Petronije S. Milojević
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SOLVABILITY OF SEMILINEAR EQUATIONS WITH STRONG NONLINEARITIES AND APPLICATIONS TO ELLIPTIC BOUNDARY VALUE PROBLEMS

P.S. MILOJEVIĆ

Abstract. Solvability of two classes of semilinear equations involving strongly nonlinear perturbations of type (M) with respect to two Banach spaces is established. An application to elliptic BV problems is also given.

Key words: Semilinear equations, noncoercive, nonlinear operators of type (M), strong nonlinearities, boundary value problems, elliptic equations.

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1. INTRODUCTION

Many problems in analysis reduce to solving operator equations of the form

\[ \lambda Cx - Ax - Nx = f, \]

where \( f \) is a given element in a Hilbert space \( H \), \( \lambda \in \mathbb{R} \), \( A \) is linear, \( C \) and \( N \) are nonlinear mappings. Motivated by applications to strongly nonlinear elliptic problems, we shall study Eq. (1) in the following setting.

(i) There is a pair \( \{V, V^*\} \) of Banach spaces in duality with \( V \subset H \subset V^* \), i.e., there is a nondegenerate continuous bilinear form \( \langle \cdot, \cdot \rangle \) on \( V \times V^* \). \( V^* \) need not be the dual of \( V \) in the usual sense.) Suppose that \( V \) is reflexive and compactly embedded in \( H \), \( |\langle x, y \rangle| \leq ||x||_V ||y||_{V^*} \) on \( V \times V^* \) and the duality \( \langle \cdot, \cdot \rangle \) is compatible with the inner product \( (\cdot,\cdot) \), i.e., \( \langle x, y \rangle = (x,y) \) for \( (x,y) \in V \times H \).

(ii) Let \( \{U, U^*\} \) be another pair of Banach spaces in duality compatible with \( (\cdot,\cdot) \) such that \( U \) is separable, \( U \subset V \) and \( V^* \subset U^* \) and the injections are continuous and dense.
(iii) $A: V \to V^*$ is a continuous "variational extension" of a closed linear mapping $A_1: D(A_1) \subset H \to H$ such that $U \subset D(A_1) \subset V$ and $< Ax, y > = (A_1x, y)$ for $x \in D(A_1)$ and $y \in V$. Moreover, let $C$, $N: D(N) \subset V \to U^*$ be such that $N - C$ is of type $(M)$ relative to $(U, V)$ with $U \subset D(N)$ and $(N - C)(U) \subset H$ (see Definition 1 below).

Under some additional conditions, we shall prove that Eq. (1) is solvable for each $\lambda \in \mathbb{R}$ and each $f \in H$. If $a$ is the quasinorm of $C$ (i.e., $a = \lim_{\|x\| \to \infty} \|Cx\|/\|x\|$) and $\lambda_1$ is the first eigenvalue of $A_1$, then the problem is not coercive when $|\lambda|a \geq \lambda_1$.

The above idea of using two pairs of Banach spaces with compatible dualities for studying (locally) coercive operator equations (with $f$ of small norm) is due to Kato [10]. Earlier, Hess [9] has also studied operator equations in a less general setting under a global coercivity condition. One importance of studying operator equations in such a setting lies in the fact that certain differential equations, which have been successfully handled earlier only by the method of Nash-Moser type (cf. Moser [15] and Rabinowitz [16]), reduce to them, and the problem of "loss of derivatives" is not present [10]. Another importance of this setting is demonstrated in the paper by an application to a class of (noncoercive) semilinear elliptic equations with strong nonlinearities (cf. also Hess [9]). Earlier, coercive quasilinear elliptic equations with strong nonlinearities have been studied by many authors using either truncation techniques and/or approximation results of Hedberg's type and generalized degree theories (e.g. [5,7,8,9,12,17]).

The second abstract problem we treat is the solvability of

$$Kz - \lambda Lz + Mz = f, \quad (z \in D(M), \quad f \in H)$$

where $L: H \to H$ is linear symmetric and compact and $K, M: D(M) \subset H \to H$ are nonlinear with $K + M$ of type $(M)$ relative to $(U, H)$. It is an extension of the problem studied by Kesavan [11] when $M: H \to H$ is completely continuous (i.e. $Mz_n \to Mz$ if $z_n \to z$ (weakly)) and $K$ is the identity.

2. SOLVABILITY OF EQ. (1) WITH $|\lambda|a < \lambda_1$

Our basic assumptions on $A_1$ and $A$ are:

(3) $A_1$ is symmetric and for some positive $c \notin \sigma(A_1)$, the spectrum of $A_1$, $B_c = A_1 + cf$ is positive, i.e., $(B_cz, z) > 0$ for $0 \neq z \in D(A_1)$ and $B_c^{-1}: H \to H$ is
compact.

