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A NOTE ON THE DIRICHLET PROBLEM FOR THE ELLIPTIC LINEAR OPERATOR IN SOBOLEV SPACES WITH WEIGHT d_M^ε

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Abstract: The Dirichlet boundary value problem for the elliptic linear operator in weighted Sobolev spaces $W^{kp}(\Omega,d_M^\varepsilon)$ is considered, M being a closed subset of $\partial\Omega$, Ω having the outer cone property on M. The existence and uniqueness of the weak solution of the problem is proved.

Key words: Dirichlet problem, elliptic linear operator, weighted Sobolev space, domain with outer cone property.

AMS Subject Classification: 35J40, 46E35

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1. Introduction to the problem

Let Ω be a domain in $\mathbb{R}^N$ and $M \subset \partial\Omega$ an arbitrary set. Let us denote

(1.1) $$d_M(x) = \mathrm{dist}(x,M) , \quad x \in \bar{\Omega} .$$

We obviously have $d_M(x) = d_{\bar{M}}(x)$ so that M is supposed from the very beginning to be closed. Let

$$L^p(\Omega,d_M^\varepsilon) = \{f, \int_\Omega |f(x)|^p \, d_M^\varepsilon(x) \, dx < \infty\} , \quad p > 1 , \quad \varepsilon \in \mathbb{R} ,$$

and let $W^{k,p}(\Omega,d_M^\varepsilon)$ stand for the space of functions from $L^p(\Omega,d_M^\varepsilon)$ whose distributional derivatives up to the order k belong again to $L^p(\Omega,d_M^\varepsilon)$. The space $W^{kp}(\Omega,d_M^\varepsilon)$ is a Banach space with respect to the norm

$$\|u\|_{kp} = \Big(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u(x)|^p \, d_M^\varepsilon(x) \, dx \Big)^{1/p} .$$

Let further

(1.2) $$b(u,v) = \sum_{|\alpha|,|\beta|\le k} \int_\Omega a_{\alpha\beta}(x) \, D^\beta u(x) \, D^\alpha v(x) \, dx$$

be an elliptic bilinear form. Let $u_0 \in W^{k,2}(\Omega, d_M^{\varepsilon})$ and $F \in [W_0^{k,2}(\Omega, d_M^{-\varepsilon})]^*$. Our goal is to find a function $u \in W^{k,2}(\Omega, d_M^{\varepsilon})$ such that

$$u - u_0 \in W_0^{k,2}(\Omega, d_M^{\varepsilon}) , \tag{1.3}$$

and

$$b(u,v) = F(v) \quad \text{for all} \quad v \in W_0^{k,2}(\Omega, d_M^{-\varepsilon}) . \tag{1.4}$$

The existence and uniqueness of the solution to the problem (1.3), (1.4) was shown by several authors: J. NEČAS [3] ($M = \partial\Omega$), A. KUFNER [2] ($M = \{x_0\}$), J. VOLDŘICH [5] under rather restricted conditions upon $\partial\Omega$ and M (namely, M being a finite union of Lipschitz images of the m-dimensional edges in $\langle 0,1\rangle^N$, $m = 1,\dots,N-1$). The previous results are extended in the following sense: M is allowed to be more general and the weaker condition than $\Omega \in C^{0,1}$ is required (see Definition 2.1).

2. The domain

For $y \in \mathbb{R}^N$, $y = [y_1,\dots,y_N]$, we write $y = [y', y_N]$.

DEFINITION 2.1. Let $\Omega \subset \mathbb{R}^N$ be the domain and $M \subset \partial\Omega$ closed. We say that

$$\Omega \in \mathcal{K}(M)$$

if the following four conditions are fulfilled:

(i) there exist m Cartesian coordinate systems (y_r', y_{rN}), $r = 1,\dots,m$, and the same number of functions a_r defined on the closures of (N-1)-dimensional cubes

$$\Delta_r = \{y_r', \ |y_{rj}| < \delta_r, \ j = 1,\dots,N-1\} \tag{2.1}$$

such that for every $y \in \partial\Omega$ there exists $1 \le r \le m$ satisfying

$$y = [y_r', y_{rN}] , \quad y_r' \in \Delta_r , \quad y_{rN} = a_r(y_r') , \tag{2.2}$$

