots,X_k\}$ of S.

Proof: Let $\{S_n; n \in FN\}$ be a generating sequence of S such that each S_n is an equivalence. Obviously it suffices to prove that there exists an $n \in FN$ such that

$$(\forall x \in A)(\exists i)(S_n \text{"} \{x\} \subseteq X_i).$$

Let us suppose it does not hold. Then there exist $x_n \in A$ with the property

$$\text{not } S_n\text{"}\{x_n\} \subseteq X_i \quad (i=1,\dots,k),$$

hence also not $S_m\text{"}\{x_n\} \subseteq X_i$ $(i=1,\dots,k \,\&\, m \leq n)$.

Let $x = \{x_\alpha; \alpha < \gamma\}$ be a prolongation of the sequence $\{x_n; n \in FN\}$ such that

$$x_\alpha \in A,$$

$$\text{not } S''_m(x_\alpha) \subseteq X_i \quad (i=1,\dots,k \;\&\; m \le \alpha \;\&\; m \in FN).$$

Now we take an infinite $\alpha < \gamma$, then clearly not $\mathrm{Mon}_S(x_\alpha) \subseteq X_i$ for $i=1,\dots$ $\dots,k$ because $\mathrm{Mon}_S(x_\alpha) = \bigcap_m S''_m\{x_\alpha\}$ and X_i are Sd-classes. This is a contradiction with the presumption $\{X_1,\dots,X_m\}$ being a covering of S. □

The idea of the proof of the following theorem is due to P. Vopěnka.

Theorem: $\mathrm{Dim}(R) \le \dim(R)$.

Proof: Let $\dim(R)=d$ and $S \subseteq R$ be the totally disconnected indiscernibility equivalence which divides R on $\le d+1$. We want to prove $\mathrm{Dim}(R) \le d$. So let $\{X_1,\dots,X_k\}$ be an Sd-covering of R. And let $\{Z_1,\dots,Z_\ell\}$ be an Sd-cover of R inscribed into the open covering $\{X_1^0,\dots,X_k^0\}$ (it can be constructed in the same way as in the proof of Proposition 4), it is then also an Sd-covering of R By the previous lemma there exists an Sd-covering $\{Y_1,\dots,Y_m\}$ of the equivalence S of order 0 inscribed into $\{Z_1,\dots,Z_\ell\}$. Then clearly $P=\{\mathrm{Fig}(Y_1),\dots$ $\dots,\mathrm{Fig}(Y_m)\}$ is a closed covering of R inscribed into $\{X_1,\dots,X_k\}$.

Let us prove that P has its order $\le d$. If the intersection of some d+2 classes $\mathrm{Fig}(Y_{i_1}) \cap \dots \cap \mathrm{Fig}(Y_{i_{d+2}})$ contained a point x then $\mathrm{Mon}(x) \cap Y_{i_k}$ would be nonempty for all $k=1,\dots,d+2$. But it would imply that Mon(x) contains more than d+1 different monads of S because $\{Y_1,\dots,Y_m\}$ is a disjoint covering of the equivalence S - a contradiction with the presumption that S divides R on $\le d+1$. Hence the order of P is $\le d$. Now from Proposition 6 and 5 it follows that $\mathrm{Dim}(R) \le d$. □

5. Relation between the covering dimension and the inner dimension. In the previous paragraph it was proved that $\mathrm{Dim}(R) \le \dim(R)$ in case R is an indiscernibility equivalence on an Sd-class A. The converse inequality we can prove till now only on condition A is a set. But this is not any essential restriction because for any indiscernibility equivalence there exists a set u such that A=Fig(u), and we can investigate properties of the equivalence only on the set u. So let R be an indiscernibility equivalence on a set a.

We say that a system v which covers a (in the sense $a=\bigcup\{x; x \in v\}$) is a partition of a system u if

(i) v is a disjoint system $(x,y \in v \;\&\; x \ne y \Longrightarrow x \cap y = 0)$,

(ii) $\bigcup\{x; x \in v\} = \bigcup\{x; x \in u\} = a$,

(iii) u, v can be written as

$$u=\{u_1,\dots,u_\alpha\},$$

$$v=\{v_1,\dots,v_\alpha\}$$

so that $v_\gamma \subseteq u_\gamma$ for $\gamma=1,\dots,\alpha$.

For a later use we abbreviate

$Fig(v)=\{Fig_R(v_1),\dots,Fig_R(v_\alpha)\}$ if $v=\{v_1,\dots,v_\alpha\}$.

The following lemma will be a key to prove the converse inequality.

Lemma: Let us have a sequence v_n of systems which all cover a such that $v_{n+1}< v_n$ (i.e. $(\forall x\in v_{n+1})(\exists y\in v_n)(x\subseteq y)$).
Then there exist partitions t_n of v_n such that $t_{n+1}< t_n$.