(4) There are constants \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[
< Ax, x > \geq c_1 \| x \|^2 - c_2 \| x \|^2 \text{ for all } x \in V.
\]

Let \( \lambda_1 \leq \lambda_2 \leq \ldots, \lambda_k \to \infty \), be the sequence of eigenvalues of \( A_1 \) and \( \{ e_k \}_1^\infty \) be the corresponding system of orthonormal eigenvectors complete in \( U \) and \( H \). Set \( H_n = \text{lin.sp.} \{ e_1, \ldots, e_n \} \) and let \( P_n : H \to H_n \) be the orthogonal projection onto \( H_n \) for each \( n \). Since \( \{ \mu_k = \lambda_k + c \} \) and \( \{ e_k \} \) are the eigenvalues and eigenvectors of \( B_c \), we have by the variational characterization of \( \{ \mu_k \} \):

(5) \( (B_c x, x) \geq \mu_1 \| x \|^2 \) and \( (B_c (I - P_k) x, (I - P_k) x) \geq \mu_{k+1} \| (I - P_k) x \|^2, \)

\( \forall x \in D(A_1). \)

Now we define the class of permissible nonlinearities.

**Definition 1.** (cf. [9]) Let \( U \subset D(N) \subset V \) and \( N : D(N) \to U^* \). Then \( N \) is said to be of type (M) relative to \( (U, V) \) if (i) \( N \) is continuous from each finite-dimensional subspace of \( U \) into the weak topology of \( U^* \) and (ii) whenever \( \{ x_n \} \subset U, x_n \to x \) in \( V \), \( N x_n \to y \) in \( U^* \) with \( y \in V^* \) and \( \limsup < N x_n, x_n > \leq < y, x > \), then \( z \in D(N) \) and \( N z = y \). If \( y \) in (ii) is given in advance, we say that \( N \) is of type (M) at \( y \) relative to \( (U, V) \).

Recall that \( N : D(N) \to U^* \) is quasibounded if, whenever \( \{ x_n \} \subset U \) is bounded in \( V \) and \( < N x_n, x_n > \leq \text{const.} \| x_n \|_V \), then \( \{ N x_n \} \) is bounded in \( U^* \). We say that \( C \) has a linear growth if there are positive constants \( a, b \) and \( \rho \) such that

(6) \( \| C x \| \leq a \| x \| + b \text{ for all } \| x \| \geq \rho, \ x \in U. \)

Our first result is:

**THEOREM 1** (cf. [14]). Let \( |\lambda| < \lambda_1, (3), (4), \) and (6) hold, \( (N - \lambda C)(U) \subset H, (N x, x) \geq 0 \) for \( x \in U \), \( N \) be quasibounded and \( N - \lambda C \) be of type (M) relative to \( (U, V) \) and \( A : V \to V^* \) be linear and continuous. Then Eq (1) is solvable in \( V \) for each \( f \in H \).
Proof. Let $f \in H$ be fixed and choose an $r \geq \rho$ such that $\| f \| + \| \psi \| < r(\lambda_1 - \| \psi \|)$. Then, for each $x \in \partial B(0,r) \cap H_n, n \geq 1$, we have

$$(\lambda P_n Cx - A_1 x - P_n N x - P_n f, x) = (\lambda Cx - A_1 x - N x - f, x)$$

$$\leq (|\lambda| a - \lambda_1) \| x \|^2 + (\| f \| + |\lambda| b) \| x \| < 0.$$ 

Hence, the homotopy $H_n(t, x) = t(\lambda P_n Cx - A_1 x - P_n N x - P_n f) - (1-t)x \neq 0$ on $[0, 1] \times \partial B(0,r) \cap H_n$, and therefore the Brouwer degree $\text{deg}(\lambda P_n C - A_1 - P_n N - P_n f, B \cap H_n, 0) \neq 0$ for each $n \geq 1$. Thus, there is an $x_n \in B(0,r) \cap H_n$ such that $\lambda P_n Cx_n - A_1 x_n - P_n N x_n = P_n f, n \geq 1$. Moreover, (4) implies that

$$c_1 \| x_n \|_V^2 - c_2 \| x_n \|_V^2 \leq (A_1 x_n, x_n) \leq a |\lambda| \| x_n \|^2 + (\| f \| + |\lambda| b) \| x_n \|, \quad \text{and consequently, } \{x_n\} \text{ is bounded in } V.
\[ \langle - < Ax + f, x > \]. \]

Hence, \( x \in D(N) \) and \( \lambda Cx - Ax - Nz = f \) by property (M).

**Remark 1.** When \( \lambda = 0 \) (< \( \lambda_1 \)), Theorem 1 is a global analogue of the result of T. Kato [10] for mappings of the form \( T = A + N \) (compare also with Hess [9]).

**3. THE CASE \( |\lambda|a \geq \lambda_1 \)**

This is a noncoercive case and a major additional difficulty is to show that the set

\[ S_\lambda(f) = \{ x \in H_n \mid \lambda P_n Cx - A_1 x - P_n (N_1 + N_2) x = P_n f, \ n = 1, 2, \ldots \} \]

is bounded in \( H \), where now \( N = N_1 + N_2 : D(N) \subset V \to U^* \).

**PROPOSITION 1.** Let (8) and (6) hold, \( N \) be such that \( N_i(U) \subset H, i=1,2, \)

\( N_1 \) be of type (M) at 0 relative to \( (U,H) \) and

(7) \( (N_i x, x) \geq 0 \) for \( x \in U, i = 1, 2, \) and \( x = 0 \) if \( N_1 x = 0 \).