(ii) the functions a_r are continuous on $\overline{\Delta}_r$,

(iii) there exists $\beta > 0$ such that for every r the set

$$B_r = \{[y_r', y_{rN}], \ y_r' \in \Delta_r, \ a_r(y_r') - \beta < y_{rN} < a_r(y_r') + \beta\}$$

allows us to write

(2.3) $$V_r = B_r \cap \Omega = \{[y_r', y_{rN}],\ y_r' \in \Delta_r,\ a_r(y_r') - \beta < y_{rN} < a_r(y_r')\}\ ,$$

(2.4) $$\Gamma_r = B_r \cap \partial\Omega = \{[y_r', y_{rN}],\ y_r' \in \Delta_r,\ a_r(y_r') = y_{rN}\}\ ,$$

and

(2.5) $$B_r \setminus \bar{\Omega} = \{[y_r', y_{rN}],\ y_r' \in \Delta_r,\ a_r(y_r') < y_{rN} < a_r(y_r') + \beta\}\ ,$$

(iv) there exists an open cone $K_{h,C}$, $h, C > 0$,

(2.6) $$K_{h,C} = \{[y', y_{rN}] \in \mathbb{R}^N,\ \ 0 < y_N < h,\ \ |y'| < Cy_N\}\ ,$$

such that for each $y = [y_r', a_r(y_r')] \in M$, $y_r' \in \bar{\Delta}_r$, there exists a cone K_y congruent to $K_{h,C}$ with the vertex y and the axis parallel to y_N , and satisfying $K_y \cap \bar{\Omega} = 0$.

Let us further denote

$$U_r = \{[y_r', y_{rN}],\ y_r' \in \Delta_r,\ a_r(y_r') - \frac{\beta}{2} < y_{rN} < a_r(y_r')\}\ .$$

REMARK 2.2.

(a) If $\Omega \in C^{0,1}$, then $\Omega \in K(M)$.

(b) If $\Omega \in K(M)$, then Ω is bounded.

LEMMA 2.3. Let $\Omega \in K(M)$. Then the Cartesian systems from Definition 2.1 can be chosen in such a manner that they additionally obey the following two conditions:

(v) for $y = [y_r', a_r(y_r')] \in M$, $y_r' \in \bar{\Delta}_r$, we have

$$\Pi_r(K_y) \supset \bar{\Delta}_r\ ,$$

Π_r standing for the orthogonal projection associating $y = [y_r', y_{rN}]$ with $y_r' \in \mathbb{R}^{N-1}$,

(vi) if $S_r = \bar{\Gamma}_r \cap M = \emptyset$, then $d_M(\bar{U}_r) > 0$.

P r o o f . Let (y_r', y_{rN}) be the r-th system and let $\tilde{\delta}_r \in (0, \delta_r)$ be chosen in such a way that $\partial\Omega$ is still covered by the new systems $(\tilde{y}_r, y_{rN})$, $\tilde{y}_r' \in \tilde{\Delta}_r$ (see (2.1)). We denote $\tilde{S}_r = \bar{\tilde{\Gamma}}_r \cap M$ and put

$$\eta_r = \frac{1}{2} \min\left(\frac{1}{3} C\,h,\ \delta_r - \tilde{\delta}_r\right)$$

and choose δ_r^* such that

$$\operatorname{diam}\left(\langle -\delta_r^*, \delta_r^* \rangle^{N-1}\right) < \eta_r\ .$$

Now we associate every $z \in G_r^1 = \Pi_r(\tilde{S}_r)$ with an open (N-1)-dimensional cube Δ_z centered at z with the side length $2\delta_r^*$ and sides parallel to the r-th system axes. As G_r^1 is compact set, there exist points $z_1, \dots, z_{n_r} \in G_r^1$ such that

$$\bigcup_{i=1}^{n_r} \Delta_{z_i} \supset \Pi_r(\tilde{S}_r) .$$

The set $G_r^2 = \bar{\bar{\Delta}}_r \setminus \bigcup_{i=1}^{n_r} \Delta_{z_i}$ is a compact set, too, and G_r^1, G_r^2 are disjoint. Thus there is $\gamma_r > 0$ satisfying

$$\operatorname{dist}(G_r^1, G_r^2) \geq \gamma_r .$$

Now we associate every $w \in G_r^2$ with an open cube Δ_w centered at w, its sides parallel to the r-th system axes and the side length ${}^*\delta_r$ where ${}^*\delta_r$ is chosen in order that

$$\operatorname{diam}\left(\langle -{}^*\delta_r, {}^*\delta_r \rangle^{N-1}\right) < \frac{1}{2}\gamma_r ;$$

