Commentationes Mathematicae Universitatis Carolinae

Martin Loebl

Hercules and Hydra

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 1, 85--95

Persistent URL: http://dml.cz/dmlcz/106600

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,1 (1988)

HERCULES AND HYDRA

Martin LOEBL

Abstract: Hercules and Hydra is a game on rooted and finite trees. L. Kirby and J. Paris proved in [1] that the statement "Every recursive strategy of Hercules is a winning strategy" is independent in Peano Arithmetic (PA). We define two simple recursive strategies MAX, MIN of Hercules and prove (in PA) that they give trajectories of maximum and minimum length.
In Section 3 we show that the statement "Strategy MAX is a winning strategy of Hercules" cannot be proved in PA. This improves the result of Kirby and Paris.

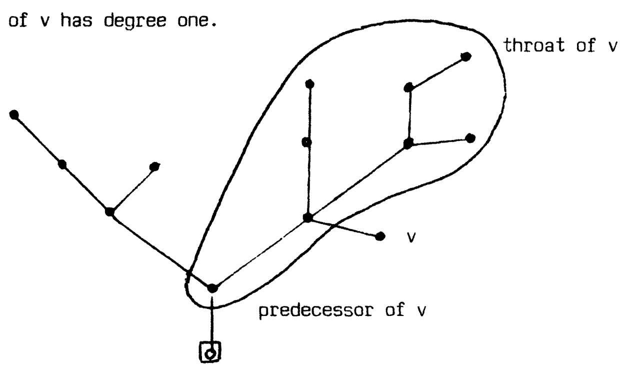
Key words: Tree, undecidability, combinatorial game.

Classification: 05C05, 90D99, 03B25

---

§ 1. Introduction. Tree is a finite connected acyclic graph. All trees will have one fixed vertex, the root. Each end-vertex different from the root is called head. The root of T will be also denoted by root(T). Let v be a head of a tree T. The vertex of the (unique) path from v to root(T), which has distance two from v, is called predecessor of v. If dist(v,root(T))=1 then the predecessor of v is defined to be the root. Throat of v is a subtree obtained by deleting v from the maximal subtree of T containing v, in which the predecessor of v has degree one.

Fig.1

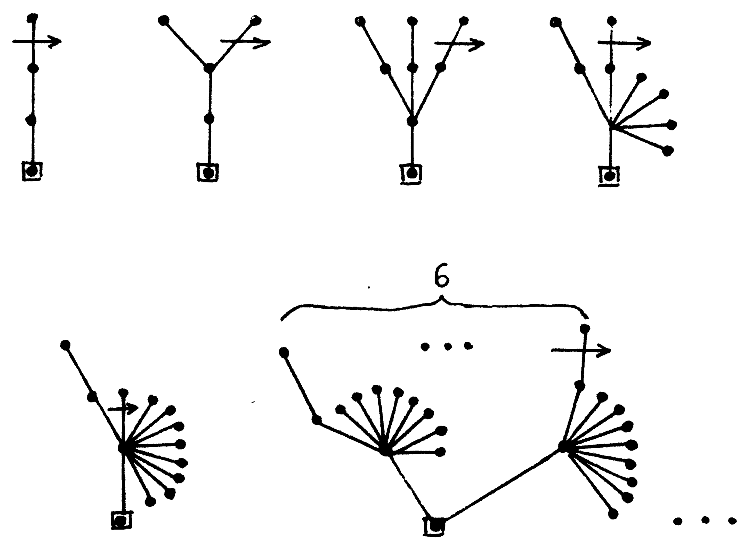


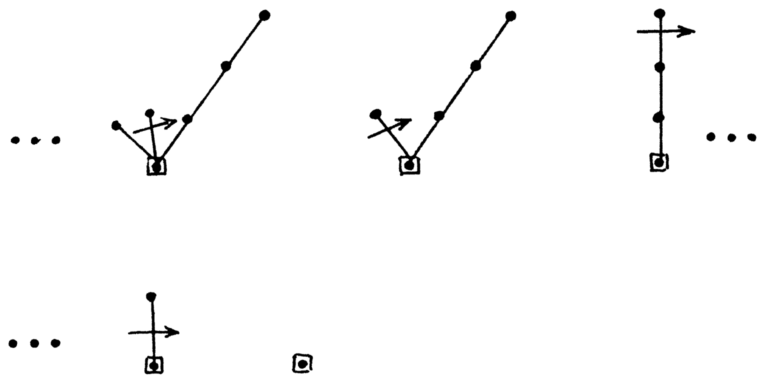
P_n will be a path on n+1 vertices, rooted in one end-point.

