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SET-LIKE EQUIVALENCE AND INNER AND OUTER CUTS

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Abstract: Our aim is to introduce a notion of the set-like equivalence (subvalence resp.) among classes and to explain it (§ 1), namely, with respect to a relation of this equivalence to a description of semisets with given inner and outer cuts (§ 2). We present, studying figures in an equivalence of indiscernibility, the compatible covering theorem which makes them more clarified (§ 2). Finally, we study (§ 3) an existence of semisets with given inner and outer cut.

Key words: Set-like equivalence, $\mathcal{Y}$-indiscernibility, inner cut, outer cut, compatible covering.

Classification: 03K10, 03K99

Introduction. The point of the AST consists in the existence of a hierarchy of variously sharp classes. We accept a so called standard system $\mathfrak{M}$ (see [2])(or a system of standard classes) as a system of the sharpest classes; such a system $\mathfrak{M}$ is, roughly speaking, a submodel, containing all sets, satisfying Gödel-Bernays axioms for finite sets and, moreover, every normal formula is absolute. For example, Sd_V, Sd_V^* are such systems of standard classes. Note that FN $\notin \mathfrak{M}$ and,, more generally, no cut is an element of $\mathfrak{M}$. We can even see that a semiset is a standard class iff it is a set.

Remember that we have two notions of equivalence among classes in the AST, i.e. $\approx$ and $\hat{\approx}$. The second one is defined among sets only and is finer than the first one. We can see that, confining the testified one-one mappings from the definition of the equivalence of two classes to the standard one, we obtain a new notion of equivalence which will be designated by $\hat{\simeq}$ (see § 1). It is finer than $\approx$ and coincides with $\hat{\approx}$ on sets. Thus, $\hat{\simeq}$ depends on a (fixed system $\mathfrak{M}$. But it is uniquely determined among semisets.

The notion of reality can be made larger. Before we do this, let us agree on the following

Convention. Throughout this paper, let capital block-letters be ranging over elements of a (fixed) system $\mathcal{M}$. The script capital letters denote classes.

The usual notation of sets, natural numbers, finite natural numbers and constants (e.g. FN, Ω, N, ...) is accepted.

A codable class with the coding pair $\langle \mathcal{Y}, \mathcal{K} \rangle$ is designated by $\{\mathcal{Y}"\{x\}; x \in \mathcal{K}\}$.

Notation. $\| x \| = \alpha \leftrightarrow \alpha \hat{\approx} x$, $[\mathcal{X}]^{\alpha} = \{u \subseteq \mathcal{X}; \|u\| = \alpha\}$, $[\mathcal{X}]^{\leq \alpha} = \{u \subseteq \mathcal{X}; \|u\| \leq \alpha\}$. Let $\mathcal{J}$ be a cut. Then $[\mathcal{X}]^{\mathcal{J}} = \{u \subseteq \mathcal{X}; \|u\| \in \mathcal{J}\}$.

By a $\pi(\mathcal{J})$-equivalence on A, where $\mathcal{J}$ is a cut, we mean an equivalence $\mathcal{E}$ on A such that there exists a relation $R \subseteq \delta \times A \times A$ with some $\delta \supseteq \mathcal{J}$ and the following holds:

$\alpha \in \delta \rightarrow R"\{\alpha\}$ is reflexive and symmetric on A, $\alpha < \beta \in \delta \rightarrow R"\{\alpha\} \supseteq R"\{\beta\}$, $1 \leq \alpha + 1 \in \delta \rightarrow R"\{\alpha+1\} \circ R"\{\alpha+1\} \subseteq R"\{\alpha\}$ and $\bigcap\{R"\{\alpha\}; \alpha \in \mathcal{J}\} = \mathcal{E}$. We designate $R"\{\alpha\}$ by R_{α}. We say that R is a <u>creating system for</u> $\mathcal{E}$.

A symmetric relation $\mathcal{R}$ on A is $\mathcal{J}$-<u>condensating</u> iff we have $(\forall u \in P(A) - [A]^{\mathcal{J}})(\exists \{x,y\} \in [u]^2)(\langle x,y\rangle \in \mathcal{R})$.

An equivalence $\mathcal{E}$ on A is called $\mathcal{J}$-<u>indiscernibility</u> (on A) iff it is a $\pi(\mathcal{J})$-equivalence, which is, moreover, $\mathcal{J}$-condensating. Note that every equivalence of indiscernibility, defined in [1], may be seen as an FN-indiscernibility (on V) under presumption that $\mathcal{M} = Sd_V^*$. We can, finally, define that a class is $\mathcal{J}$-<u>real</u> iff it is a figure in an $\mathcal{J}$-indiscernibility.

Note yet the following. Let R be a symmetric relation on A which is $\mathcal{J}$-condensating. Then there exists a set $u \in [A]^{\mathcal{J}}$ such that $(\forall x \in A)(\exists y \in u)$ $\langle x,y\rangle \in R$. Indeed, such a u can be found as a maximal (w.r.t. $\subseteq$) set-R-net, where a set $v \subseteq A$ is an R-net iff $(\forall x,y \in v)(x \neq y \rightarrow \langle x,y\rangle \notin R)$ holds; namely, we can see that $(\exists \alpha \in \mathcal{J})(\forall v \subseteq A)$ (v is an R-net $\rightarrow \|v\| < \alpha$) and, consequently, the existence of the u in question follows from this immediately.

§ 1. Set-like equivalence. Two classes $\mathcal{X}$, $\mathcal{Y}$ are <u>set-like equivalent</u> iff there holds $(\exists F)$(F is a one-one function & dom(F) $\supseteq \mathcal{X}$ & $F"\mathcal{X} = \mathcal{Y}$).
We denote this relation by $\mathcal{X} \hat{\approx} \mathcal{Y}$.

$\mathcal{X}$ is said to be <u>set-like subvalent</u> to $\mathcal{Y}$, $\mathcal{X} \hat{\preceq} \mathcal{Y}$, iff
$(\exists F)$(F is a one-one function & dom(F) $\supseteq \mathcal{X}$ & $F"\mathcal{X} \subseteq \mathcal{Y}$).

The following proposition is a list of some elementary properties of the relations in question.

Proposition. 1) $\hat{\approx}$ is an equivalence.
2) $\hat{\preceq}$ is transitive.

3) $\mathfrak{X} \approx \mathfrak{Y} \rightarrow P(\mathfrak{X}) \hat{\approx} P(\mathfrak{Y})$, $\mathfrak{X} \hat{\trianglelefteq} \mathfrak{Y} \rightarrow P(\mathfrak{X}) \hat{\trianglelefteq} P(\mathfrak{Y})$.

4) Let $\mathfrak{J}$ be a cut. Then $\mathfrak{X} \hat{\trianglelefteq} \mathfrak{J} \rightarrow P(\mathfrak{X}) \subseteq [V]^{\mathfrak{J}}$..

Theorem. (Cantor-Bernstein.) Let ξ, ν be two semisets such that $\xi \hat{\trianglelefteq} \nu \;\&\; \nu \hat{\trianglelefteq} \xi$. Then $\xi \hat{\approx} \nu$.

