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A NEW WAY TO FIND COMPACT ZERO-DIMENSIONAL FIRST COUNTABLE PREIMAGES OF FIRST COUNTABLE COMPACT SPACES
V.V. TKAČUK

Abstract: Any compact first countable space X possesses a base B such that the family $P_B = \{\text{Fr}(U): U \in B\}$ has the order less than $\mathbb{C}$ at every $x \in X$. Therefore CH implies X has a peripherally point-countable base. We prove also that every first countable compact space with a peripherally point-countable base is a continuous image of a zero-dimensional first countable compact space, giving thus a new easier way to prove A.V. Ivanov's theorem [1].

Key words: First countable compact space, order, peripherally point-countable base.

Classification: 54A25, 54C35

It is not yet known within ZFC if for any first countable compact space X there exists a first countable zero-dimensional compact Y and a continuous onto mapping $f: Y \to X$. A.V. Ivanov proved using inverse spectra technique that such a Y exists in case $\omega(X) = \omega_1$ [1]. Hence it follows from CH that the answer to the above question (which is actually V.I. Ponomarev's problem [21]) is positive. In this paper we extend the result of A.V. Ivanov over the classes of Corson and linearly ordered first countable compact spaces. Thereby some new properties of a first countable compact X come into consideration and seem to be interesting in themselves. For example, if CH is assumed, then any X as above has a base B such that the family $P_B = \{\text{Fr}(U): U \in B\}$ is point-countable (we will call such a base peripherally point-countable).

It is relevant to mention hereby A.S. Mishchenko's theorem [3]: any compact space having a point-countable base is metrizable and B.E. Shapirovskii's result [4]: if the tightness of a compact X is countable, then X has a point-countable $\pi$-base. Unfortunately, the author did not succeed to clear up whether it is true in ZFC that any first countable compact X has a peripherally point-countable base.

Our notations and terminology are standard. All spaces under considera-
tion are Tychonoff (and in fact compact). For a space $X$ by $T(X)$ is denoted its topology and $T^*(X) = T(X) \setminus \{\emptyset\}$. The boundary $\text{Fr}(A)$ of a set $A \subset X$ is the set $\overline{A} \setminus \text{int}\ A$. If $X = \prod_{\alpha} X_{\alpha}$ is a Tychonoff product of the spaces $X_{\alpha}$ and $A \subset X$, then $\pi^0_{\alpha} : \prod_{\alpha} X_{\alpha} : \alpha \in \mathcal{A} \rightarrow T(X_{\alpha} : \alpha \in \mathcal{A})$ is the natural projection and $\pi^*_{\alpha} = \pi^0_{\alpha}$. Functions are treated as their graphs so that $f = U \{f_{\alpha} : \alpha \in \mathcal{T}\}$ means $f$ is the common extension of $f_{\alpha}$'s. For $x \in X$, the cardinal number $\chi(x) = \#X$ is the weight of $X$ at $x$, and $\chi(X) = \sup \{\chi(x) : x \in X\}$. By the order $\text{ord}(x)$ of a family $\mathcal{S}$ of subsets of $X$ at the point $x$ is meant the power of the set $\{ U \in \mathcal{S} : x \in U \}$. The expression $x_n \rightarrow x$ means the sequence $\{x_n : n \in \omega\}$ converges to $x$. The space $\mathbb{R}$ is the real line with its natural topology and $D = \{0,1\}$ - the discrete two-point space.

1. Theorem. Any compact $X$ with $\chi(X) = \omega$ has a base $B$ such that $\text{ord}(B,x) < C$ for all $x \in X$ (recall that $P_B = \{\text{Fr}(U) : U \in B\}$).

Proof. It is possible by the well known A.V. Arhangel’skii's theorem [5] to faithfully index all points of $X$ by the ordinals from $C : x = \{x_{\alpha} : \alpha < C\}$. Of course $|X| < C$ implies $|x| = \omega$ and the theorem is trivial in this case, so we assume from now on that $|X| = C$.

Fix a family $F = \{f_{\alpha} : \alpha < C\}$ of real-valued continuous functions on $X$ satisfying the following conditions:

(1) $f_{\alpha}(x) \in [0,1]$;

(2) $\{x_{\alpha}\} = f_{\alpha}^{-1}(0)$

for all $\alpha < C$.