Making use of compactness of G_r^2 we can choose $w_1, \dots, w_{k_r}$ such that

$$\bigcup_{j=1}^{k_r} \Delta_{w_j} \supset G_r^2$$

Now Δ_{z_i}, $i = 1, \dots, n_r$, and Δ_{w_j}, $j = 1, \dots, k_r$, form the (N-1)-dimensional bases for the new Cartesian systems (with common y_{rN} axis) which obviously satisfy (i) - (vi). ⊠

The following lemma, in particular, the inequality (2.7), is a substantial tool for dealing with the density of smooth functions in the weighted Sobolev space. The inequality (2.8) is of crucial importance for the imbedding theorem, and, consequently, employing the standard argument (see [2], Chaps. 7, 8, 14), for solving the Dirichlet problem (1.3), (1.4).

<u>LEMMA 2.4.</u> Let $\Omega \in \mathcal{K}(M)$, $\partial\Omega$ described by the coordinate systems from Lemma 2.3. Let $S_r = M \cap \bar{\Gamma}_r \neq \emptyset$. Then

(i) there exists $C_1 > 0$ such that for $y, z \in U_r$ of the form $y = [y_r', y_{rN}]$, $z = [y_r', z_{rN}]$, $y_r' \in \Delta_r$, $y_{rN} > z_{rN}$ we have

$$d_{S_r}(y) \leq C_1\, d_{S_r}(z) , \tag{2.7}$$

(ii) there exists $C_2 > 0$ such that

$$d_{S_r}(y) \leq a_r(y_r') - y_{rN} + d_{S_r}(y^*) \leq C_2\, d_{S_r}(y) \tag{2.8}$$

is satisfied for each $y = [y_r', y_{rN}] \in U_r$ and $y^* = [y_r', a_r(y_r')]$.

P r o o f . For $y, z \in U_r$ let $y_1, z_1 \in S_r$ be such that

$$d_{S_r}(y) = |y - y_1| , \quad d_{S_r}(z) = |z - z_1| .$$

(Recall that S_r is the compact set.)

Clearly,

$$\frac{|y - y_1|}{|z - z_1|} \le \frac{|y - z_1|}{|z - z_1|} .$$

Let $z_1 = [w', w_N]$ and denote $\tilde{y} = [y_r', w_N]$. If $y_{rN} \le w_N$, then the assumption $y_{rN} > z_{rN}$ yields

$$\frac{|y - z_1|}{|z - z_1|} \le 1$$

If $y_{rN} > w_N$ and 2α is the vertex angle of the cone K_{z_1} , we denote by t the intersection of the line $\overline{zy}$ with the envelope of the cone K_{z_1} . Existence of such a point is guaranteed by Lemma 2.3, (v). Now

$$\frac{|y - z_1|}{|z - z_1|} \le \frac{|z_1 - t|}{|z_1 - \tilde{y}|} = \frac{1}{\sin \alpha} .$$

We prove the assertion (ii). The compactness of S_r guarantees the existence of $w, z \in S_r$ such that

$$d_{S_r}(y) = |w - y| \quad \text{and} \quad d_{S_r}(y^*) = |z - y^*| .$$

Using the triangle inequality we get

$$d_{S_r}(y) = |w - y| \le |z - y| \le |y - y^*| + |y^* - z| =$$
$$= a_r(y_r') - y_{rN} + d_{S_r}(y^*) ,$$

which is the first estimate in (2.8). We prove the remaining one. First, we show that

$$(2.9) \qquad |y - y^*| \le \frac{1}{\sin \alpha} |y - w| ,$$

2α being the vertex angle in K_w . Let us denote with s the intersection of the line $\overline{yy^*}$ with the envelope of K_w . If $w = s$, then also $w = y^*$, and (2.9) holds trivially. Otherwise we denote with y^{**} the foot of the perpendicular to the line $\overline{sw}$ from y . Then

$$|y - y^{**}| \le |y - w|$$

and

$$|y - y^*| \le |y - s| .$$

Hence

$$|y - y^*| \le |y - s| = \frac{|y - s|}{|y - y^{**}|} |y - y^{**}| \le \frac{1}{\sin \alpha} |y - w| ,$$

which gives (2.9).