Proof. Let f, g be one-one functions such that $\mathrm{dom}(f) \supseteq \xi$, $\mathrm{dom}(g) \supseteq \nu$ and $f''\xi \subseteq \nu$, $g''\nu \subseteq \xi$ hold.

Put $x = \mathrm{dom}(f)$, $y = \mathrm{dom}(g)$. The function $h: P(x) \rightarrow P(y)$ is defined by the relation $f(u) = x - g''(y - f''u)$. It is monotonic (w.r.t. $\subseteq$) and, consequently, there is a $c \subseteq x$ such that $h(c) = c$. Indeed, let $C = \{u \subseteq x; u \subseteq h(u)\}$; then $c = \bigcup C$ has the required properties. We deduce that $c = x - g''(y - f''c)$ and $x - c \subseteq g''(y - f''c)$ hold. Assume that $a \in x - c$. Then $a \in \mathrm{rng}(g)$ and $g^{-1}{}''(a) \in f''c$. Thus the mapping $t: x \rightarrow V$, defined by the formulas $t(a) = f(a)$ iff $a \in c$ and $t(a) = g^{-1}(a)$ iff $a \in x - c$, is one-one. To finish our proof we prove $t''\xi \subseteq \nu$. Choose $b \in \nu - f''\xi$. Then $g(b) \in \xi - c$. Consequently, $t(g(b)) = g^{-1}(g(b)) = b$ holds.

Assume that $\xi \subseteq X$, $\nu \subseteq Y$. We define

$$\binom{X}{\xi} \Rightarrow \binom{Y}{\nu} = \{f \neq 0; f \text{ is a function} \;\&\; \xi \subseteq \mathrm{dom}(f) \;\&\; \nu \subseteq \mathrm{rng}(f) \subseteq Y\}.$$

Writing

$$X \Rightarrow Y$$

we mean $\binom{X}{0} \Rightarrow \binom{Y}{0}$.

Assuming $X \neq 0$ we have $X \Rightarrow Y = {}^{X}Y$.

We define the mapping $F: X \Rightarrow (Y \Rightarrow A) \rightarrow (X \times Y) \Rightarrow A$ as follows: Let $f \in X \Rightarrow (Y \Rightarrow A)$. Then $F(f)$ is a function defined on $D(f) = \bigcup\{\{x\} \times \mathrm{dom}(f(x)); x \in \mathrm{dom}(f)\}$ by the relation

$$F(f)(x,y) = f(x)(y).$$

We can see that F is a one-one mapping onto $(X \times Y) \Rightarrow A$. Let us prove that F maps $\binom{X}{\xi} \Rightarrow \left(\binom{Y}{\nu} \Rightarrow A\right)$ onto $\binom{X \times Y}{\xi \times \nu} \Rightarrow A$. First, F maps the class in question into the second one. Indeed, let $f \in \binom{X}{\xi} \Rightarrow \left(\binom{Y}{\nu} \Rightarrow A\right)$. We have $(\forall x \in \xi)(\exists y \in \nu)(x \in \mathrm{dom}(f) \;\&\; y \in \mathrm{dom}(f(x)))$ and, consequently, $F(f) \in \binom{X \times Y}{\xi \times \nu} \Rightarrow A$. Choose $g \in \binom{X \times Y}{\xi \times \nu} \Rightarrow A$. Let f be defined by the relation $f(x)(y) = g(x,y)$ where $\langle x,y \rangle \in \mathrm{dom}(g)$. Then $f \in X \Rightarrow (Y \Rightarrow A)$ and $F(f) = g$. We have $\xi \times \nu \subseteq \mathrm{dom}(g)$, thus $(\forall x \in \xi)(f(x) \in \binom{Y}{\nu} \Rightarrow A)$, i.e. $f \in \binom{X}{\xi} \Rightarrow \left(\binom{Y}{\nu} \Rightarrow A\right)$. Consequently, F is onto and we just have proved

Proposition. $\binom{X}{\xi} \Rightarrow \left(\binom{Y}{\nu} \Rightarrow A\right) \approx \binom{X \times Y}{\xi \times \nu} \Rightarrow A.$

Proposition. Let $\mathcal{J}$, $\mathcal{Y}$ be two cuts. Then

$$\mathcal{J} \approx \mathcal{Y} \text{ iff } \mathcal{J} = \mathcal{Y}.$$

Proof. Assume that f is a one-one mapping such that $\mathrm{dom}(f) \supseteq \mathcal{J}$ and $f''\mathcal{J} = \mathcal{Y}$. Take $\alpha \in \mathcal{J}$. Then $f''\alpha \subseteq \mathcal{Y}$ and, consequently, $\max f''\alpha \in \mathcal{Y}$. We deduce from this that $\alpha \in \mathcal{Y}$ holds.

Now, we shall study the sum $\bigcup \mathcal{Z}$ under presumption that $\mathcal{Z} \approx \mathcal{Y}$ and $\mathcal{Z} \subseteq [V]^{\mathcal{J}}$, where $\mathcal{J}$ is a cut. The required results will be obtained, namely, under presumption that $\mathcal{J}$ is <u>regular cut</u>, i.e. the formula $(\forall u)(u \cap \mathcal{J}$ is unbounded in $\mathcal{J} \rightarrow u \cap \mathcal{J} \approx \mathcal{J})$ holds.

Lemma. A cut $\mathcal{J}$ is regular iff $(\forall u \subseteq N)(\forall \alpha, d)$ ($u \cap \mathcal{J}$ is unbounded in $\mathcal{J}$ & d is an isomorphism of $\langle \alpha, \in \rangle$ and $\langle u, \in \rangle \rightarrow d^{-1}{}''(\mathcal{J} \cap u) = \mathcal{J}$).

Proof. The implication from right to left is trivial; let us prove the converse one. Let u, α, d be such sets as is required and put $\mathcal{Y} = d^{-1}{}''(u \cap \mathcal{J})$. Then $\mathcal{Y}$ is a cut. We conclude, by using the regularity of $\mathcal{J}$, that $u \cap \mathcal{J} \approx \mathcal{J}$. We have $\mathcal{Y} \approx u \cap \mathcal{J}$ and $\mathcal{Y} \approx \mathcal{J}$ holds.

To formulate the required results, we use the following definitions: A function $h \in \bigcup_{\alpha \in N} {}^{\alpha}V$ is called <u>$\mathcal{J}$-function</u> iff the formula $\alpha \in \mathcal{J} \cap \mathrm{dom}(h) \rightarrow$ $\rightarrow h(\alpha) \in [V]^{\alpha}$ holds. h is a <u>total $\mathcal{J}$-function</u> if, moreover, $\mathrm{dom}(h) \supseteq \mathcal{J}$.

Theorem. Let $\mathcal{J}$ be a regular cut, closed under $\cdot$ (multiplication), and let h be an $\mathcal{J}$-function. Then 1) $\bigcup h''\mathcal{J} \approx \mathcal{J} \vee (\exists \gamma \in \mathcal{J})(\bigcup h''\mathcal{J} \approx \gamma)$,
2) $\bigcup h''\mathcal{J} \approx \mathcal{J} \leftrightarrow \neg(\exists \alpha \in \mathcal{J})(\bigcup h''\mathcal{J} = \bigcup h''\alpha)$.