(8) \( N_1 \) \( x_n \to 0 \) for some \( \{ x_n \} \subset U \) bounded in \( H \), then \( N_1 x_n \to 0 \)

in \( U^* \).

(9) There is a \( \delta > 1 \) such that \( N_1(t x) = t^\delta N_1(x) \) for all \( x \in U, t \geq 0 \).

(10) There are positive constants \( a_1, b_1, \) and \( \delta_1 < \delta \) such that

\[ \| N_2 x \| \leq a_1 \| x \| + b_1 \text{ for all } x \in U \text{ with } \| x \| \text{ large.} \]

Then \( S_\lambda(f) \) is bounded in \( H \) for each \( \lambda \) with \( |\lambda|a \geq \lambda_1 \) and each \( f \in H \).

**Proof.** Let \( |\lambda|a \geq \lambda_1 \) be fixed and suppose that \( S_\lambda(f) \) is not bounded in \( H \)

for some \( f \in H \). Let \( x_{n_k} \in S_\lambda(f) \) be such that \( \| x_{n_k} \| \to \infty \) as \( k \to \infty \), and

set \( u_n = \frac{x_{n_k}}{\| x_{n_k} \|} \). Then

\[ (N_1 u_{n_k}, u_{n_k}) = \frac{1}{\| x_{n_k} \|^{\delta - 1}} |c \| u_{n_k} \|^2 \]

\[ -(B_c u_{n_k}, u_{n_k}) - \| x_{n_k} \|^{-1}((N_2 - \lambda_c) x_{n_k} - f, u_{n_k}) \to 0 \text{ as } k \to \infty \]

- 739 -
and \( N_1 u_{n_k} \to 0 \) in \( U^* \) by (8). Since we may assume that \( u_{n_k} \to u \) in \( H \), the \((M)\)-property of \( N_1 \) implies that \( u \in D(N_1) \) and \( N_1 u = 0 \). Hence, \( u = 0 \) by (7).

Next, let \( \alpha \in (0,1) \) and \( \epsilon > 0 \) small be fixed, \( \bar{a} = a + \epsilon \) and \( m \geq 1 \) be such that \( \lambda_{m+1} - |\lambda| a > \alpha \) and \( \| I - P_m \| f \| \leq \alpha \). Then, for each \( n_k > m \) large and fixed, (6) and (7) imply that

\[
(\| P_m x_{n_k} \|^2 + \| (I - P_m) x_{n_k} \|^2) \geq ((\lambda P_{n_k} C + c) x_{n_k}, x_{n_k})
\]

\[
= (B_c x_{n_k}, x_{n_k}) + (P_{n_k} (N_1 + N_2) x_{n_k}, x_{n_k}) + (P_{n_k} f, x_{n_k})
\]

\[
\geq (B_c P_m x_{n_k}, P_m x_{n_k}) + (B_c (I - P_m) x_{n_k}, (I - P_m) x_{n_k}) + (P_m f, P_m x_{n_k})
\]

\[
+ ((I - P_m) f, (I - P_m) x_{n_k}) \geq \mu_1 \| P_m x_{n_k} \|^2 + \mu_{m+1} \| (I - P_m) x_{n_k} \|^2
\]

\[
- \| P_m f \| \| P_m x_{n_k} \| - \| (I - P_m) f \| \| (I - P_m) x_{n_k} \|,
\]

or after rearranging,

\[
(\lambda_{m+1} - |\lambda| a) \| (I - P_m) x_{n_k} \|^2 \geq \| P_m x_{n_k} \|^2 + \| f \| \| P_m x_{n_k} \|. \]

Since \( \| (I - P_m) f \| \leq \alpha \), we get, after dividing by \( \| x_{n_k} \|^2 \),

\[
(\lambda_{m+1} - |\lambda| a) \| (I - P_m) u_{n_k} \|^2 \geq \| P_m u_{n_k} \|^2 - \alpha \| x_{n_k} \|^{-1} \| (I - P_m) u_{n_k} \|
\]

or

\[
\leq (\| \bar{a} - \lambda_1 \| P_m u_{n_k} \|^2 + \| x_{n_k} \|^{-1} \| f \| \| P_m u_{n_k} \|. \]

or

\[
12 \quad (\lambda_{m+1} - |\lambda| a) \| (I - P_m) u_{n_k} \|^2 \geq \| P_m u_{n_k} \|^2 - \alpha \| (I - P_m) u_{n_k} \|
\]

\[
\leq (\| \bar{a} - \lambda_1 \| P_m u_{n_k} \|^2 + \| f \| \| P_m u_{n_k} \|. \]

On the other hand, we may assume that \( P_m u_{n_k} \to v_0 \in H_m \) as \( k \to \infty \) and \( (I - P_m) u_{n_k} \to -v_0 \in H_m^1 \). Hence, \( v_0 = 0 \) and \( \| (I - P_m) u_{n_k} \| \to 1 \) as \( k \to \infty \) since
Finally, passing to the limit in (12) we obtain $\lambda_{m+1} - |\lambda|^i \leq \alpha$, which contradicts our choices of $\alpha$ and $m$. Hence, $S_\lambda(f)$ is bounded in $H$ for all $\lambda$ with $|\lambda|^a \geq \lambda_1$ and $f \in H$. 