Hercules and Hydra is a battle of two players, Hercules and Hydra. A path $P_k = T_0$ is given on the beginning and it is modified to other trees T_1, T_2,... by moves of Hercules and Hydra in the following manner:

In his n-th move, where n is a positive integer, Hercules chops off a head v of the tree T_{n-1}. On return, Hydra adds n new replicas of the throat of v growing from the predecessor of v, creating T_n. Hercules wins if after some finite number of moves m (the length of the battle), the tree T_m is just the root. If Hercules does not win, then the length of the battle is not defined.

Fig. 2: Example of a battle





Every battle defines a sequence of trees $T_0, T_1, \ldots$, ; we call such a sequence trajectory.

The length of a trajectory is the length of a corresponding battle.

Theorem 1 [1]. The statement "Every recursive strategy of Hercules is a winning strategy" cannot be proved in PA. $\square$

Let us refer to [1] for the fact that the statement from Theorem 1 is valid in the infinite set theory.

§2. Optimal Strategies. Let us draw the trees $T_0=P_k$, $T_1, T_2, \ldots$ from a trajectory on the paper so that the original throat and the new replicas of the throat are put on the right in the same manner as the original throat. This drawing defines naturally an ordering of heads of every tree T_i: $a < b$ iff a is drawn more left than b.

Strategy MAX says: always chop off the rightmost head.

Strategy MIN says: always chop off the leftmost head.

Theorem 2. Let $\tau(k)$ be a class of all trajectories starting with P_k.

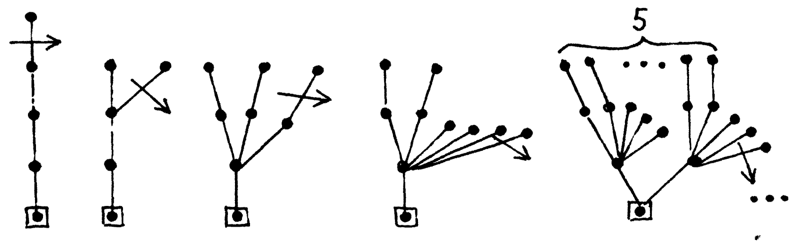
A) MAX trajectory has maximum length among all trajectories from $\tau(k)$.

B) MIN trajectory has minimum length among all trajectories from $\tau(k)$.

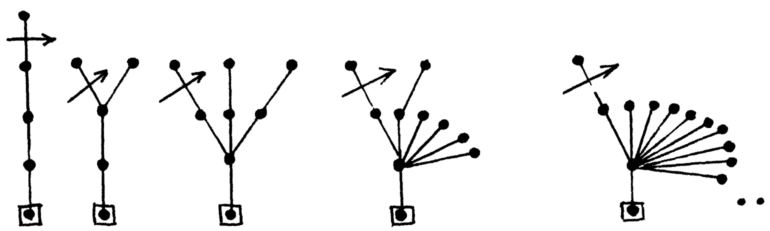
To prove Theorem 2, we use the following definitions: let $a \neq b$ be distinct heads of a tree T. Let x be the first common vertex of the paths from a to root (T) and from b to root (T). Then T(a,b) is the subtree of T which is maximal among all subtrees T' of T, satisfying

Fig.3: Examples of MAX and MIN

MAX:

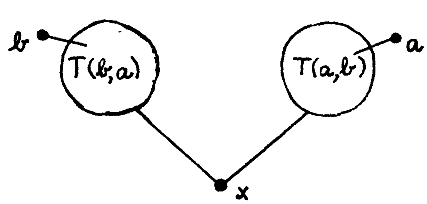


MIN:



i) a and x are vertices of T´,

ii) x is the root of T´ and x has degree one in T´.