Proof. Note that 2) is an easy consequence of 1) and the equivalence $\bigcup h''\mathcal{J}$ is a set $\leftrightarrow (\exists \gamma \in \mathcal{J})(\bigcup h''\gamma = \bigcup h''\mathcal{J})$. We prove the assertion 1) in two steps (A), (B).

(A) Assume that h is, moreover, an <u>exact function</u>, i.e. h is a function such that

$$\alpha \in \mathrm{dom}(h) \rightarrow h(\alpha) \neq 0$$

and

$$\alpha \neq \beta \rightarrow h(\alpha) \cap h(\beta) = 0.$$

hold. Then $\bigcup h''\mathcal{J} \approx \mathcal{J} \vee (\exists \gamma \in \mathcal{J})(\bigcup h''\mathcal{J} \approx \gamma)$ is satisfied.

Proof: Put, for $\gamma \in \mathrm{dom}(h)$, $g(\gamma) = \| h(\gamma) \|$ and let $\bar{g}(\alpha) = \sum \{ g(\gamma);$ $\gamma \leq \alpha \}$. Then $\bar{g}$ is an increasing function. Let, for $\alpha \in \mathrm{dom}(\bar{g})$, $\alpha \geq 1$, I_{α}

be the interval $[\bar{g}(\alpha-1)+1,\bar{g}(\alpha)]$ and $I_0=[0,\bar{g}(0)]$. We assume that $\mathcal{J}$ is closed under $\bullet$; thus, $\alpha\in\mathcal{J}\rightarrow\bar{g}(\alpha)\in\mathcal{J}$ holds. We have $\bigcup_{\alpha\in\mathcal{J}} I_\alpha \subsetneq \mathcal{J}\leftrightarrow \mathrm{dom}(h)\in\mathcal{J}$ and $\bigcup I_\alpha=\mathcal{J}\leftrightarrow\mathrm{dom}(h)\notin\mathcal{J}$. Let $\{t_\alpha\}_{\alpha\in\mathrm{dom}(h)}$ be a set such that $t_\alpha : I_\alpha \overset{1-1}{\longleftrightarrow} h(\alpha)$ holds for every $\alpha\in\mathrm{dom}(H)$. We define the function $\bar{h}$ on $\bigcup_{\alpha\in\mathrm{dom}(h)} I_\alpha$ by the relation: $\bar{h}\restriction I_\alpha=t_\alpha$, $\alpha\in\mathrm{dom}(h)$. We obtain immediately

$$\bar{h}''\bigcup_{\alpha\in\mathcal{J}} I_\alpha=\bigcup_{\alpha\in\mathcal{J}} t_\alpha'' I_\alpha=\bigcup_{\alpha\in\mathcal{J}} h(\alpha)=\bigcup h''\mathcal{J}.$$

The function $\bar{h}$ is one-one, thus, consequently, $\bigcup h''\mathcal{J}\approx\bigcup_{\alpha\in\mathcal{J}} I_\alpha$ holds. We can conclude that the proposition in question is true. (Note that we have not used the presumption that $\mathcal{J}$ is regular.)

(B) **Lemma.** Let $\mathcal{J}$ be a regular cut and let $f\in\mathcal{U}^{\alpha}V$ be a disjoint function. Then there exists an exact function $g\in\mathcal{U}^{\alpha}V$ such that

(1) $f''\mathcal{J}=g''\mathcal{J}$,

(2) if f is an $\mathcal{J}$-function then g is, too

Proof. Put $u=\{\alpha\in\mathrm{dom}(f);f(\alpha)\neq 0\}$.

(i) $u\cap\mathcal{J}$ is bounded in $\mathcal{J}$, put $v=u\cap\mathcal{J}$. Let d,δ be such that d is an isomorphism of $\langle\delta,\in\rangle$ and $\langle v,\in\rangle$. We define, for $\alpha\in\delta$, $g(\alpha)=f(d(\alpha))$. The function g has the required properties.

(ii) Let $u\cap\mathcal{J}$ be unbounded in $\mathcal{J}$, let d,δ be such that d is an isomorphism of $\langle\delta,\in\rangle$ and $\langle u,\in\rangle$. Put, for $\alpha\in\mathrm{dom}(d)$, $g(\alpha)=f(d(\alpha))$. We can see that $g''\mathcal{J}=g''(d^{-1}{}''(\mathcal{J}\cap u))=f''(\mathcal{J}\cap u)=f''\mathcal{J}$. Thus (1) and (2) hold.

To finish our proof, we put, for $\alpha\in\mathrm{dom}(h)$, $f(\alpha)=h(\alpha)-h''\alpha$. Then f is a disjoint $\mathcal{J}$-function such that $f''\alpha=h''\alpha$. Let g be a function, guaranteed by the preceding lemma. Then $\bigcup h''\mathcal{J}=\bigcup g''\mathcal{J}$ and we can use the part (A).

Corollary. Let $\mathcal{J}$ be a regular cut, closed under $\bullet$. Then

$$\mathcal{Z}\approx\mathcal{J}\ \&\ \mathcal{Z}\subseteq[V]^{\mathcal{J}}\rightarrow(\bigcup\mathcal{Z}\approx\mathcal{J}\vee\bigcup\mathcal{Z}\in[V]^{\mathcal{J}}).$$

Proof. Then there exists a one-one total $\mathcal{J}$-function f such that $\mathcal{Z}=f''\mathcal{J}$. The assertion is a consequence of the previous theorem.

We say that a function is $\mathcal{U}$-<u>unbounded in</u> $\mathcal{J}$ iff $\neg(\exists\alpha\in\mathcal{J})(\bigcup f''\mathcal{J}=\bigcup f''\alpha)$ holds.

Remark. If the function f, presented in the previous proof, is $\mathcal{U}$-unbounded in $\mathcal{J}$ then $\bigcup\mathcal{Z}\approx\mathcal{J}$.

Proposition. Let $\mathcal{J}$ be a cut, closed under $\bullet$. $\mathcal{J}$ is closed under the function 2^x iff every one-one total $\mathcal{J}$-function is $\bigcup$-unbounded in $\mathcal{J}$.

Proof. 1) Suppose that there exists $\delta \in \mathcal{J}$ such that $2^\delta \notin \mathcal{J}$. Let $h: 2^\delta \xleftrightarrow{1-1} P(\delta)$ such that $h(\alpha) = \alpha$ holds for every $\alpha \leq \delta$. We have $\bigcup rng(h) = \delta$. Thus h is a one-one total $\mathcal{J}$-function which is not $\bigcup$-bounded in $\mathcal{J}$.

2) Assume that $\mathcal{J}$ is closed under 2^x; let f be a one-one total $\mathcal{J}$-function. Suppose that f is not $\bigcup$-unbounded in $\mathcal{J}$. Then there exists $\alpha \in \mathcal{J}$ such that $v = \bigcup f''\alpha = \bigcup f''\mathcal{J}$ and, consequently, $P(v) \subseteq f''\mathcal{J}$ holds. We can conclude, using the presumption that $\mathcal{J}$ is closed under $\cdot$ and 2^x, that $v \in [V]^{\mathcal{J}}$ and $2^{\|v\|} \in \mathcal{J}$. Further, $f''\gamma \subseteq P(v)$ holds for some $\gamma \supseteq \mathcal{J}$, and, by using the fact that f is one-one, we see that $\|v\| \geq \gamma$, which is a contradiction.