Our basic result in this case is:

**Theorem 2** (cf. [14]). Let $|\lambda|^a \geq \lambda_1$, (9) - (4) hold, $N = N_1 + N_2$ be such that $N_i(U) \subset H$, $i=1,2$, $N_1$ be of type (M) at 0 relative to $(U,H)$, $u = 0$ if $(N_1u,u) = 0$ and (6) - (10) hold. Suppose that $N : D \to U^*$ is quasibounded, $N - \lambda C$ is of type (M) relative to $(U,V)$ and $A : V \to V^*$ is continuous. Then Eq. (1) is solvable in $V$ for each $f \in H$.

**Proof.** Let $f \in H$ be fixed. We will show first that each finite dimensional equation in $S_\lambda(f)$ is solvable. For each $n \geq 1$, we claim that there is a constant $c_n > 0$ such that

\[ (N_1z,x) \geq c_n \| x \|^{1+\delta} \text{ for each } x \in H_n. \]

If not, then there is a sequence $\{x_k\} \subset H_n$ for some $n$ such that

\[ (N_1x_k,x_k) \leq \frac{1}{k} \| x_k \|^{1+\delta} \text{ for each } k, \]

and, setting $u_k = \frac{x_k}{\| x_k \|}$, we get

\[ 0 \leq (N_1u_k,u_k) \leq \frac{1}{k} \to 0 \text{ as } k \to \infty. \]

We may assume that $u_k \to u$ in $H_n$ and, passing to the limit in (14), we get $(N_1u,u) = 0$. Hence, $u = 0$ in contradiction to $\| u \| = 1$, and therefore (13) holds for each $n$ and some $c_n > 0$.

Next, we choose $r_n \geq \rho$ such that $\frac{\| f \|_{1+\delta} + |\lambda|^i}{r_n} < \lambda_1 - |\lambda|^a + c_n r_n^{\delta-1}$ and note that for each $x \in \partial B(0,r_n) \cap H_n,$

\[ (\lambda Cz - A_1z - N_1z - N_2z - f, x) \leq (|\lambda|^a - \lambda_1 - c_n r_n^{\delta-1} + \frac{\| f \|_{1+\delta}}{r_n} + \frac{\| f \|_{1+\delta}}{r_n}) r_n^2 < 0. \]
Hence, as before, there is an \( x_n \in H_n \) such that \( \lambda Cx_n - A_1x_n - P_n(N_1 + N_2)x_n = P_nf \) for each \( n \geq 1 \). Moreover, \( S_\lambda(f) \) is bounded in \( H \) by Proposition 1, and is also bounded in \( V \) by (4). Finally, the completion of the theorem can be carried out as in Theorem 1.

4. SOLVABILITY OF EQ. (2)

We assume that \( K : D(M) \subset H \to H \) has a linear growth and is coercive, i.e.,

(15) There are positive constants \( a, b, c, \) and \( \rho \geq 0 \) such that

(i) \( \|Kx\| \leq a\|x\| + b \) for all \( \|x\| \geq \rho \),

(ii) \( (Kx, x) \geq c\|x\|^2 \), for all \( x \in D(M) \).

Again, the noncoercive case is harder and a result analogous to Proposition 1 holds.

PROPOSITION 2. Let \( L : H \to H \) be a linear, symmetric, positive and compact mapping, \( L\kappa = \lambda_k\kappa \) for \( k \geq 1 \) with \( \{\kappa_k\} \subset U \) and complete in \( H \), and \( \{H_n, P_n\} \) as before. Suppose that \( M = M_1 + M_2 : D(M) \subset H \to H \) is such that \( M_1 \) is quasibounded and of type (M) at 0 relative to \( (U,H), M_1, M_2, \) and \( K \) satisfy (7), (9), (10), and (15) on \( U \), respectively. Then, for each \( \lambda \geq c\lambda^{-1}_1 \) and each \( f \in H \), the set \( S_\lambda(f) = \{x \in H_n \mid P_nKx - \lambda Lx + P_nMx = P_nf, n = 1, 2, \ldots\} \) is bounded in \( H \).

Proof. Let \( \lambda \geq c\lambda^{-1}_1 \) be fixed and suppose that \( S_\lambda(f) \) is not bounded in \( H \) for some \( f \in H \). Let \( x_{n_k} \in S_\lambda(f) \) be such that \( \|x_{n_k}\| \to \infty \) and \( u_{n_k} = \frac{x_{n_k}}{\|x_{n_k}\|} \).

Then, \( (M_1u_{n_k}, u_{n_k}) \to 0 \) as in (11), and therefore \( \{M_1u_{n_k}\} \) is bounded in \( H \) by the quasiboundedness of \( M_1 \). Thus, we may assume that \( u_{n_k} \to u \) and \( M_1u_{n_k} \to y \) in \( H \) with \( y = 0 \), since \( L \) is injective and

\[
L\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) = \lambda^{-1}_1\frac{P_{n_k}Kx_{n_k}}{\|x_{n_k}\|} - P_{n_k}M_1u_{n_k} - \frac{P_{n_k}(M_2x_{n_k} - f)}{\|x_{n_k}\|} \to y.
\]

Moreover, \( M_1u = 0 \) since \( M_1 \) is of type (M) at 0, and consequently \( u = 0 \).