Suppose we have a family $S = \{S_{\alpha} : \alpha < C\}$ where $S_{\alpha} = \{r_{\alpha}^n : n \in \omega\}$ is a decreasing sequence of positive elements of $I$ converging to zero. It is straightforward to verify that

$$B_S = \{f_{\alpha}^{-1}([0,r_{\alpha}^0])) : n \in \omega, \alpha < C\}$$

is a base of $X$. To find a base promised in the theorem we will construct an appropriate $S$ by recursion along $\alpha < C$.

Assume that the sequences $S_{\alpha} = \{r_{\alpha}^n : n \in \omega\}$ have been constructed for all $\alpha < \beta < C$. As $\{f_{\beta}(x_{\beta}) : \alpha < \beta\} < C$ there exists a decreasing sequence $\{r_{\alpha}^n : n \in \omega\} \subset (0,1) \setminus \{f_{\beta}(x_{\beta}) : \alpha < \beta\}$ converging to 0. Let $S_\beta = \{r_{\beta}^n : n \in \omega\}$.

The family $S = \{S_{\alpha} : \alpha < C\}$ being at hand let us prove that the base $B = B_S$ is as required. Since $\text{Fr}(f_{\alpha}^{-1}([0,r_{\alpha}^0])) \subset f_{\alpha}^{-1}(r_{\alpha}^0)$, it suffices to prove that $\text{ord}(\mathcal{S},x) < C$ for $\mathcal{S} = \{f_{\alpha}^{-1}(r_{\alpha}^n) : \alpha < C, n \in \omega\}$ and any point $x \in X$.

Indeed, there is an $\alpha < C$ with $x_{\alpha} = x$. For every $\beta > \alpha$ and $n \in \omega$ it is impossible that $f_{\beta}(x) = r_{\beta}^n$, so $\{E \in \mathcal{S} : x \in E\}$ and $x \in \bigcap_{\beta \leq \alpha} \{f_{\beta}^{-1}(r_{\beta}^n) : n \in \omega, \beta < \alpha\}$ and this finishes our proof.
2. Corollary. (CH). Any first countable compact $X$ has a peripherally point-countable base.

The peripherally point-countable base (abbr.: PPC-base) seems to be an interesting notion in itself. It is hereditary and looks like a dimensional property since all zero-dimensional spaces have a PPC-base. Any compact space is a continuous image of a zero-dimensional compact space. Therefore our following example shows that the PPC-base property is not invariant with respect to perfect mappings.

3. Example. The space $X=I^\omega_1$ has no PPC-base.

Proof. Let $B$ be a base in $X$, $A \subset I^\omega_1$ - a countable set and $z \in I^A$. Then there is a $U \in B$, a countable $A_1 \supset A$, $A_1 \subset I^\omega_1$ and $z_1 \in I^{A_1}$ such that $p_{A_1}^{-1}(z_1)=z$ and $p_{A_1}^{-1}(z_1) \in \text{Fr}(U)$. To prove this, pick any $U \in B$ with $U \cap \bigcup_{\alpha} p_{A_1}^{-1}(x) \notin p_{A_1}^{-1}(z) \setminus \bigcup_{\alpha}$. There is a countable $A_1 \supset A$ for which $p_{A_1}^{-1}(U)=U$ holds. The set $p_{A_1}(U)$ is open in $I^{A_1}$ and $\emptyset \neq p_{A_1}(U) \cap (p_{A_1}^{-1}(z) \cap (p_{A_1}^{-1}(z))$ for if $p_{A_1}(U) \supset (p_{A_1}^{-1}(z)$, then $\exists \alpha \neq p_{A_1}(U) \cap p_{A_1}^{-1}(p_{A_1}^{-1}(z)) = p_{A_1}^{-1}(z)$.

The space $(p_{A_1}^{-1}(z)$ being connected, there is a point $z_1 \in \text{Fr}(p_{A_1}(U) \cap (p_{A_1}^{-1}(z)) \cap \text{Fr}(p_{A_1}(U))$.