Corollary. Let $\mathcal{J}$ be a regular cut, closed under 2^x. Then
$\mathcal{X} \precsim \mathcal{J} \,\&\, \mathcal{X} \subseteq [V]^{\mathcal{J}} \rightarrow \bigcup \mathcal{X} \precsim \mathcal{J}$.

The function $f: \delta \times \delta \rightarrow V$ is called $\mathcal{J} \times \mathcal{J}$-<u>function</u> iff $\alpha \in \mathcal{J} \rightarrow$ $\rightarrow f(\alpha)''\mathcal{J} \mathrel{\hat{\preceq}} \mathcal{J}$ holds.

Theorem. Let $\mathcal{J}$ be a regular cut, closed under $\cdot$. Let f be an $\mathcal{J} \times \mathcal{J}$-function. Then

1) $\bigcup_{\alpha \in \mathcal{J}} f(\alpha)''\mathcal{J} \mathrel{\hat{\preceq}} \mathcal{J}$.

2) If $f(\beta)''\mathcal{J} \approx \mathcal{J}$ for some $\beta \in \mathcal{J}$ then $\bigcup_{\alpha \in \mathcal{J}} f(\alpha)''\mathcal{J} \approx \mathcal{J}$.

Proof. Put, for γ such that $\langle \gamma, \gamma \rangle \in dom(f)$,

$$\hat{f}(\gamma) = \bigcup_{\alpha \in \mathcal{J}} f(\alpha)''\gamma .$$

Then $\bigcup \hat{f}''\mathcal{J} = \bigcup_{\alpha \in \mathcal{J}} f(\alpha)''\mathcal{J}$. Indeed, $x \in \bigcup \hat{f}'' \leftrightarrow (\exists \gamma \in \mathcal{J})(x \in \hat{f}(\gamma)) \leftrightarrow$ $\leftrightarrow (\exists \gamma \in \mathcal{J})(\exists \alpha < \gamma)(x \in f(\alpha)''\gamma) \leftrightarrow (\exists \alpha \in \mathcal{J})(x \in f(\alpha)''\mathcal{J}) \leftrightarrow x \in$ $\in \bigcup_{\alpha \in \mathcal{J}} f(\alpha)''\mathcal{J}$ holds. We have, moreover, $\langle \alpha, \beta \rangle \in (\mathcal{J} \times \mathcal{J}) \cap dom(f) \rightarrow$ $\rightarrow \|f(\alpha)''\beta\| \in \mathcal{J}$. Thus $\gamma \in \mathcal{J} \cap dom(\hat{f}) \rightarrow \|\hat{f}(\gamma)\| \leq \gamma \cdot \max\{\|f(\alpha)''\gamma\|; \alpha < \gamma\} \in \mathcal{J}$.
Consequently, $\hat{f}$ is an $\mathcal{J}$-function and the proof can be finished by using the previous theorem.

§ 2. Figures in an $\mathcal{J}$-indiscernibility. Our intention is to study, with respect to the set-like subvalence (requivalence resp.) to $\mathcal{J}$, a figure $\mathcal{X}$ in an $\mathcal{J}$-indiscernibility, submitted to the condition $\mathcal{P}(\mathcal{X}) \subseteq [V]^{\mathcal{J}}$.

First, our aim is to prove

Theorem. Let $\mathcal{J}$ be a cut closed under $\bullet$. Let $\mathfrak{X}$ be a figure in an $\mathcal{J}$-indiscernibility $\mathcal{E}$ on A such that $\mathcal{P}(\mathfrak{X}) \subseteq [A]^{\mathcal{J}}$.

Then there exists a (total) $\mathcal{J}$-function f such that $\mathfrak{X} \subseteq \bigcup f''\mathcal{J}$ holds.

To do this, we shall study the situation in question more generally.

Let $R=\{R_\alpha\}_{\alpha\in\eta}$ be a creating system of an $\mathcal{J}$-indiscernibility $\mathcal{E}$ on A. By an $\langle R,\mathcal{J}\rangle$-<u>system</u> we mean every system

$$\mathcal{W}=\{R_\alpha''\{\alpha\} ; x\in u(\alpha) \,\&\, \alpha \in \mathcal{J}\},$$

where u is a total $\mathcal{J}$-function such that $\mathrm{rng}(u)\subseteq A$.

Let T be a relation with $\mathrm{dom}(T)\subseteq N$. T is called <u>$\mathcal{J}$-chain</u> iff $\mathrm{dom}(T)\supseteq\mathcal{J}$ and $T''\{\alpha+1\}\subseteq T''\{\alpha\}$ holds for every $\alpha+1\in\mathrm{dom}(T)$. We designate $T''\{\alpha\}$ by T_α.

$\mathcal{W}$ <u>is compatible with</u> T <u>w.r.t.</u> a property $\varphi(x,\mathfrak{X})$ iff

$(\forall\alpha\in\mathcal{J})(\forall z\in u(\alpha))\ \{y;\varphi(y,R_\alpha''\{z\})\}\subseteq T_\alpha$.

Theorem (on compatible covering of figures). Let R be a creating system for an $\mathcal{J}$-indiscernibility $\mathcal{E}$ on A. Assume that $\mathfrak{X}$ is a figure in $\mathcal{E}$ and let $\varphi(x,\mathfrak{X})$ be a normal formula (possibly with standard parameters) which is monotonic w.r.t. $\mathfrak{X}$ (i.e. $\varphi(x,\mathfrak{X})\,\&\,\mathfrak{X}\subseteq\mathfrak{X}'\rightarrow\varphi(x,\mathfrak{X}')$ holds. Let T be an $\mathcal{J}$-chain such that

$$\{x;\varphi(x,\mathfrak{X})\}\subseteq\bigcup_{\alpha\in\mathcal{J}}T''\{\alpha\}.$$

Then there exists an $\langle R,\mathcal{J}\rangle$-system $\mathcal{W}$ which is compatible with T w.r.t. $\varphi(x,\mathfrak{X}($ and covers $\mathfrak{X}$ (i.e. $\mathfrak{X}\subseteq\bigcup\mathcal{W}$).

Proof of the first theorem of this section by using the just presented one. Let R be creating for $\mathcal{E}$. Put $T''\{\alpha\}=[A]^{\leq\alpha}$, $\varphi(x,\mathfrak{X})\leftrightarrow x\subseteq\mathfrak{X}$. Let u be such that the $\langle R,\mathcal{J}\rangle$-system $\mathcal{W}=\{R_\alpha''\{z\};z\in u(\alpha)\,\&\,\alpha\in\mathcal{J}\}$ is compatible with T w.r.t. $\varphi(x,\mathfrak{X})$ and covers $\mathfrak{X}$. Then $\alpha\in\mathcal{J}\,\&\,z\in u(\alpha)\longrightarrow$ $\longrightarrow P(R_\alpha''\{z\})\subseteq[A]^{\leq\alpha}$, i.e. $\alpha\in\mathcal{J}\,\&\,z\in u(\alpha)\longrightarrow R_\alpha''\{z\}\in[A]^{\leq\alpha}$. Especially, $\alpha\in\mathcal{J}\longrightarrow\bigcup\{R_\alpha''\{z\};z\in u(\alpha)\}\in[A]^{\mathcal{J}}$. Thus, there is a $\delta\supseteq\mathcal{J}$ such that

$\alpha\in\delta\longrightarrow\bigcup\{R_\alpha''\{z\};z\in u(\alpha)\}$ is a set.