Let M, N be trees and $j \geq 0$. $M \geq_j N$ if there exists a trajectory H=H(1),H(2), ... from $\tau(k)$ which satisfies:

1) H(j)=M,

2) H has not a finite length or there exists $i \geq j$ such that H(i)=N.

Let b be a head of M. $M >_{j,b} N$ if there exists a trajectory H from $\tau(k)$ which satisfies:

1) $H(j)=M$,
2) b is choped off in the (j+1)-th move, i.e. $b \notin H(j+1)$,
3) H has not a finite length or there exists $i>j$ such that $H(i)=N$.

Proof of A). Let m be a positive integer. A trajectory defined by a strategy of Hercules which coincides with MAX on the first m moves is called m-MAX trajectory (MAX trajectory $=\infty$ - MAX trajectory).

1) If MAX trajectory has an infinite length, then we are done.

2) Let MAX trajectory have length n. For a contradiction let $m_0 < n$ be the maximum integer such that some m_0-MAX trajectory has a biiger length than MAX trajectory. The following claim is proved in Appendix.

Claim 1. Let m be a positive integer. Let $T_0, T_1, \dots$ be m-MAX trajectory from $\tau(k)$. Let $S_0, S_1, \dots$ be (m+1)-MAX trajectory from $\tau(k)$. Then $S_{m+1} \geq (m+1) T_{m+1}$.

Claim 1 gives a contradiction with the choice of m_0. $\square$

Remark. It follows from the proof that MAX is the only strategy of Hercules with the maximum length. However, this fact cannot be proved in PA as one can find other recursive strategies with an unprovably finite length.

The proof of B) is analogous to the proof of A).

Sketch of Proof of B). Let m be a positive integer. A trajectory defined by a strategy of Hercules which coincides with MIN on the first m moves us called m-MIN trajectory.

Let m be a positive integer. Let $T_0, T_1, \dots$ be m-MIN trajectory from $\tau(k)$. Let a, b be heads of T_m.

1) $T_m(a,b)$ are stars: this is proved by induction on m.
2) Let $S_0, S_1, \dots$ be (m+1)-MIN trajectory from $\tau(k)$. Then $T_{m+1} \geq \geq (m+1) S_{m+1}$: this follows from 1).
3) Let MIN trajectory have length n. For a contradiction let $m_0 < n$ be the maximum integer such that some m_0-MIN trajectory has less length than MIN trajectory. Then 2) gives a contradiction with the choice of m_0. $\square$

§ 3. Unprovability

Theorem 3. The statement "Strategy MAX is a winning strategy" cannot be proved in PA.

We say that a function f is total, if f(x) is defined for every positive integer x. Denote by l(n) the length of MAX trajectory starting on P_n. We show below that the statement "l is total" is unprovable in PA. We use the method of Fast Growing Hierarchy (FGH) of provably total recursive functions [4]. If some function f exceeds all functions from this hierarchy (we call f large), then the statement "f is total" cannot be proved in PA [4].

Definition 1 (of fundamental sequence). Let α be an ordinal number, $\alpha < \varepsilon_0 = \omega^{\omega^{\cdot^{\cdot^{\cdot}}}}\}\omega$. Then α can be represented in a unique Cantor normal form

$$\alpha = \omega^{\beta_k} + \omega^{\beta_{k-1}} + \ldots + \omega^{\beta_1} + \omega^{\beta_0};\ \alpha > \beta_k \geq \beta_{k-1} \geq \ldots \geq \beta_0.$$

If $\beta_0 = 0$, then α is a successor. Otherwise α is a limit and we can assign to it a fundamental sequence $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$ with supremum α as follows:

$\alpha_n = \omega^{\beta_k} + \omega^{\beta_{k-1}} + \ldots + \omega^{\beta_1} + \omega^{\delta}(n+1)$ if $\delta + 1 = \beta_0$.

