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ON THE NUMBER OF COMPACT SUBSETS
IN TOPOLOGICAL GROUPS

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Abstract: Results on the number of compact subsets in topological groups are proved. Examples are provided.

Key words: Pseudocharacter, boundedness number, weak Lindelöf number.

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Notation and terminology. Let (G, τ) be a nondiscrete Hausdorff group, e be its neutral element and \mathcal{K} denote the set of all compact subsets of G . For any set X , $|X|$ denotes the cardinality of X ; and, for any topological space X , $\psi(X)$, $\chi(X)$, $w(X)$, $c(X)$, $wL(X)$ denote the pseudocharacter, character, weight, cellularity, weak Lindelöf number of X , respectively.

1. Number of compact subsets

Definition (due to I. Juhász). The boundedness number of (G, τ) - denoted by $bo(G)$ - is the smallest infinite cardinal number \aleph such that for any open neighborhood V of e , there is a subset A of G , with $|A| \leq \aleph$, so that $V \cdot A = G$.

Notice that this notion is different from total- β -boundedness introduced by Comfort in [3].

Theorem 1. The following inequalities hold $\psi(G) \leq |G| \leq |\mathcal{K}| \leq bo(G)^{\psi(G)}$.

Proof. There is a collection of open symmetric neighborhoods of e , \mathcal{V} , such that $|\mathcal{V}| = \psi(G)$ and $\bigcap \{V \mid V \in \mathcal{V}\} = \{e\}$. For each $V \in \mathcal{V}$ fix a subset A_V of G such that $V \cdot A_V = G$ and $|A_V| \leq bo(G)$. Now the proof follows the one which appears in [1], since $\bigcap_{V \in \mathcal{V}} (\bigcup_{x \in A_V} Vx \cdot Vx) = \Delta$, the diagonal of $G \times G$,

Partially supported by CNPq.

$|G| \leq \text{bo}(G)^{\psi(G)}$ and any compact subset of G has density not bigger than $\psi(G)$.

Remarks. 1) As a matter of fact, the proof above shows that the set of all closed subsets of G whose densities do not exceed $\psi(G)$ has cardinality not bigger than $\text{bo}(G)^{\psi(G)}$.

2) It is easy to see that $\text{bo}(G) \leq \text{wL}(G) \leq \text{c}(G)$ (hence, $\text{bo}(G)^{\psi(G)} = \text{wL}(G)^{\psi(G)} = \text{c}(G)^{\psi(G)}$) and $\text{w}(G) = \text{bo}(G) \cdot \chi(G)$.

3) If $\text{bo}(G)$ is either a successor cardinal or a singular cardinal, then $\text{o}(G)^{\text{bo}(G)} = \text{o}(G)$, where $\text{o}(G)$ denotes the number of open sets in G .

Corollary 1. If $\text{bo}(G) \leq 2^{\psi(G)}$, then $\psi(G) \leq |\mathcal{K}| \leq 2^{\psi(G)}$.

Lemma. If K is a nonempty compact subset of G , then $\psi(K, G) \leq \psi(G)$.

Proof. Let \mathcal{V} be a collection of symmetric open neighborhoods of e , closed under finite intersections. Furthermore we shall assume that $|\mathcal{V}| = \psi(G)$ and $\bigcap \{\text{cl}(V) \mid V \in \mathcal{V}\} = \{e\}$. Then $\bigcap \{V, K \mid V \in \mathcal{V}\} = K$; indeed, let $y \notin K$, then there is $V \in \mathcal{V}$ such that $V \cap K = \emptyset$ (otherwise, $V \cap K \neq \emptyset, \forall V \in \mathcal{V}$ and since K is compact, $\bigcap \{\text{cl}(V) \cap K \mid V \in \mathcal{V}\}$ would be nonempty, which is impossible). But if $V \cap K = \emptyset$, then $y \notin V, K$.

Corollary 2. If $\text{bo}(G) \leq 2^{\psi(G)}$ and there is a compact subset K of G such that $\psi(K, G) < \psi(G)$, then $|\mathcal{K}| = 2^{\psi(G)}$.

Proof. If there is a nonempty compact subset K of G such that $\psi(K, G) < \psi(G)$, then for each $x \in K$, $\psi(G) = \psi(x, G) \leq \psi(x, K) \cdot \psi(K, G)$. It follows from Čech-Pospíšil's theorem that $|K| \geq 2^{\psi(G)}$, hence $|\mathcal{K}| = 2^{\psi(G)}$.

Theorem 2. (GCH) If G is pseudocompact, then $|\mathcal{K}|^{\aleph_0} = |\mathcal{K}|$.

Proof. Since G is infinite and pseudocompact, then $|G| \geq 2^{\aleph_0}$. We may assume that if α is a cardinal number such that $\alpha \geq 2^{\aleph_0}$, $\text{cf}(\alpha) \neq \aleph_0$, then $\alpha^{\aleph_0} = \alpha$.