Put, for $\alpha\in\delta$, $f(\alpha)=\bigcup\{R_\alpha''\{z\};z\in u(\alpha)\}$. Then f is a total $\mathcal{J}$-function and $\mathfrak{X}\subseteq\bigcup f''\mathcal{J}$.

Proof of the last theorem. We can assume that $\mathrm{dom}(R)=\mathrm{dom}(T)=\eta$ for some $\eta\supseteq\mathcal{J}$ and that $s:\eta\longrightarrow P(A)$ is a function such that $s(\alpha)$ is a maximal

R_{α}-net. Put, for $\alpha \in \eta$,

$$u(\alpha)=\{x; x\in s(\alpha) \& \{y; \varphi(y, R''_{\alpha}\{x\})\}\subseteq T_{\alpha}\}.$$

Then $\mathcal{W}=\{R''_{\alpha}\{z\}; z\in u(\alpha) \& \alpha\in \mathcal{J}\}$ is compatible with T w.r.t. $\varphi(x,\mathcal{X})$. We must only prove that $\mathcal{W}$ covers $\mathcal{Z}$. Let $z\in\mathcal{Z}$ be arbitrary. Assume that there exists an $\alpha\in\mathcal{J}$ such that

(1) $$\{y; \varphi(y, R''_{\alpha}\{z\})\}\subseteq T_{\alpha+1}.$$

Choose $\hat{z}\in s(\alpha+1)$ with $\langle z,\hat{z}\rangle\in R_{\alpha+1}$. We have $R''_{\alpha+1}\{\hat{z}\}\subseteq R''_{\alpha}\{z\}$. We obtain, by using the fact that $\varphi(x,\mathcal{X}($ is monotonic in $\mathcal{X}$, that

$$\{y; \varphi(y)R''_{\alpha+1}\{\hat{z}\})\}\subseteq\{y; \varphi(y, R''_{\alpha}\{z\})\}\subseteq T_{\alpha+1}.$$

Thus $\hat{z}\in u_{\alpha+1}$ and $z\in R''_{\alpha+1}\{\hat{z}\}\subseteq R''_{\alpha+1}u(\alpha+1)$.

We must prove yet that there exists $\alpha\in\mathcal{J}$ such that (1) holds. Assume the contrary that (1) is false for every $\alpha\in\mathcal{J}$. Thus, there exists $\gamma\notin\mathcal{J}$ such that $\neg(\{y; \varphi(y, R''_{\gamma}\{z\})\}\subseteq T_{\gamma+1})$; choose y with $\varphi(y, R''_{\gamma}\{y\}) \& y\notin T_{\gamma+1}$. φ is monotonic, thus $\varphi(y, \mathcal{E}''\{y\})$ holds and $\varphi(y,\mathcal{Z})$ is satisfied, too. We have $y\in\bigcup_{\alpha\in\mathcal{J}}T_{\alpha}\subseteq T_{\gamma+1}$, which is a contradiction.

Theorem. Let $\mathcal{J}$ be a regular cut, closed under $\bullet$. Suppose that $\mathcal{Z}$ is a figure in an $\mathcal{J}$-indiscernibility.

Then $P(\mathcal{Z})\subseteq[V]^{\mathcal{J}} \leftrightarrow (\mathcal{Z}\stackrel{\sim}{\rightarrow}\mathcal{J}\vee(\exists u\in[V]^{\mathcal{J}})(\mathcal{Z}\in u))$.

Proof. We have an $\mathcal{J}$-function f such that $\mathcal{Z}\subseteq\bigcup f''\mathcal{J}$ (see the first theorem of this section). Thus the implication from left to right follows from the theorem of the first section. The converse implication is trivial.

We define, for a class $\mathcal{X}$, inner cut $\rho^{-}(\mathcal{X})$ by

$$\rho^{-}(\mathcal{X})=\{\alpha; (\exists u\subseteq\mathcal{X})(u\stackrel{\sim}{\rightarrow}\alpha)\}.$$

We can see that $\rho^{-}(\mathcal{X})$ is closed under $\preceq$ and $\rho^{-}(\mathcal{X})$ is a cut iff $\mathcal{X}$ is no set.

Assume that ξ is a semiset. Outer cut $\rho^{+}(\xi)$ is defined by

$$\rho^{+}(\xi)=\bigcap\{\alpha; (\exists u\supseteq\xi)(u\stackrel{\sim}{\rightarrow}\alpha)\}.$$

$\rho^{+}(\xi)$ is closed under $\preceq$; it is a cut iff ξ is no set. We have, for every set x, $\rho^{-}(x)=\rho^{+}(x)=\|x\|$.

Note that $\rho^{-}(\xi)\subseteq\rho^{+}(\xi)$ holds for every semiset ξ and $\rho^{-}(\mathcal{J})=\mathcal{J}$ is true for every cut $\mathcal{J}$.

Theorem. Let $\mathfrak{I}$ be a regular cut, closed under $\bullet$. Let $\mathfrak{Z}$ be a figure in an $\mathfrak{I}$-indiscernibility.

Then the following are equivalent:

1) $\wp^{-}(\mathfrak{Z})=\mathfrak{I}$.

2) $\mathfrak{Z}$ is a semiset $\&\ \wp^{+}(\mathfrak{Z})=\mathfrak{I}$.

3) $\mathfrak{Z}\,\hat{\approx}\,\mathfrak{I}$.

Proof. At first, $\wp^{\square}(\mathfrak{Z})=\mathfrak{I}\rightarrow P(\mathfrak{Z})\subseteq[V]^{\mathfrak{I}}$ and $\wp^{\square}(\mathfrak{Z})=\mathfrak{I}\rightarrow\neg(\exists u\in\in[V]^{\mathfrak{I}})(\mathfrak{Z}\in u)$ hold for $\square=+$ and $\square=-$. We can see, by using the previous theorem, that $\wp^{-}(\mathfrak{Z})=\mathfrak{I}\rightarrow\mathfrak{Z}\,\hat{\approx}\,\mathfrak{I}$ and, consequently, $(1)\rightarrow(2)$ holds. We deduce quite analogously that $(2)\rightarrow(3)$ holds, too. The implication $(3)\rightarrow\rightarrow(1)$ is trivial.

§ 3. **Some properties of inner and outer cuts.** In this last section, we present some elementary properties of cuts in question and we discuss the existence of semisets with the prescribed inner and outer cut.

Throughout this section, $\mathfrak{s},\mathfrak{v},\varsigma,\mathfrak{s}_0,\dots$ range over semisets.