Next, let \( \alpha \in (0,1) \) be fixed and \( m \geq 1 \) be such that \( \| (I - P_m)f \| \leq \alpha \) and \( c - \lambda\lambda_{m+1} > \alpha \). Then, using the variational characterization of the
eigenvalues of $L$:

$$(Lx, x) \leq \lambda_1 \|x\|^2 \quad \text{and} \quad (L(I-P_n)x, (I-P_n)x) \leq \lambda_{n+1} \| (I-P_n)x \|^2, \quad x \in H,$$

we obtain, as in the proof of Proposition 1, that for each $n_k > m$

$$(c - \lambda \lambda_{m+1}) \| (I-P_m)u_{n_k} \|^2 - \alpha \| (I-P_m)u_{n_k} \| \leq (\lambda \lambda_1 - c) \| P_m u_{n_k} \|^2 - \| f \| \| P_m u_{n_k} \|.$$

Again, $\| (I-P_m)u_{n_k} \| \to 1$ and $\| P_m u_{n_k} \| \to 0$ as $k \to \infty$, and therefore passing to the limit in the last inequality we get that $c - \lambda \lambda_{m+1} \leq \alpha$, which contradicts our choices of $m$ and $\alpha$. Hence, $S(\lambda)$ is bounded in $H$.

Our main solvability result for Eq. (2) reads:

**THEOREM 3.** (cf. [14]) Let $L : H \to H$ be linear, symmetric, positive, and compact, $\{H_n, P_n\}$ be as in Proposition 2, $K, M = M_1 + M_2 : D(M) \subset H \to H$ be such that (15) holds and $K + M$ is of type $(M)$ relative to $(U,H)$.

(a) If $M$ is quasibounded and $(Mx, x) \geq 0$ for $x \in D(M)$, then Eq. (2) is solvable for each $f \in H$ and each $A < c \lambda_1^{-1}$.

(b) If $M_1$ is quasibounded and of type $(M)$ at 0 relative to $(U,H)$, $M_1$ and $M_2$ satisfy (7), (9), and (10) on $U$, respectively, and $u = 0$ if $(M_1u, u) = 0$, then Eq. (2) is solvable for each $f \in H$ and each $\lambda \geq c \lambda_1^{-1}$.

**Proof.** Let $f \in H$ be fixed. We will show first that each equation $P_nKx - \lambda Lx + P_nMx = P_nf$ is solvable in $H_n$. Suppose that $\lambda < c \lambda_1^{-1}$. If $\lambda > 0$, then choosing $r > 0$ such that $\| f \| < (c - \lambda \lambda_1)r$, we get that for $x \in B(0, r) \cap H_n$,

$$(P_nKx - \lambda Lx + P_nMx - P_nf, x) \geq (c - \lambda \lambda_1) \| x \|^2 - \| x \| \| f \| > 0.$$

If $\lambda < 0$, then taking $r > 0$ with $\| f \| < cr$, we get that for $x \in B(0, r) \cap H_n$

$$(P_nKx - \lambda Lx + P_nMx - P_nf, x) \geq c \| x \|^2 - \| f \| |x| > 0.$$

Hence, using the homotopy $H_n(t, x) = t(P_nKx - \lambda Lx + P_nMx - P_nf) + (1-t)x$ on $[0,1] \times \bar{B}(0, r) \cap H_n$, we get that $\text{deg}(P_nK - \lambda L + P_nM, B \cap H_n, P_nf) \neq 0$.
for each $n \geq 1$. Thus, there is an $x_n \in B(0,r) \cap H_n$ such that $P_n K x_n - \lambda L x_n + P_n M x_n = P_n f$ with $n \geq 1$.

Next, if $\lambda \geq c \lambda_1^{-1}$, then (13) holds for $M_1$ and each $n$. Now, we choose $r_n > 0$ such that $\frac{\|f\|}{r} < c - \lambda_1 + c_n r_n^{\delta-1}$, and note that for $x \in \partial B(0,r_n) \cap H_n$

$$(P_n K x - \lambda L x + P_n M x - P_n f, x) \geq (c - \lambda_1) \|x\|^2 + c_n \|x\|^{1+\delta} - \|f\| \|x\| > 0.$$ 

Hence, as above, $P_n K x_n - \lambda L x_n + P_n M x_n = P_n f$ for some $x_n \in B(0,r_n) \cap H_n$ and each $n$, and $\{x_n\}$ is bounded in $H$ by Proposition 2.

Now, since $\{x_n\}$ is bounded in either case, some subsequence $x_{nk} \rightarrow x$ in $H$. It remains to show that $K x - \lambda L x + M x = f$. Since $M$ is quasibounded in either case and

$$(M x_n, x_n) = (P_n M x_n, x_n) \leq -c \|x_n\|^2 + \lambda(L x_n, x_n) + (f, x_n) \leq \text{const.} \|x_n\|,$$

it follows that $\{M x_n\}$ is bounded and a subsequence $(K + M)x_{nk} \rightarrow y$. Moreover,

$$P_n (K + M)x_{nk} = P_n f + \lambda L x_{nk} \rightarrow f + \lambda L x = y$$

and

$$\limsup((K + M)x_{nk}, x_{nk}) \leq (\lambda L x + f, x) = (y, x).$$

Hence, $x \in D(M)$ and $(K + M)x = y$ by property (M), and therefore, $K x - \lambda L x + M x = f$.