$\alpha_n = \omega^{\beta_k} + \omega^{\beta_{k-1}} + \ldots + \omega^{\beta_1} + \omega^{(\beta_0)_n}$ if β_0 is a limit.

Definition 2 (of FGH). For every $\alpha < \varepsilon_0$ we define a function f_α:

$f_0(n) = n+1$,

$f_{\alpha+1}(n) = f_\alpha^n(n)$ (the exponent n means n-fold application of f_α),

$f_\alpha(n) = f_{\alpha_n}(n)$ if α is a limit.

Definition 3. A tree is called hanged if its root has degree one. The vertex joined to the root in a hanged tree is called join.

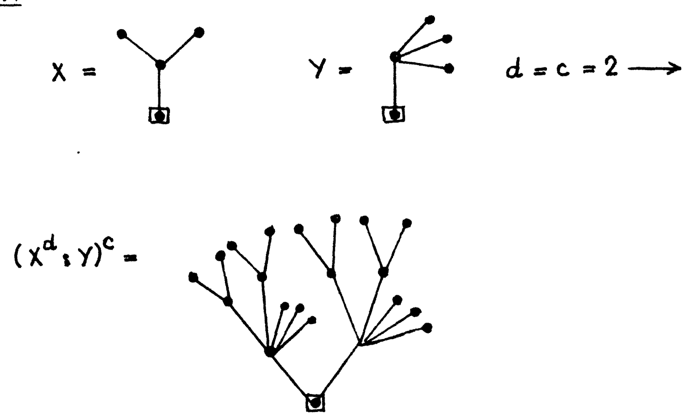
Definition 4. Let S, T be trees. Then the tree S.T is created as follows: take one copy of S and one disjoint copy of T and identify root(S) with root(T). Put root(S.T)=root(S). (We write $\underbrace{S \ldots S}_{d} = S^d$.)

Definition 5. Let S be a tree and T be a hanged tree. Then the tree S:T is created as follows: take one copy of S and one disjoint copy of T and identify root(S) with join(T). Put root(S:T)=root(T).

Remark. The symbols defined above can be composed to get more complicated trees, e.g. $(X^d:Y)^c$.

S_n will be a star on n+1 vertices, rooted in one endpoint.

Fig.4



Definition 6. The trees T_α, $\alpha < \varepsilon_0$, are defined by induction:

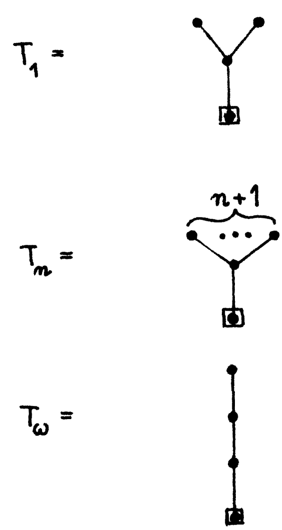
1) $\alpha \in \omega$. Then $T_\alpha = S_{\alpha+2}$.

2) $\alpha = c \cdot \omega^a + b$, where $c \in \omega$, $a > 0$ and $b < \omega^a$. Then $T_\alpha = R^c : S$, where

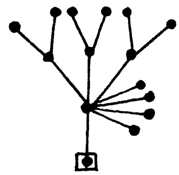
i) if a is an integer, then $R = T_{a-1}$, otherwise $R = T_a$;

ii) if $b=0$ then $S = S_1$. If b is a positive integer, then $S = T_{b-1}$. If $b \geq \omega$, then $S = T_b$.

Fig.5



$T_{3\omega^2+4} =$



T_{ω^ω}

$T_{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}\}n} = P_{n+2}$

Definition 7. Let x be an integer. Let T be a tree. If there exists a trajectory $T_0, T_1, \ldots$ of undefined length, then put $s(x,T)=\infty$. Otherwise put $s(x,T)=\sup\{y-x;\ y$ is the length of a trajectory $T_0, T_1, \ldots$ such that $T_x=T\}$.

Next Lemma 1 and Lemma 2 follow simply from the definition of s(x,T) and from the definition of the strategy MAX. Therefore we omit their proofs.