Proposition. 1) $\mathfrak{s}\,\hat{\trianglelefteq}\,\mathfrak{v}\rightarrow(\wp^{-}(\mathfrak{s})\subseteq\wp^{-}(\mathfrak{v})\ \&\ \wp^{+}(\mathfrak{s})\subseteq\wp^{+}(\mathfrak{v}))$,

2) $\mathfrak{s}\,\hat{\approx}\,\mathfrak{v}\rightarrow(\wp^{-}(\mathfrak{s})=\wp^{-}(\mathfrak{v})\ \&\ \wp^{+}(\mathfrak{s})=\wp^{+}(\mathfrak{v}))$.

3) $(\wp^{-}(\mathfrak{s})\subsetneqq\wp^{-}(\mathfrak{v})\ \&\ \wp^{+}(\mathfrak{v})\subsetneqq\wp^{+}(\mathfrak{s}))\rightarrow\neg(\mathfrak{s}\,\hat{\trianglelefteq}\,\mathfrak{v})\ \&\ \neg(\mathfrak{v}\,\hat{\trianglelefteq}\,\mathfrak{s})$.

Proof. 1) Let f be a one-one mapping with $\mathrm{dom}(f)\supseteq\mathfrak{s}$ and $f''\mathfrak{s}\subseteq\mathfrak{v}$. If $u\in[\mathfrak{s}]^{\wp^{-}(\mathfrak{s})}$ then $f''u\,\hat{\approx}\,u$ and $f''u\in P(\mathfrak{v})$ holds. We conclude that $\|u\|\in\in\wp^{-}(\mathfrak{v})$. Assuming $v\supseteq\mathfrak{v}$ we can see that $f^{-1}{}''v=w\supseteq\mathfrak{s}$ and, consequently, $\|w\|\notin\wp^{+}(\mathfrak{s})$. Thus, $\|v\|\notin\wp^{+}(\mathfrak{s})$ holds, too. 2) and 3) are immediate consequences of 1).

Proposition. $\wp^{+}(\mathfrak{s})\subsetneqq\wp^{-}(\mathfrak{v})\rightarrow\mathfrak{s}\,\hat{\trianglelefteq}\,\mathfrak{v}$.

Proof. Suppose that $\mathfrak{s}\subseteq u$ and $\|u\|\in\wp^{-}(\mathfrak{v})$. We have a $v\subseteq\mathfrak{v}$ such that $\|u\|=\|v\|$. Thus $\mathfrak{s}\subseteq u\,\hat{\approx}\,v\subseteq\mathfrak{v}$ and, consequently, $\mathfrak{s}\,\hat{\trianglelefteq}\,\mathfrak{v}$.

Let $\mathfrak{J}\subseteq\eta$ be a cut, $\eta\in N$. We define

$$\eta\div\mathfrak{J}=\bigcap\{\eta-\alpha;\ \alpha\in\mathfrak{J}\}.$$

Proposition. Let $\mathfrak{I},\mathfrak{J}\subseteq\eta$ be cuts, $\eta\in N$.

1) $\eta\div\mathfrak{J}=\{\eta-\gamma;\ \gamma\in\eta-\mathfrak{J}\}$,

2) $\eta\div\mathfrak{J}\,\hat{\approx}\,\eta-\mathfrak{J}$,

3) $\mathfrak{I}\subseteq\mathfrak{J}\rightarrow(\eta\div\mathfrak{J}\subseteq\eta\div\mathfrak{I})$,

4) $\eta \div (\eta \div \mathcal{J}) = \mathcal{J}$,

5) $\mathcal{I} = \eta \div \mathcal{J} \leftrightarrow \mathcal{J} = \eta \div \mathcal{I}$

6) $\alpha \in \eta \rightarrow (\eta - \alpha \in \mathcal{J} \leftrightarrow \alpha \in \eta \div \mathcal{J})$.

Proof. Let us prove 1) and 2) only. 1) $\sigma \in \bigcap\{\eta - \alpha, \alpha \in \mathcal{J}\}$
$\leftrightarrow (\forall \alpha \in \mathcal{J})(\sigma < \eta - \alpha) \leftrightarrow (\forall \alpha \in \mathcal{J})(\alpha < \eta - \sigma) \leftrightarrow \mathcal{J} \leq \eta - \sigma$
$\leftrightarrow (\exists \gamma \in \eta - \mathcal{J})(\gamma < \eta - \sigma) \leftrightarrow (\exists \gamma \in \eta - \mathcal{J})(\sigma < \eta - \gamma) \leftrightarrow$
$\leftrightarrow (\exists \gamma \in \eta - \mathcal{J})(\sigma = \eta - \gamma)$.

2) Put, for $\gamma \in \eta$, $f(\gamma) = \eta - \gamma$. Then $f''(\eta - \mathcal{J}) = \eta \div \mathcal{J}$.

Proposition. Assume $\xi \leq \eta \in N$. Then

1) $\rho^-(\eta - \xi) = \eta \div \rho^+(\xi)$,

2) $\rho^+(\xi) = \eta \div \rho^-(\eta - \xi)$.

Proof. Only 1) must be proved; 2) is a consequence of 1) and the previous proposition. We have
$\alpha \in \rho^-(\eta - \xi) \leftrightarrow (\exists u \leq \eta - \xi)(\|u\| = \alpha) \leftrightarrow (\exists v)(\xi \leq v \leq \eta \,\&\, \|v\| = \eta - \alpha) \leftrightarrow$
$\leftrightarrow \eta - \alpha \geq \rho^+(\xi) \leftrightarrow \alpha \in \eta \div \rho^+(\xi)$ (see 6) of the previous proposition).

Here and down, let η denote a fixed number from N-FN.

Now, our aim is the following: let $\mathcal{J}^- \leq \mathcal{J}^+ \leq \eta$ be two cuts. We are looking for a semiset $\mathcal{Z} \leq \eta$ such that $\rho^-(\mathcal{Z}) = \mathcal{J}^-$ and $\rho^+(\mathcal{Z}) = \mathcal{J}^+$ holds. Our problem can be reduced to an anlogous one concerning the inner cuts only. Indeed, let $\mathcal{J}_0 = \mathcal{J}^-$, $\mathcal{J}_1 = \eta \div \mathcal{J}^+$ and suppose that $\mathcal{Z} \leq \eta$ satisfies the conditions:

$$\rho^-(\mathcal{Z}) = \mathcal{J}_0,\ \rho^-(\eta - \mathcal{Z}) = \mathcal{J}_1.$$

Then

$$\rho^+(\mathcal{Z}) = \eta \div \rho^-(\eta - \mathcal{Z})$$

and, consequently,

$$\rho^+(\mathcal{Z}) = \eta \div (\eta \div \mathcal{J}^+) = \mathcal{J}^+.$$

Further, note that $\mathcal{Z}$ can be found by such a way that $\mathcal{J}_0 \leq \mathcal{Z}$ and $\eta - \mathcal{J}^+ \leq \leq \eta - \mathcal{Z}$. These two relations guarantee that

$$\rho^-(\mathcal{Z}) \geq \mathcal{J}_0 \text{ and } \rho^-(\eta - \mathcal{Z}) \geq \eta \div \mathcal{J}^+ = \mathcal{J}_1.$$