From Theorem 1 and since $\text{bo}(G) = \aleph_0$, either $|\mathcal{K}| = 2^{\psi(G)}$ or $|G| = |\mathcal{K}| = \psi(G)$. In the first case there is nothing to prove; let us consider that $|G| = |\mathcal{K}| = \psi(G)$. From van Douwen's theorem 1.1 ([4]) if $\text{cf}(|G|) = \aleph_0$, there is a cardinal $\mu < |G|$, such that $\psi(G) \leq \text{w}(G) \leq 2^{\mu}$. But $|G| \geq 2^{\mu}$, hence $|G| = |\mathcal{K}| = 2^{\mu}$ and the proof is completed.

Lemma. If V is an open symmetric neighborhood of e and $\mathcal{K}(\text{cl}(V))$ denotes the set of all compact subsets of $\text{cl}(V)$, then $|\mathcal{K}| = \text{bo}(G) \cdot |\mathcal{K}(\text{cl}(V))|$.

Proof. It is immediate that $|\mathcal{K}| \geq \text{bo}(G)$ and $|\mathcal{K}| \geq |\mathcal{K}(\text{cl}(V))|$: On the other hand, let B be a subset of G such that $\text{bo}(G) \geq |B|$ and $V \cdot B = G$. For each nonempty finite subset F of B let \mathcal{K}_F denote the set of all compact subsets of G contained in $V \cdot F$. The function from \mathcal{K}_F into $\prod \{\mathcal{K}(\text{cl}(V)y) \mid y \in F\}$ which assigns to each $K \in \mathcal{K}_F$ the point $(\text{cl}(V) \cap K)_{y \in F}$ is injective. But $\mathcal{K} = \cup \{\mathcal{K}_F \mid \emptyset \neq F \subset B, \text{ finite}\}$ and $|\mathcal{K}(\text{cl}(V))| = |\mathcal{K}(\text{cl}(V))|$, hence $|\mathcal{K}| \leq \text{bo}(G) |\mathcal{K}(\text{cl}(V))|$, which completes the proof.

Remark. The GCH cannot be avoided in Theorem 2, since I. Juhász, under CH and using forcing arguments, obtained an HFD subgroup of $\{0,1\}^{\omega_1}$, such that $|\mathcal{K}|^{\aleph_0} \neq |\mathcal{K}|$.

2. Examples

Example 1. ([5] or [2], page 1170.) Under $\aleph_1 = 2^{\aleph_0}$ and $\aleph_2 < 2^{\aleph_1}$, there is a hereditarily separable pseudocompact group G with $|G| = |\mathcal{K}| = \aleph_2$ (which is not a power of 2, but $\aleph_2^{\aleph_0} = \aleph_2$).

Example 2. Let G be the topological subgroup of $\{0,1\}^{\omega_1}$ whose members are the $(x_\alpha)_{\alpha \in \omega_1}$ such that $\{\alpha \in \omega_1 \mid x_\alpha = 1\}$ is countable. G is countably compact, $\psi(G) = \aleph_1$, $|G| = 2^{\aleph_0}$ and $|\mathcal{K}| = 2^{\aleph_1}$. (Notice that the set $\{(x_\alpha)_{\alpha \in \omega_1} \in G \mid x_\alpha = 1 \text{ for at most one } \alpha \in \omega_1\}$ is compact and has just one accumulation point.)

Example 3. Let us consider $\{0,1\}^{\omega_1}$ with the G_σ -topology (each factor with the discrete topology). For each $\beta \in \omega_1$, let y_β be the point $(x_\alpha)_{\alpha \in \omega_1}$ such that $x_\alpha = 1, \forall \alpha < \beta$ and $x_\alpha = 0$, otherwise. Denote by G the topological subgroup generated by the y_β . Then $\text{bo}(G) = \aleph_0$ and $w(G) = \aleph_1$. (Notice that no countable subcollection of $\{\text{pr}_\xi^{-1}(\{0\}) \mid \xi \in \omega_1\}$, where pr_ξ denotes the projection for each $\xi \in \omega_1$, has its union dense in G .)

Example 4. Let α be an infinite cardinal such that $\alpha^{\aleph_0} = \alpha$. Comfort proved that there is a dense countably compact subgroup G_* of $\{0,1\}^{2^\alpha}$ such that $|G_*| = \alpha$. Denoting by G the topological product group $\sum \times G_*$, where \sum denotes the subgroup of $\{0,1\}^\alpha$ such that its elements have at most countably many coordinates different from 0, we have that $|G| = \alpha$, $\psi(G) = \alpha$ and $w(G) = |\mathcal{K}| = 2^\alpha$. Notice that G is countably compact.

Example 5. Let α be an infinite cardinal number, whose cofinality is β and let $(\alpha_i)_{i \in \beta}$ be a strictly increasing family of cardinals such that $\alpha = \sup_{i \in \beta} \alpha_i$. For each $i \in \beta$ let G_i be a discrete topological group with $|G_i| = \alpha_i$. The topological product group $G = \prod_{i \in \beta} G_i$ has a boundedness number equal to α , $\psi(G) = \beta$ and $|K| = \alpha^\beta$.

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