Now, using the estimate just obtained and the triangle inequality we have

$$|y^* - z| \le |y^* - w| \le |y^* - y| + |y - w| \le \left(1 + \frac{1}{\sin \alpha}\right) |y - w| .$$

Therefore

$$a_r(y_r') - y_{rN} + d_{S_r}(y^*) = |y - y^*| + |y^* - z| \le \left(1 + \frac{2}{\sin \alpha}\right) |y - w| .$$

So, both assertions of the Lemma are proved. ⊠

LEMMA 2.5. Let $\Omega \in \mathcal{K}(M)$, and (y_r', y_{rN}) be the r-th coordinate system from Lemma 2.3. For every fixed $\tilde{\delta}_r \in (0, \delta_r)$ we define $\tilde{\Delta}_r$, $\tilde{U}_r$ in a similar way as in Definition 2.1. Let $S_r = M \cap \overline{\Gamma}_r$ be nonempty.

Then there is $C > 0$ such that

$$(2.10) \qquad d_M(y) \le d_{S_r}(y) \le C\, d_M(y)$$

holds for all $y \in \tilde{U}_r$.

P r o o f . Clearly, it suffices to demonstrate just the second inequality. We shall omit the subscript r . If $\gamma = \min (\delta - \tilde{\delta}, \frac{1}{2} \beta)$, then

$$\text{dist}\, (\overline{\tilde{U}}, \overline{\Omega} \setminus V) \ge \gamma > 0 .$$

Further, let

$$K = \max d_S(y) , \quad y \in \overline{\tilde{U}} ,$$

$$G_1 = \Big(\bigcup_{y \in S} B_{\gamma/2}(y)\Big) \cap \tilde{U} ,$$

where $B_\lambda(y) = \{z, |y - z| < \lambda\}$, and

$$G_2 = \overline{\tilde{U}} \setminus G_1 .$$

Obviously, $\text{dist}\,(y,M) \ge \frac{1}{2} \gamma$ for $y \in G_2$. Therefore

$$\frac{d_S(y)}{d_M(y)} \le \frac{2K}{\gamma} , \quad y \in G_2 .$$

As d_M equals d_S on G_1 , the second inequality in (2.10) is proved. ⊠

3. Density of smooth functions

THEOREM 3.1. If $\Omega \in \mathcal{K}(M)$, $\varepsilon \ge 0$, $p > 1$, $k \in \mathbb{N}_0$, then $C^\infty(\overline{\Omega})$ is dense in the space $W^{kp}(\Omega, d_M^\varepsilon)$.

P r o o f . Having the crucial inequality (2.7), the proof proceeds in a standard way (cf. the comments preceding Lemma 2.4). ⊠

4. Imbedding theorems

THEOREM 4.1. If $\Omega \in K(M)$, $1 < p < \infty$, $\varepsilon > p-1$, then
$$W^{1,p}(\Omega,d_M^\varepsilon) \to L^p(\Omega,d_M^{\varepsilon-p}) .$$

P r o o f . Since $\varepsilon > 0$, it suffices according to Theorem 3.1 to find $C > 0$ such that for all $u \in C^\infty(\bar\Omega)$

$$(4.1) \qquad \|u\|_{p,d_M^{\varepsilon-p}} \le C\, \|u\|_{1,p,d_M^\varepsilon} .$$

We assume $\partial\Omega$ being described by Cartesian systems from Lemma 2.3. Let
$$H_r = (B_r \setminus \Omega) \cup U_r , \quad r = 1,\dots,m_0 ,$$
and $H_0 \subset \bar H_0 \subset \Omega$ be an open set such that
$$\bigcup_{r=0}^{m_0} H_r \supset \bar\Omega .$$
Let $\psi_0,\dots,\psi_{m_0}$ be the corresponding partition of unity. We put

$$(4.2) \qquad v_r(x) = u(x)\,\psi_r(x) , \quad x \in \Omega , \quad r = 0,\dots,m_0 .$$

Let us fix r. If $S_r = \emptyset$ (including the case $r = 0$), it follows from Lemma 2.3, (vi), that
$$\operatorname{dist}(U_r,M) \ge \gamma > 0 .$$
Hence

$$(4.3) \qquad W^{1p}(U_r,d_M^\varepsilon) = W^{1p}(U_r) \to L^p(U_r) = L^p(U_r,d_M^{\varepsilon-p}) .$$

Now, let $S_r \ne \emptyset$. Since $\operatorname{supp}\psi_r \subset H_r$, v_r is in $W^{1p}(U_r,d_M^\varepsilon)$ iff it is in $W^{1p}(U_r,d_{S_r}^\varepsilon)$. Moreover, due to Lemma 2.5, the norms in this spaces are equivalent.