Thus $z_1 \in p_{A_1}(U) \cap p_{A_1}(U) \cap p_{A_1}(U) \cap p_{A_1}(U)$ so that $p_{A_1}^{-1}(z_1) \in \text{Fr}(U)$. Of course $p_{A_1}^{-1}(z_1)=z$.

Now it is not difficult to construct a transfinite sequence $(\omega_1, A, U_{\alpha})$ with the following properties:

3. $A \subset A_1^\omega_1$, $|A_1^\omega_1|=\omega_1$;
4. $A \subset A_2$, if $\alpha < \beta < \omega_1$;
5. $x_\alpha \in I^{A_\alpha}$, $U_{\alpha} \subset B$ and $p_{A_\alpha}^{-1}(x_\alpha) \in \text{Fr}(U_{\alpha})$;
6. $p_{A_\alpha}(x_\beta)=x_\alpha$ for $\alpha < \beta$;
7. $U_{\alpha} \cap \text{Fr}(U_{\alpha}) \neq \emptyset$ for all $\alpha < \beta < \omega_1$.

Once this is done, let $A=\bigcup_{\alpha} A_\alpha: \alpha < \omega_1$ and $x=\bigcup_{\alpha} x_\alpha: \alpha < \omega_1$. Then $U_{\alpha} \neq U_{\beta}$ for different $\alpha, \beta: x \in I^A$ and any $y \in p_{A_\alpha}^{-1}(x)$ belongs to the set $\bigcap_{\alpha} \text{Fr}(U_{\alpha}) \cap x < \omega_1$ which shows that $B$ is not peripherally point-countable.
4. **Main technical result.** Given a first countable compact $X$ and a base $B$ in $X$, one can produce a zero-dimensional compact $Y$ and a continuous onto mapping $f:Y \to X$ such that $\chi(y,Y) \leq \ord(p_B,f(y))$ for any $y \in Y$.

**Proof.** Let $q_y(0) = U$, $q_y(1) = X \setminus U$ for all $U \in B$. Define $Y$ to be the subset of $D^B$ consisting of those points $y = \langle y_U : U \in B \rangle$ for which the family $\{q_y(y_U) : U \in B\}$ has the finite intersection property. It is straightforward that $Y$ is closed in $D^B$. For $y = \langle y_U : U \in B \rangle \in Y$ let $f(y) = x$, where $x = \bigcap \{q_y(y_U) : U \in B\}$. To prove the consistency of our definition we must check that

$$|\bigcap \{q_y(y_U) : U \in B\}| \leq 1.$$

Take any $z \neq x$. There is a $U \in B$ with $x \notin U \cup z$. Therefore $q_y(y_U) \not\supset X \setminus U$ and $z \notin q_y(y_U)$ which is what we needed. That $f$ is continuous and onto is routine. To verify the inequality $\chi(y,Y) \leq \ord(p_B,f(y))$ let $C = \{U \in B : f(y) \notin Fr(U)\}$. Prove that $p_C$ is one-to-one on $f^{-1}(f(y))$. Pick $y_1, y_2 \in f^{-1}(f(y))$, $y_1 = \langle y_U^1 : U \in B \rangle$, $y_2 = \langle y_U^2 : U \in B \rangle$. If $p_C y_1 = p_C y_2$ then for any $U \notin C$ either $f(y) \notin U$ or $f(y) \notin U$. We have $q_y(y_U^1) \supseteq f(y)$ for $U \in B$ and $i = 1, 2$. The set $q_y(y_U^1)$ contains $f(y)$ iff $q_y(y_U^1) \supseteq f(y)$ for $U \in B \setminus C$, so there is a single possibility to choose a set $W$ out of the couple $\{U, X \setminus U\}$ with $f(y) \in W$. Hence $y_1, y_2$ for $U \in B \setminus C$ and $y_1 = y_2$.

Therefore $w(f^{-1}(f(y))) \leq |C|$ and our proof is complete.

Let us list some consequences of 4.

5. **Theorem.** For any first countable compact $X$ there is a zero-dimensional compact $Y$ and a continuous onto mapping $f:Y \to X$ with $\chi(y,Y) \leq \omega$ for all $y \in Y$.