Let us describe the structure of our problem more generally: A list $\langle \mathcal{Q}, \mathcal{Z}_0, \mathcal{Z}_1 \rangle$ is said to be <u>doublet in</u> $\mathcal{Q}$ iff $\mathcal{Z}_0 \cup \mathcal{Z}_1 \leq \mathcal{Q}$ and $\mathcal{Z}_0 \cap \mathcal{Z}_1 = 0$. It is $\langle \mathcal{J}_0, \mathcal{J}_1 \rangle$-<u>doublet</u> (<u>in</u> $\mathcal{Q}$), where $\mathcal{J}_0$, $\mathcal{J}_1$ are two cuts, iff $\mathcal{Z}_i \approx \mathcal{J}_i$ holds for i=0,1. Assume that $\{\mathcal{Y}_i^{\alpha}\}_{i=0,1}^{\alpha \in \mathcal{Q}}$ is a system of subclasses of $\mathcal{Q}$. A doublet $\langle \mathcal{Q}, \mathcal{Z}_0, \mathcal{Z}_1 \rangle$ is called $\langle \mathcal{J}_0, \mathcal{J}_1 \rangle$ - $\{\mathcal{Y}_i^{\alpha}\}_{0,1}^{\mathcal{Q}}$-<u>determined</u> iff the following holds for i=0,1: $\rho^-(\mathcal{Z}_i) = \mathcal{J}_i \,\&\, (\forall \alpha \in \mathcal{Q})(\Delta(\mathcal{Y}_i^{\alpha}, \mathcal{Z}_i)$ is infinite) (we have $\Delta(\mathcal{X},\mathcal{Y})=(\mathcal{X}-\mathcal{Y})\cup(\mathcal{Y}-\mathcal{X})$). A doublet $\langle \mathcal{A}, \mathcal{Z}_0, \mathcal{Z}_1\rangle$ is <u>fully</u> <u>$\langle \mathcal{J}_0, \mathcal{J}_1\rangle$-$\{\mathcal{Y}_i^\alpha\}_{0,1}^\Omega$-determined</u> iff every <u>larger doublet</u> (i.e. a doublet $\langle \mathcal{A}, \tilde{\mathcal{Z}}_0, \tilde{\mathcal{Z}}_1\rangle$, where $\mathcal{Z}_0 \subseteq \tilde{\mathcal{Z}}_0$ and $\mathcal{Z}_1 \subseteq \tilde{\mathcal{Z}}_1$ holds) is $\langle \mathcal{J}_0, \mathcal{J}_1\rangle$-$\{\mathcal{Y}_i^\alpha\}_{0,1}^\Omega$-determined.

Here and down, let $\mathcal{A}$, $\{\mathcal{Y}_i^\alpha\}_{0,1}^\Omega$, $\mathcal{J}_0$, $\mathcal{J}_1$ have the meaning introduced above.

A doublet $\langle \mathcal{A}, \mathcal{S}_0, \mathcal{S}_1\rangle$ is said to be <u>$\langle \mathcal{J}_0, \mathcal{J}_1\rangle$-normal</u> iff it is an $\langle \mathcal{J}_0, \mathcal{J}_1\rangle$-doublet and the following holds for i=0,1. Put $\Theta_i = P(\mathcal{A}-\mathcal{S}_{\bar{i}}) - [\mathcal{A}]^{\mathcal{J}_i}$. Then we have $(\Theta_i=0 \,\&\, \mathcal{A}-\mathcal{S}_i \approx \Omega) \vee \Theta_i \neq 0 \,\&\, (\forall v \in \Theta_i)(v-\mathcal{S}_i \approx \Omega))$. We have used the notation: $\bar{0}=1$, $\bar{1}=0$; this one will be used further on.

Theorem. Let $\langle \mathcal{A}, \mathcal{S}_0, \mathcal{S}_1\rangle$ be a $\langle \mathcal{J}_0, \mathcal{J}_1\rangle$-normal doublet in $\mathcal{A}$. Then there exists a larger fully $\langle \mathcal{J}_0, \mathcal{J}_1\rangle$-$\{\mathcal{Y}_i^\alpha\}_{0,1}^\Omega$-determined doublet.

Proof. We define, for i=0,1, the relations $\mathcal{U}_i \subseteq \Omega \times \mathcal{A}$ with $\mathrm{dom}(\mathcal{U}_i) = \Omega$ as follows:

if $\Theta_i=0$ then $\mathcal{U}_i''\{\alpha\} = \mathcal{A}$,

if $\Theta_i \neq 0$ then $\{\mathcal{U}_i''\{\alpha\}; \alpha \in \Omega\} = \Theta_i$.

Let, for i=0,1, $\mathcal{F}_i$ be the function and $\mathcal{Q}_i$ the relation defined by induction on Ω as follows:

$$\mathcal{F}_i(\alpha) \in \mathcal{U}_i''\{\alpha\} - \mathcal{S}_i - (\mathcal{F}_i''(\alpha\cap\Omega) \cup \mathcal{F}_{\bar{i}}''(\alpha\cap\Omega) \cup \mathcal{Q}_i''(\alpha\cap\Omega) \cup \mathcal{Q}_{\bar{i}}''(\alpha\cap\Omega)),$$
$$\mathcal{Q}_i''\{\alpha\} \subseteq \mathcal{Y}_i^\alpha - \mathcal{S} - (\mathcal{F}_{\bar{i}}''(\alpha\cap\Omega) \cup \mathcal{Q}_{\bar{i}}''(\alpha\cap\Omega))$$

is a countable class iff $\mathcal{Y}_i^\alpha - \mathcal{S}_i \approx \Omega$,

$\mathcal{Q}_i''\{\alpha\}=0$ iff $\mathcal{Y}_i^\alpha - \mathcal{S}_i \trianglelefteq \mathrm{FN}$.

Put, for i=0,1, $\mathcal{Z}_i = \mathcal{S}_i \cup \mathcal{F}_{\bar{i}}''\Omega \cup \mathcal{Q}_{\bar{i}}''\Omega$. We have

$$\mathcal{Z}_0 \cap \mathcal{Z}_1 = 0$$

because of the following relations holding for i=0,1:

$$\mathcal{F}_i''\Omega \cap \mathcal{Q}_{\bar{i}}''\Omega=0,\ \mathcal{F}_i''\Omega \cap \mathcal{S}_i=0,\ \mathcal{Q}_i''\Omega \cap \mathcal{S}_i=0$$

and

$$\mathcal{F}_0''\Omega \cap \mathcal{F}_1''\Omega=0,\ \mathcal{Q}_0''\Omega \cap \mathcal{Q}_1''\Omega=0.$$

Further, we can see, by using the fact that $\mathcal{F}_0, \mathcal{F}_1$ are one-one functions, for i=0,1,

$$\mathcal{Z}_i - \mathcal{S}_i \approx \Omega$$

holds. Assume $\mathcal{Y}_i^\alpha - \mathcal{S}_i \trianglelefteq \mathrm{FN}$. Then $\mathcal{Z}_i - \mathcal{Y}_i^\alpha \approx \Omega$ and, moreover, $\tilde{\mathcal{Z}}_i - \mathcal{Y}_i^\alpha \approx \Omega$

holds for every $\tilde{\mathcal{Z}}_i \supseteq \mathcal{Z}_i$.