Lemma 1. Let M be a subtree of N (roots coincide) and $x \leq y$. Then $s(x,M) \leq s(y,N)$.

Lemma 2. Let $T_0, T_1, T_2, \ldots$ be a MAX trajectory with $T_0=P_x$. Let i, j be integers and X be a tree such that $T_i=X^j$. Let $j' \leq j$ and N be a maximum rooted path of X. Then $l(x)$ is not defined or there exists $m \geq i$ such that $T_m=N^{j'}$.

The following claim is proved in the Appendix.

Claim 3. $s(x,T^x_\alpha) \geq f_{\alpha+1}(x)$, for all $\alpha < \varepsilon_0$, $x \leq \omega$.

Lemma 4. Let x be an integer. Let $P_x=T_0$, $T_1, T_2, \ldots$ be MAX trajectory. If there exists $n \geq x$ such that $T_n=T^x_\alpha$, then $l(x) \geq s(x,T^x_\alpha) \geq f_{\alpha-1}(x)$.

Proof. It follows from Theorem 2 and Claim 3. □

Proof of Theorem 3. Let x be an integer, let $T_0=P_x$, $T_1, T_2, \ldots$ be MAX trajectory. Let $n \leq x-3$. Let m be the first integer such that the rightmost head of T_m has distance two from the root (if m does not exist, then $l(x)=$ $=\infty$ and we are done). It holds that $m \geq x$. Then $T_{m+1}=X^{m+2}$ and a maximum rooted path of X has length x-1. We have from Lemma 2 that either $l(x)=\infty$ or there exists $m' \geq m$ such that $T_{m'}=P^{X}_{n+2}$. Hence from Lemma 4 $l(x) \geq f_{\delta_n}(x)$, where $\delta_n = \omega^{\cdot^{\cdot^{\cdot^{\omega}}}} \}n$. □

§ 4. Appendix

Proof of Claim 1. Let $T_0, T_1, T_2, \ldots$ be an m-MAX trajectory. Let $a > b$ be distinct heads of T_m. We prove

1) $T_m(b,a) \geq m \; T_m(a,b)$;

2) If $T_m(b,a)$ is not isomorphic to $T_m(a,b)$ and b is the rightmost head of $T_m(b,a)$, then $T_m(b,a) > m,b \; T_m(a,b)$.

3) Let us denote by M the m+1-th stage after choping off a and by N the m+1-th stage after choping off b. Then $M \geq m+1 \; N$.

i) First we prove together 1) and 2) by induction on m. It obviously holds for m=1,2. Let $m > 2$: If a, b are heads of distinct replicas which grow after the m-th move of Hercules, then $T_m(b,a)$ is isomorphic to $T_m(a,b)$. Otherwise we may assume without loss of generality that a, b are vertices of T_{m-1}. Clearly b is a head of T_{m-1}. Let v be the head of T_{m-1} choped off in the m-th move (according to MAX).

We distinguish two cases denoted A and B.

A Let a be a head of T_{m-1}. v cannot be a vertex of $T_{m-1}(b,a)$, thus $T_{m-1}(b,a)$ is equal to $T_m(b,a)$. If v is not a vertex of $T_{m-1}(a,b)$, then also $T_m(a,b)$ is equal to $T_{m-1}(a,b)$. Hence 1) and 2) follow from the induction assumption.

If v is a head of $T_{m-1}(a,b)$, then $T_{m-1}(a,b) \geq m-1 \; T_m(a,b)$. Thus $T_m(b,a) = T_{m-1}(b,a) \geq m-1 \; T_{m-1}(a,b) \geq m \; T_m(a,b)$. Let b be the rightmost head of $T_m(b,a)$. If $T_{m-1}(b,a)$ is not isomorphic to $T_{m-1}(a,b)$, then 2) follows from the induction assumption. Otherwise there exists an isomorphism which takes v to b. As $T_{m-1}(a,b) > m-1,v \; T_m(a,b)$, 2) holds.

B Let a fail to be a head of T_{m-1}. Then $T_m(b,a) = T_{m-1}(b,v) \geq m-1 \; T_{m-1}(v,b) \geq m \; T_m(a,b)$.