We shall omit r again. Lemma 2.4, (ii) implies for $y = [y',y_N] \in U_r$

$$(4.4) \qquad d_S^{\varepsilon-p}(y) \le c_1\big(a(y') - y_N + d_S(y^*)\big)^{\varepsilon-p} ,$$

and

$$(4.5) \qquad \big(a(y') - y_N + d_S(y^*)\big)^\varepsilon \le c_2\, d_S^\varepsilon(y) .$$

Thus, using (4.4), (4.5), the Hardy inequality (see [1], Theorem 330, and [2], Lemma 5.3), and setting $v(y',y_N) = 0$ for $y_N \le a(y') - \beta/2$, we obtain

$$\|v\|^p_{p,U,d_S^{\varepsilon-p}} = \int_U |v(y)|^p \, d_S^{\varepsilon-p}(y)\, dy \le$$

$$\le c_1 \int_U |v(y)|^p \left(a(y') - y_N + d_S(y^*)\right)^{\varepsilon-p} dy =$$

$$= c_1 \int_\Delta \int_{a(y')-\beta/2}^{a(y')} |v(y',y_N)|^p \left(a(y') - y_N + d_S(y^*)\right)^{\varepsilon-p} dy_N\, dy' =$$

$$(4.6) \quad = c_1 \int_\Delta \int_0^\infty |v(y',\, a(y') - t)|^p \left(d_S(y^*) + t\right)^{\varepsilon-p} dt\, dy' \le$$

$$\le \left(\frac{p}{\varepsilon-p+1}\right)^p c_1 \int_\Delta \int_0^\infty \left|\frac{\partial v}{\partial t}(y',\, a(y') - t)\right|^p \left(d_S(y^*) + t\right)^{\varepsilon} dt\, dy' =$$

$$= \left(\frac{p}{\varepsilon-p+1}\right)^p c_1 \int_\Delta \int_{a(y')-\beta/2}^{a(y')} \left|\frac{\partial v}{\partial y_N}(y',y_N)\right|^p \left(a(y') - y_N + d_S(y^*)\right)^{\varepsilon} dy_N\, dy' \le$$

$$\le \left(\frac{p}{\varepsilon-p+1}\right)^p c_1 c_2 \int_U \left|\frac{\partial v}{\partial y_N}(y',y_N)\right|^p d_S^{\varepsilon}(y)\, dy \le c_3 \|v\|^p_{1,U,d_S^\varepsilon} .$$

According to Lemma 2.5 we can take M instead of S in (4.6). Combining (4.3) and (4.6), and using the partition of unity we finish the proof. ⊠

<u>THEOREM 4.2.</u> Let $\Omega \in K(M)$, $M \subset \partial\Omega$ closed, $1 < p < \infty$, $\varepsilon \ne p - 1$. Then

$$W_0^{1p}(\Omega, d_M^\varepsilon) \to L^p(\Omega, d_M^{\varepsilon-p}) .$$

The p r o o f is similar to the preceding one. ⊠

5. The Dirichlet problem

The following theorem is well known (cf. [2], Chaps. 13 and 14).

<u>THEOREM 5.1.</u> If Ω, M, ε are such that

(i) $\quad W_0^{12}(\Omega, d_M^\varepsilon) \to L^2(\Omega, d_M^{\varepsilon-2})$, $\varepsilon \ne 1$,

and

(ii) there exists a function $\rho \in C^\infty(\Omega) \cap C(\bar\Omega)$ such that

$$c_1 d_M(y) \le \rho(y) \le c_2 d_M(y) , \quad y \in \bar\Omega ,$$

and

$$|D^{\alpha}\rho(y)| \le C_{\alpha}[\rho(y)]^{1-|\alpha|} , \quad |\alpha| \le k , \quad y \in \Omega ,$$

then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the Dirichlet problem from Sec. 1 possesses precisely one solution u. Further, there is $C > 0$ such that

$$\|u\|_{k,2,d_M^{\varepsilon}} \le C(\|u_0\|_{k,2,d_M^{\varepsilon}} + \|F\|_Z) ,$$

Z standing for $[W_0^{k,2}(\Omega, d_M^{-\varepsilon})]^*$. ⊠

In the case considered, for $\Omega \in \mathcal{K}(M)$, M closed, the validity of (i) is guaranteed by Theorem 4.2. The existence of the function ρ with the properties required in (ii) follows from theorem due to STEIN, see [4], Chap. 6, § 2.

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