**Proof.** Apply Theorem 1 and Result 4.

6. **Corollary.** For any first countable compact $X$ with a PPC-base there is a zero-dimensional compact $Y$ with $\chi(Y) = \omega$ which can be mapped continuously onto $X$.

7. **Corollary.** (A.V. Ivanov [1].) If CH is assumed, then any first countable compact space is a continuous image of a zero-dimensional first countable compact space.

We are going to prove in ZFC that first countable compact spaces have a PPC-base in the case they belong to some wide classes extending thus the theorem of A.V. Ivanov within ZFC.
8. Theorem. If a first countable compact $X$ belongs to one of the classes below:
(i) Corson (Eberlein) compact spaces;
(ii) linearly ordered spaces,
then $X$ has a PPC-base.

Proof. For (i) it is sufficient to prove that the $\Sigma$-product of real lines has a PPC-base. Let $\Sigma = \{ x \in \mathbb{R}^\omega : \text{supp}(x) : \alpha < \omega \}$ is countable and $B = \{ M(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Sigma, 0_1, \ldots, 0_n \in \mathbb{T}(\mathbb{R}) \}$ are rational intervals, $Fr(0_1) \not= 0$, $i = 1, \ldots, n$. Here $M(\alpha_1, \ldots, \alpha_n) = \{ x \in \Sigma : x(\alpha_i) \in 0_i, i = 1, \ldots, n \}$ - the standard open set in $\Sigma$. If $\text{ord}(P_B, x) > \omega$ for some $x \in \Sigma$, then by $\Delta$-argument there is an uncountable $A \subset \Sigma$ such that $\text{supp}(x) \supset A$ contradicting $x \in \Sigma$. Thus (i) is proved.

As to (ii) we shall establish even more, namely that every first countable compact LOTS $X$ has a peripherally disjoint (in an obvious sense) base.

Note first that for any $x \in X$ either $X$ is locally countable at $x$, or $| (a, b) | = \mathcal{C}$ for each interval $(a, b)$ containing $x$. Fix a numeration $\{ x_\alpha : \alpha < \mathcal{C} \}$ of the set $X$. Suppose intervals $(a_\alpha^\beta, b_\alpha^\beta)$ are chosen for $\alpha < \beta < \mathcal{C}$ and $n \in \omega$ so that

(8) $\{ (a_\alpha^n, b_\alpha^n) : n \in \omega \}$ is a base of $X$ at the point $x_\alpha$;
(9) if $X$ is locally countable at $x_\alpha$, then $(a_\alpha^n, b_\alpha^n)$ are clopen for all $n \in \omega$;
(10) the family of boundaries of chosen intervals is disjoint.

If $X$ is locally countable at $x_\alpha$, then pick any clopen interval base $B_\beta$ at $x_\alpha$ : $B_\beta = \{ (a_\alpha^n, b_\alpha^n) : n \in \omega \}$. If not, then let $A_\beta = \{ (a_\alpha^n, b_\alpha^n) : \alpha < \beta, n \in \omega \}$. We will consider only the case when $X$ is locally countable from the left at $x_\alpha$ (there is an $x < x_\alpha$ with $| (x, x_\alpha) | = \omega$). All other possible cases are similar or simpler.

As $| A_\beta | < \mathcal{C}$ reasoning as in proof of Theorem 1, we obtain a sequence $\{ b_\alpha^n : n \in \omega \} \subset X \setminus A_\beta$ with $b_\alpha^n \rightarrow x_\alpha$ for all $n \in \omega$ and $b_\alpha^n \not= x_\alpha$ such that $a_\alpha^n \not= \text{Fr}(a_\alpha^n, x_\alpha)$ and $\{ (a_\alpha^n, b_\alpha^n) : n \in \omega \}$ is a base at $x_\alpha$. The inductive step being done, we have got a base $B = \{ (a_\alpha^n, b_\alpha^n) : \alpha < \mathcal{C}, n \in \omega \}$ which is as promised, so our proof is complete.

9. Corollary. If a first countable space $X$ belongs to one of the following classes:
(i) Corson compact spaces;
(ii) Eberlein compact spaces;
(iii) continuous images of first countable compact LOTS,
then there exists a zero-dimensional first countable compact space which can be mapped onto $X$ continuously.

References


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