Remark 2. Analyzing the above proof we see that $x_{nk} \rightarrow x$ if either $K + M$ is of type $(S_+) \text{ (i.e. } x_n \rightarrow x \text{ if } x_n \rightarrow x \text{ and } \limsup((K + M)x_n, x_n - x) \leq 0),$ or $K + M$ is compact on $H$. When $M_1$ and $M_2$ are completely continuous on $H$, and $K = I$, Theorem 3 has been proved by Kesavan [11] using different type of arguments.

5. AN APPLICATION

Let $Q \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary $\partial Q$, $H = L_2(Q)$ and $W_2^k(Q)$ be the usual real Sobolev space with norm $\| \cdot \|_k$, $k \geq 1$ an integer.
Let $F = F_1 + F_2$, $G : Q \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions and $V$ be a closed subspace of $W_2^m(Q)$ containing $W_2^m(Q)$.

In this section we shall establish the weak solvability in $V$ of the semilinear elliptic equation

(16) \[ \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)D^\beta u(x)) + F(x, u(x)) - \lambda G(x, u(x)) = f(x), \quad x \in Q \]

where the coefficients $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$ are real valued, smooth and bounded, $f \in L_2$, $\lambda \in \mathbb{R}$, $F$ is strongly nonlinear, and $G$ has linear growth.

We begin by specifying conditions on the linear part.

(H1) The bilinear form $a(u, v) = \sum_{|\alpha|,|\beta| \leq m} (D^\alpha a_{\alpha\beta}D^\beta u)_L$ is coercive on $V$, i.e., there are constants $c_1 > 0$ and $c_2 \geq 0$ such that

\[ a(u, u) \geq c_1 \|u\|^2 - c_2 \|u\|^2, \quad \text{for } u \in V. \]

Using the Lax-Milgram theorem, one can show (see, e.g., [2]) that $a(u, v)$ generates a linear, closed, and densely defined mapping $A_1 : D(A_1) \subset L_2 \to L_2$, with compact resolvent, characterized by $D(A_1) = \{ u \in V \mid \text{for some } h \in L_2, a(u, v) = (h, v) \text{ for all } v \in V \} \subset \overset{\wedge}{W}_2^m$ and $a(u, v) = (A_1 u, v)$ $u \in D(A_1)$ and $v \in V$. Let $\{B_j\}_{j=1}^m$ be boundary differential operators of orders $m_j \leq 2m$, $1 \leq j \leq m$, such that the problem

\[ \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)D^\beta u) = f(x) \quad \text{in } Q \]

\[ B_j u(x) = \sum_{|\alpha| = m_j} b_{j\alpha}(x)D^\alpha u(x) = 0 \quad \text{on } \partial Q \]

is regularly elliptic (cf., e.g., [2]). Set $\overset{\wedge}{W}_2^m = \{ u \in \overset{\wedge}{W}_2^m(Q) \mid B_j u = 0 \text{ on } \partial Q, \; j = 1, \ldots, m \}$. We assume (cf. [1]):

(H2) $V$ is such that $D(A_1) = \overset{\wedge}{W}_2^m$, $A_1$ is symmetric in $L_2$ and possesses an orthonormal basis of eigenfunctions $\{u_k\}$ in $L_2$; $A_1 u_k = \lambda_k u_k$.

Let $H_n = \text{lin.sp.}\{u_1, \ldots, u_n\}$, $W = W_2^m(Q) \cap \overset{\wedge}{W}_2^m$ with $k \geq \max\{1 + \frac{m}{2}, 2m\}$ and note that $W \subset C(Q)$ by the Sobolev embedding theorem. If
$a_{\alpha \beta}, B_{j \alpha}$ and $\partial Q$ are sufficiently smooth, then $\overline{H_n} = U$ for some closed subspace $U$ of $W$. Indeed, write $k = 2m + 2rm + s$ for some $r \geq 0$ and $0 \leq s < 2m$, and note that $B_c : W_2^{2m+2im+s}(Q) \cap \overline{W_2^{2m}} \rightarrow W_2^{2im+s}(Q)$ is a homeomorphism for each integer $i \in [0, r]$. Let $i = 0$ and note that $\overline{H_n} = \overline{W_2^s}$ since $W_2^s$ is dense in $W_2^s$ and $\overline{H_n} = \overline{W_2^{2m}}$ (cf. [1]). Since $\overline{W_2^s}$ is a closed subspace of $W_2^s$, $U_0 = B_c^{-1}(\overline{W_2^s})$ is closed subspace of $W_2^{2m+s} \cap \overline{W_2^{2m}}$ and $\overline{H_n} = U_0$. To see this, let $f \in U_0, g = B_c f \in \overline{W_2^s}$ and $g_n \in H_n$ be such that $g_n \rightarrow g$ in $\overline{W_2^s}$. Then, $B_c^{-1}g_n \rightarrow f$ in $U_0$ with $B_c^{-1}g_n \in H_n$, and therefore $\overline{H_n} = U_0$. Next, let $i = 1$ and note that $U_1 = B_c^{-1}(U_0)$ is closed in $W_2^{2m+s}(Q) \cap \overline{W_2^{2m}}$ and $\overline{H_n} = U_1$ as above. Proceeding in this way, we get that $U = U_r$ is a closed subspace of $W$ with $\overline{H_n} = U$.