Assume $\mathcal{Y}_i^{\alpha} - \varsigma_i \approx \Omega$. Then $\mathcal{Q}_{\bar{i}}''\{\alpha\}$ is a countable subclass of $\mathcal{Z}_{\bar{i}} \cap (\mathcal{Y}_i^{\alpha} - \varsigma_i)$. Thus, assuming $\tilde{\mathcal{Z}}_i \cap \mathcal{Z}_{\bar{i}} = 0$, we obtain $FN \mathrel{\hat{\leq}} \mathcal{Y}_i^{\alpha} - \tilde{\mathcal{Z}}_i$.

Especially, the relation

$$\Delta(\mathcal{Y}_i^{\alpha}, \tilde{\mathcal{Z}}_i) \text{ is infinite}$$

holds for every $\alpha \in \Omega$, i=0,1 and $\langle \mathcal{a}, \tilde{\mathcal{Z}}_0, \tilde{\mathcal{Z}}_1 \rangle$ larger than $\langle \mathcal{a}, \mathcal{Z}_0, \mathcal{Z}_1 \rangle$.

It remains to prove the following:

Let $\langle \mathcal{a}, \tilde{\mathcal{Z}}_0, \tilde{\mathcal{Z}}_1 \rangle$ be a doublet, larger than $\langle \mathcal{a}, \mathcal{Z}_0, \mathcal{Z}_1 \rangle$. Then $\wp^{-}(\tilde{\mathcal{Z}}_i) = \mathcal{I}_i$ holds for i=0,1.

First, the relation $\wp^{-}(\tilde{\mathcal{Z}}_i) \supseteq \mathcal{I}_i$ follows from the fact that $\varsigma_i \subseteq \tilde{\mathcal{Z}}_i$. Thus, we must only prove for i=0,1 that $u \in P(\mathcal{a}) - [\mathcal{a}]^{\mathcal{I}} \rightarrow \neg(u \subseteq \tilde{\mathcal{Z}}_i)$ holds. Assume $u \in P[\mathcal{a}] - [\mathcal{a}]^{\mathcal{I}}$. If $u \cap \varsigma_{\bar{i}} \neq 0$ then $u \cap \tilde{\mathcal{Z}}_{\bar{i}} \neq 0$ holds trivially. Suppose $u \cap \varsigma_{\bar{i}} = 0$. We have $u \in \Theta$ and $\mathcal{F}_i(\alpha) \in u$ for some $\alpha \in \Omega$. Thus $\mathcal{F}_i(\alpha) \in u \cap \mathcal{Z}_{\bar{i}}$, i.e. $\neg(u \subseteq \tilde{\mathcal{Z}}_i)$ holds, too.

Let us introduce one special type of $\langle \mathcal{I}_0, \mathcal{I}_1 \rangle$-normal doublets, being connected immediately with the problem, we have started with.

We say that a cut $\mathcal{J}$ is Ω-complementary iff we have $(\forall \alpha \notin \mathcal{J})$ $(\alpha - \mathcal{J} \approx \Omega)$.

Proposition. Let $\mathcal{J}$ be an Ω-complementary cut and suppose that $\mathcal{J} \approx \varsigma \subseteq$ $\subseteq$ a. Then $a - \varsigma \approx \Omega$.

Proof. Let f be a one-one function such that $dom(f) = \gamma \supseteq \mathcal{J}$, $f''\gamma \subseteq a$ and $f''\mathcal{J} = \varsigma$. Then $\gamma - \mathcal{J} \mathrel{\hat{\approx}} f''\gamma - \varsigma \mathrel{\hat{\leq}} a - \varsigma$ holds. We have $\gamma - \mathcal{J} \approx \Omega$ and, consequently, $a - \varsigma \approx \Omega$.

Proposition. Let $\mathcal{I}_0, \mathcal{I}_1$ be Ω-complementary cuts. Then:

1) Every $\langle \mathcal{I}_0, \mathcal{I}_1 \rangle$ doublet in a set is $\langle \mathcal{I}_0, \mathcal{I}_1 \rangle$-normal.

2) Assume $\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \eta$ (for some $\eta \in N$) and let $\mathcal{I}_0, \eta \div \mathcal{I}_1$ be Ω-complementary.

Then $\langle \eta, \mathcal{I}_0, \eta - \mathcal{I}_1 \rangle$ is an $\langle \mathcal{I}_0, \mathcal{I}_1 \rangle$-normal doublet.

A proof follows immediately from the previous one. (Remember, for the case 2) that $\eta - \mathcal{I}_1 \mathrel{\hat{\approx}} \eta \div \mathcal{I}_1$ holds.)

Corollary. Let $\mathcal{I}^{-} \subseteq \mathcal{I}^{+} \subseteq \eta$ be two cuts so that $\mathcal{I}^{-}$ and $\eta \div \mathcal{I}$ are Ω-complementary. Let $\{\mathcal{Y}_i^{\alpha}\}_{0,1}^{\Omega}$ be a codable system of subclasses of η.

Then there exists a semiset $\xi \subseteq \eta$ such that

$$\rho^{-}(\xi)=\mathcal{J}^{-},\ \rho^{+}(\xi)=\mathcal{J}^{+}$$

and, moreover,

$$\Delta(y_0^{\alpha},\xi) \text{ and } \Delta(y_1^{\alpha},\eta-\xi)$$

are both infinite for every $\alpha\in\Omega$.

Example. Let $\mathcal{J}\subseteq\eta$ be an Ω- complementary cut such that $\mathcal{J}\subseteq\eta\div\mathcal{J}$. Then there exists $\xi\subseteq\eta$ with

$$\rho^{-}(\xi)=\mathcal{J},\ \rho^{+}(\xi)=\eta\div\mathcal{J}.$$

Moreover, suppose that $\mathcal{J}\subsetneq\eta\div\mathcal{J}$; put $\eta\div\mathcal{J}=\mathcal{Y}$. Then we have $\rho^{+}(\xi)=\mathcal{Y}\ \&\ \neg(\xi\approx\mathcal{Y})$.

We finish this section by a short investigation of elementary properties of Ω-complementary cuts.

Proposition. A cut $\mathcal{J}$ is Ω-complementary iff $(\forall\alpha\notin\mathcal{J})(\exists\gamma\notin FN)$ $(\mathcal{J}\subseteq\alpha-\gamma)$ holds.

Proposition. Every cut closed under + is Ω-complementary.

Proof. Assume that a cut $\mathcal{J}\subseteq\eta$ is closed under +. Then, under presumption that $\eta=2\cdot\gamma$, we have $\gamma\notin\mathcal{J}$. Thus $\eta-\mathcal{J}\trianglerighteq\gamma\approx\Omega$.

Proposition. Let $\mathcal{J}$ be such a cut that $N-\mathcal{J}$ is revealed.
Then $\mathcal{J}$ is Ω-complementary.

Proof. Assume that $\mathcal{J}\subseteq\eta$. We have $\{\eta-n;n\in FN\}\subseteq N-\mathcal{J}$ and, consequently, there exists $\gamma\notin FN$ such that $\eta-\gamma\in N-\mathcal{J}$, i.e. $\mathcal{J}\subseteq\eta-\gamma$.

Example. Assume $\delta\notin FN$. Then $\bigcup\{\delta+n;n\in FN\}$ is an Ω-complementary cut which is not closed under +.

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