Proof: Let $\{v_\alpha;\alpha\leq\beta\}$ be a prolongation of the sequence v_n such that $\bigcup\{x;x\in v_\alpha\}=a$ and $v_{\alpha+1}< v_\alpha$ for $\alpha<\beta$. Let us have all these systems orderd

$$v_\alpha=(a_1^\alpha,\dots,a_{\delta_\alpha}^\alpha).$$

Put $b_\varepsilon^\beta=a_\varepsilon^\beta-(a_1^\beta\cup\dots\cup a_{\varepsilon-1}^\beta)$ for $\varepsilon=1,\dots,\delta_\beta$ and

$$t_\beta=(b_1^\beta,\dots,b_{\delta_\beta}^\beta).$$

Obviously t_β is a partition of v_β. Let $t_{\alpha+1}$ be a partition of $v_{\alpha+1}$, we inductively define a partition t_α of the system v_α. Put

$$b_\varepsilon^\alpha=\bigcup\{b\in t_{\alpha+1};b\subseteq a_\varepsilon^\alpha \;\&\; (\forall\varepsilon_0<\varepsilon)\mathrm{not}(b\subseteq a_{\varepsilon_0}^\alpha)\},$$

$$t_\alpha=\{b_1^\alpha,\dots,b_{\delta_\alpha}^\alpha\}.$$

Considering that $t_{\alpha+1}< v_{\alpha+1}< v_\alpha$ and that $t_{\alpha+1}$ covers a we see that t_α also covers a. Because it is a set-theoretically defined construction, the t_α is constructed for each $\alpha\leq\beta$ and consequently also for $\alpha\in FN$. $\{t_n;n\in FN\}$ fulfil our requirements. $\square$

Theorem: If $Dim(R)\leq d$ then $dim(R)\leq d$.

Proof: We will prove that there exist relations r_n and equivalences s_n so that

$R=\bigcap\{r_n;n\in FN\}$,

$s_{n+1}\subseteq s_n$,

$s_n\subseteq r_n$

and s_n does not divide any R-monad into more than d+1 parts.

If we have this we will put $S=\bigcap s_n$. Evidently $S\subseteq R$ is a totally disconnected indiscernibility equivalence. In the same way as in the proof of the theorem 3.1 we can prove that S divides R on $\leq d+1$, consequently $dim(R)\leq d$.

Let $\{r_n^1;n\in FN\}$ be a generating sequence of R. Because the relation R is compact, a finite R-subcovering u_n can be chosen from the R-covering $\{r_n^1\text{"}\{x\};x\in a\}$. Let us define r_n in the following way:

$$r_n=\{\langle x,y\rangle;(\exists c\in u_n)(\{x,y\}\subseteq c)\}.$$

We prove that $R=\bigcap_n r_n$. Plainly $R\subseteq\bigcap_n r_n$ because each u_n is an R-covering. On the other hand let $\langle y,z\rangle\in r_n$, then there exists $x\in a$ such that $\{y,z\}\subseteq r_n^1=\{x\}$, hence $\langle y,z\rangle\in r_{n-1}^1$. It proves $r_n\subseteq r_{n-1}^1$ which implies $\bigcap_n r_n\subseteq R$.

From the presumption $\mathrm{Dim}(R)\leq d$ it follows that a finite set R-covering q can be inscribed into any finite set R-covering p so that Fig(q) has order $\leq d$. More precisely: by the proposition 4.5 a set-covering p' of order $\leq d$ can be inscribed into p, into it by the proposition 4.3 an open covering p" of order $\leq d$ and into it by the proposition 4.4 a set covering q such that Fig(q) has order $\leq d$. The R-covering q is obviously also inscribed into p.

So let v_1 be an R-covering such that $v_1 < u_1$ and the order of $\mathrm{Fig}(v_1)$ is $\leq d$. Inductively take v_{n+1} an R-covering such that $\mathrm{Fig}(v_{n+1})$ has order $\leq d$ and

$$v_{n+1} < \{x\cap y; x\in u_{n+1} \,\&\, y\in v_n\},$$

hence $v_{n+1}< u_{n+1}$ and $v_{n+1}< u_n$.

We constructed a sequence of R-coverings v_n such that $v_{n+1}< v_n$ and, in addition, the order of $\mathrm{Fig}(v_n)$ is $\leq d$. Let t_n be partitions of v_n guaranteed by the lemma. Obviously no monad is intersected by more than d+1 sets from t_n.

Finally set

$$s_n=\{\langle x,y\rangle;(\exists c\in t_n)(\{x,y\}\subseteq c)\}.$$

These are exactly the desired equivalences. □

6. A local characterization of the dimension. When we study the dimension of indiscernibility equivalences, there naturally arises a question whether it is possible to determine the dimension in a point x of an equivalence from the structure of the monad of x, or if it is necessary to know the structure of some class containing x (as in the definition of the local dimension). It turns out that it depends on the kind of information about the monad.