Now, denote by $<,>$ the usual duality between $V$ and its dual $V^*$ or $U$ and $U^*$ and note that $<,>$ is compatible with the inner product $(,)$ on $H$ in either case. Since $a(u,.)$ is a continuous linear functional on $V$ for each $u \in V$, it defines a continuous linear mapping $A : V \rightarrow V^*$ such that $a(u,v) = <Au,v>$ for $u,v \in V$, and $<Au,v> = (A_1u,v)$ for $u \in D(A_1), v \in V$.

Regarding the nonlinear part, we assume:

$(F1)$ $F_1(x,0) = 0$ and $F_1(x,.)$ is increasing in a neighborhood of $0$ for a.e. $x \in Q$, and for each $s \geq 0$ there is a function $h_s \in L_2$ such that

$$\sup_{|t| \leq s} |F_1(x,t)| \leq h_s(x) \quad \text{and} \quad F_1(x,t)t \geq 0 \quad \text{for a.e.} \ x \in Q, t \in R.$$ 

$(F2)$ $|F_2(x,t)| \leq a(x) + b(x) |t|$ for a.e. $x \in Q, t \in R$ and some $a,b \in L_2$.

$(F3)$ $s = 0$ if $F_1(x,s) = 0$ for some $x \in Q$, and $F_1(x,st) = s^\delta F_1(x,t)$ for a.e. $x \in Q, t \in R, s \geq 0$ and some $\delta > 1$.

$(G1)$ $|G(x,t)| \leq c(x) + d(x)|t|$ for a.e. $x \in Q, t \in R$ and some $c,d \in L_2$.

Let $D(N_1) = \{u \in V \mid F_1(x,u) \text{ and } F_1(x,u)u \text{ are in } L_1\}$, and $C,N = N_1 + N_2 : D(N_1) \rightarrow U^*$ be defined by $<Cu,v> = (G(x,u),v)$ and $<N_1u + N_2u,v> = (F_1(x,u) + F_2(x,u),v)$ for $u \in D(N_1)$ and $v \in U$. By $(F1)$, $U \subset D(N_1)$, $N$ is well defined and $N(U) \subset H$. Moreover, $(6)$ holds for some constants $a$ and $b$, by $(G1)$. 

- 746 -
PROPOSITION 3. (a) If (F1) holds, then \( N^t : D(N_1) \to U^* \) is of type (M) at 0 relative to \((U, L_2)\) and (8) holds.

(b) If (F1), (F2), and (G1) hold, then \( N^t : D(N_1) \to U^* \) is quasibounded and \( N - \lambda C \) is of type (M) relative to \((U, V)\).

Proof. (a) Suppose that \( \{u_n\} \subset U, u_n \rightharpoonup u \) in \( L_2, N_1u_n \to 0 \) in \( U^* \) and \( \limsup(N_1u_n, u_n) \leq 0 \). Then Fatou's lemma and (F1) imply that \( (N_1u_n, u_n) \to 0 \), and therefore we may assume that \( F_1(x, u_n(x))u_n(x) \to 0 \) a.e. in \( Q \).

Since \( F_1(x, t)t \) is also increasing in \( t \) in a neighborhood of zero for a.e. \( x \in Q \), it follows that \( u_n(x) \to 0 \) a.e. in \( Q \). To show that \( u_n \to 0 \) in \( L_1 \), let \( \epsilon > 0 \) be fixed and, for any \( n \geq 1 \), define \( Q_1 = \{x \in Q \mid |u_n(x)| \leq \frac{1}{2}\} \) and \( Q_2 = Q \setminus Q_1 \).

Then, for any measurable subset \( A \subset C \),

\[
\int_A |u_n(x)| \, dx \leq \int_{A \cap Q_1} |u_n(x)| \, dx + \epsilon \int_{A \cap Q_2} u_n^2(x) \, dx \leq \frac{m(A)}{\epsilon} + \text{const.} \epsilon.
\]

Hence, \( u_n \to 0 \) in \( L_1 \) by Vitali's theorem, and \( u = 0 \) with \( N_10 = 0 \) since \( u_n \rightharpoonup u \) in \( L_1 \).

To see that (8) holds, let \( \{u_n\} \subset U \) be bounded in \( L_2 \) and \( (N_1u_n, u_n) \to 0 \) as \( n \to \infty \). We get, as above, that \( u_n \rightharpoonup u \) in \( L_2, u_n \to 0 \) in \( L_1 \) and therefore \( u = 0 \). On the other hand, for any \( \epsilon > 0 \),

\[
|F_1(x, u_n(x))| \leq \sup_{|t| \leq \frac{1}{t}} |F_1(x, t)| + \epsilon F_1(x, u_n(x))u_n(x)
\]

and, for any measurable subset \( A \subset Q \),

\[
\int_A |F_1(x, u_n(x))| \, dx < \|h_1\|_{L_1(A)} + \text{const.} \epsilon.
\]

Hence, by Vitali's theorem, \( F_1(., u_n) \to F_1(., u) = 0 \) in \( L_1 \), and therefore \( N_1u_n \to 0 \) in \( U^* \).