Suppose that there is a given R on a set a with a prolongation of a generating sequence $r=\{r_\alpha;\alpha<\gamma\}$. An information about the structure of the monad of x can be

(a) the class Mon(x),

(b) the sequence $r_\alpha''\{x\}$, $\alpha<\gamma$, $\alpha\notin FN$,

(c) the sequence $r_\alpha\cap(\mathrm{Mon}(x))^2$, $\alpha<\gamma$.

We show that the information under (a) and (b) is not sufficient even to decide whether the dimension is 0 or 1, but that it is possible to determine the dimension from the information under (c) (Theorem 1).

For the first question it is sufficient to use the example from the paragraph 2 (the indiscernibility equivalence of real numbers). It is obvious that dim(R,0)=1 and dim(S,0)=0, but $R_n''\{0\}=S_n''\{0\}=(-1/n,1/n)$, hence also $R_\alpha''\{0\}=S_\alpha''\{0\}$ (for a suitable prolongation) and the information under (a) and (b) is the same in both cases.

Now we are going to find a local characterization of the dimension by means of generating sequences using results from the part 3.

Lemma: Let $\varphi(\alpha,\beta)$ be a set-theoretical formula monotonous in β (i.e. $\varphi(\alpha,\beta)\Longrightarrow\varphi(\alpha,\beta+1)$). Then $(\forall\alpha\notin FN)(\forall\beta\notin FN)\varphi(\alpha,\beta)\equiv$
$\equiv(\exists n_0\in FN)(\forall\alpha\notin FN)\varphi(\alpha,n_0)$.

Proof: The implication $\Longleftarrow$ is obvious, $\Longrightarrow$ will be proved by contradiction.

Suppose we have a sequence $\alpha_n\notin FN$ such that not $\varphi(\alpha_n,n)$. We prolong the sequence and take $\beta\notin FN$ such that $\alpha_\beta\notin FN\,\&$ not $\varphi(\alpha_\beta,\beta)$ (similarly as in the lemma in the part 1) - a contradiction. $\square$

Now we again restrict ourselves to equivalences on a set. We denote

$M=Mon(x)=R''\{x\}$,

$r=\{r_\alpha;\ \alpha<\gamma\}$

a prolongation of a generating sequence of an indiscernibility equivalence R,

$S=\{s_\alpha;\ \alpha<\gamma\}$

a prolongation of a generating sequence of a totally disconnected indiscernibility equivalence S.

Theorem 1. Let $d\in FN$ and r be given. Then $\dim(R,x)\leq d$ iff

$(\exists s)(\exists\bar{r}$ selected from $r)(\forall\alpha<\gamma)$

$(s_{\alpha+1}\cap M^2\in\bar{r}_\alpha\cap M^2\,\&\,s_\alpha\cap M^2$ divides $\bar{r}_\alpha\cap M^2$ on $\leq d+1)$,

Proof: Let us denote

$\varphi_1(\alpha,X)\equiv(s_{\alpha+1}\cap X^2\subseteq\bar{r}_\alpha\cap X^2\,\&\,s_\alpha\cap X^2$ divides $\bar{r}_\alpha\cap X^2$ on $\leq d+1)$,

$\varphi(\alpha,\beta)\equiv\varphi_1(\alpha,r_\beta''\{x\})$ or $\alpha\geq\gamma$ or $\beta\geq\gamma$.

The formula $\varphi(\alpha,\beta)$ is obviously set-theoretical and monotonous in β. By the theorem 2.1 and 3.1 we have (for suitably short prolongations s and $\bar{r}$)

$\dim(R,x)\leq d\equiv(\exists s)(\exists\bar{r}$ selected from $r)(\exists n_0\in FN)(\forall\alpha)\varphi(\alpha,n_0)$.

Because φ is set-theoretical and finitely many members of s and $\bar{r}$ are irrelevant, we have

$\dim(R,x,)\leq d\equiv(\exists s)(\exists\bar{r}$ selected from $r)(\exists n_0\in FN)(\forall\alpha\notin FN)\varphi(\alpha,n_0)$.

From the lemma it follows

$\dim(R,x)\leq d\equiv(\exists s)(\exists\bar{r}$ selected from $r)(\forall\alpha\notin FN)(\forall\beta\notin FN)\varphi(\alpha,\beta)$

and because $M=R''\{x\}=\bigcup\{r_\beta''\{x\};\ \beta\in(\gamma-FN)\}$ we have

$$\dim(R,x) \leq d \equiv (\exists s)(\exists \bar{r} \text{ selected from } r)(\forall \alpha \notin FN, \alpha < \gamma)\, \varphi_1(\alpha, M)$$

which is the required statement. $\square$

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R e f e r e n c e

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