(b) Note first that \( C, N_2 : V \to L_2 \) are completely continuous since \( V \) is compactly embedded in \( L_2 \). Let \( i : U \to V \) be the natural injection. Next, let \( \{u_n\} \subset U, u_n \rightharpoonup u \) in \( V \), \( (N - \lambda C)u_n \rightharpoonup \ast u \) in \( U^* \) for some \( \ast u \in V^* \) and \( \limsup <(N - \lambda C)u_n, u_n > \leq < \ast u, u > \). Hence, in view of (17), Vitali's theorem and Fatou's lemma imply that \( F_1(., u_n) \to F_1(., u) \) in \( L_1 \) and

\[
\int_0 F_1(x, u) \, dx \leq \liminf \int_0 F_1(x, u_n) \, dx \leq \text{const.}
\]
Thus, \( u \in D(N) \), \( N_1 u_1 \rightarrow N_1 u \) in \( U^* \) and \( (N - \lambda C)u_n = N_1 u_n + N_2 u_n - \lambda C u_n \rightarrow N_1 u + N_2 u - \lambda C u = (N - \lambda C)u = \star v \), proving that \( (N - \lambda C) : D(N) \rightarrow U^* \) is of type \((M)\) relative to \((U, V)\). Moreover, using (17) as above, we see that \( N_1 \) is quasibounded and therefore such is \( N = N_1 + N_2 \) by the boundedness of \( N_2 \).

Now, let \( \lambda \in \mathbb{R} \) and \( f \in L_2 \). We are looking for a solution \( u \) of the following variational problem:

\[
\begin{align*}
\frac{a(u, v)}{\mathcal{F}_Q F(x, u)v dx} - \frac{\lambda}{\mathcal{F}_Q G(x, u)v dx} = (f, v) \quad \forall v \in W^k_2 \cap V, \\
u \in D(N) \subset W^k_2
\end{align*}
\]

which can be considered as weak formulation of Eq. (16). We have:

**THEOREM 4.** Let \( a_{\alpha \beta}, b_{j \alpha} \) and \( \partial Q \) be sufficiently smooth, \((H1), (H2), (F1), (F2)\), and \((G1)\) hold. Then BVP (18) has a solution for each \( |\lambda|a < \lambda_1 \) and each \( f \in L_2 \). If, in addition, \((F3)\) holds, then the same conclusion is also valid for \( |\lambda|a \geq \lambda_1 \).

**Proof.** Let \( i : U \rightarrow V \) be the natural injection and \( i^* : V^* \rightarrow U^* \) be its dual mapping. Define a bilinear form on \( V \times i^*(V^*) \) by \( < u, i^* v > = < u, v > \) for \( u \in V, v \in V^* \), and note that \( < i^* Au, v > = < Au, v > \) for \( u, v \in V \). Since BVP (18) is equivalent to the operator equation \( \lambda i^* C u - i^* A u - Nu = -i^* f \), the conclusions of the theorem follow, in view of Proposition 3, from Theorems 1 and 2 with \( V^*, \lambda C - A \) and \( f \) replaced by \( i^*(V^*), i^*(\lambda C - A) \) and \( i^* f \), respectively.

For the sake of comparison, consider the BVP

\[
\begin{align*}
\{ -\Delta u = \pm |u|^{p-1} u + \lambda u + f \quad & \text{in } Q \subset \mathbb{R}^n \\
u = 0 \quad & \text{on } \partial Q.
\end{align*}
\]

Theorem 4 implies that BVP (19-) has a weak solution for each \( \lambda \in \mathbb{R}, f \in L_2 \) and \( p > 1 \). However, the situation is quite different for BVP (19+) and has been studied by many authors. Many existence results on positive solutions of (19+) with \( p < \frac{n+2}{n-2} \) are known (see the review article by P.L. Lions [13] and...
the references in there). In the critical case, when \( p = \frac{n+2}{n-2} \), Brezis-Nirenberg [6] have shown that BVP (19+), with \( f = 0 \), has a positive solution only for \( \lambda \in (0, \lambda_1) \) provided \( n \geq 4 \) and \( Q \) is starshaped. If, in addition, \( Q \) is not contractable and \( n \geq 3 \), Bahri-Coron [3] have established this fact also for \( \lambda = 0 \) (using the methods of algebraic topology). For the existence of infinitely many solutions of (19+) with \( \lambda = 0 \), we refer to Bahri-Lions [4] and the references therein.

**Remark 3.** When \( 1 < p < \frac{n+2}{n-2} \) (\( p > 1 \) if \( n < 2 \)), the weak solvability of (19−) was proved by Kesavan [11] using different methods. When \( F_2 = 0, \lambda = 0 \) and \( A \) is coercive, Theorem 4 is contained in Hess [9] with \( m = 1 \), and in Webb [17] and Brezis-Browder [5] (under an additional condition on \( F \)) with \( m > 1 \). For an application of Theorem 3, with \( M : H \to H \) completely continuous, to the Von Kármán Equations, we refer to [11].

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Department of Mathematics,
New Jersey Institute of Technology,
Newark, New Jersey 07102
USA

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