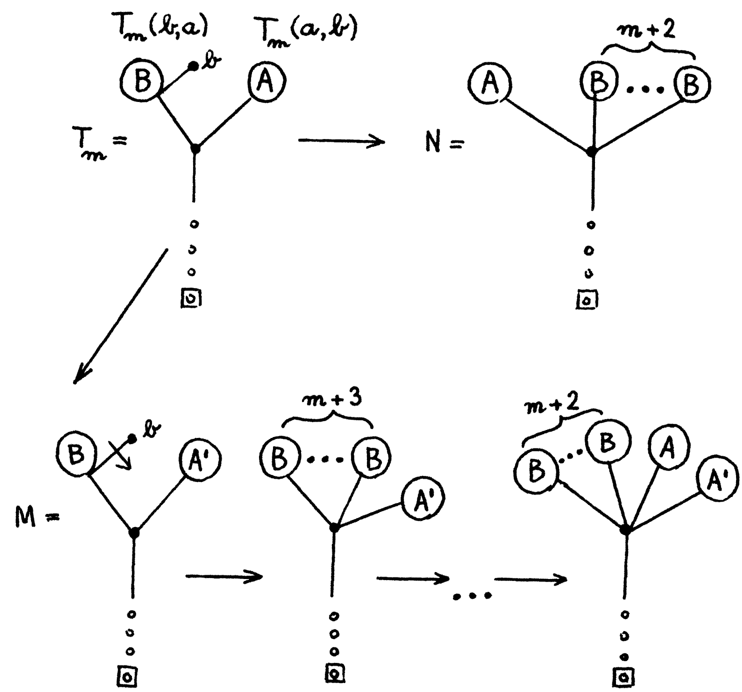
2) follows as above.

ii) To prove 3), we proceed by induction on the distance of a and b. If the rightmost head of $T_m(b,a)$ is $v \neq b$, then 3) holds from the induction assumption for heads a, v and v, b.

Let b be the rightmost head of $T_m(b,a)$. If $T_m(a,b)$ is isomorphic to $T_m(b,a)$, then also M is isomorphic to N. Otherwise 3) follows from 2) by an easy discussion (see Fig. 6).

To prove Claim 1 take a equal to the rightmost head of T_m. □

Fig.6



Proof of Claim 3. We proceed by induction on α. If $\alpha=\beta+1$, then we prove that $s(x,T_\alpha^x)\geq f_{\alpha+1}(x)$. If α is a limit, then we prove that $s(x,T_\alpha)\geq$ $\geq s(x,T_{\alpha_x})$, where $\{\alpha_x\}$ is the fundamental sequence for α.

1) $\alpha=\beta+1$. By the definition of the game, $s(x,T_\alpha)\geq s(x,T_\beta^x)\geq f_\alpha(x)$. Thus $s(x,T_\alpha^2)\geq f_\alpha\circ f_\alpha(x)$ and finally $s(x,T_\alpha^x)\geq \underbrace{f_\alpha\circ\ldots\circ f_\alpha}_{x}(x)$.

2) $\alpha=c\cdot\omega^a+b$ is a limit. If $b\neq 0$, then $\alpha_x=c\cdot\omega^a+b_x$ and we use the induction assumption for b. If $c>1$, then $\alpha_x=(c-1)\omega^a+(\omega^a)x$ and we use

the inductive assumption for ω^a. If a is a limit, then $\alpha_x = \omega^{\omega_x}$ and we use the inductive assumption for a. If $a=a'+1$, then $\alpha_x = X\omega^{a'}$. $T_\alpha = S_{a+1}:S_1$ by the definition. We have $s(x,T_\alpha)=s(x,S_{a+1}:S_1) \geq s(x,S_\alpha^{x+1}:S_1)=s(x,T_{\alpha_x})$. □

References

[1] L. KIRBY, J. PARIS: Accessible independence results for Peano Arithmetic, Bulletin of the London Math. Soc. 14(1982).

[2] M. LOEBL, J. MATOUŠEK: On undecidability of the weakened Kruskal theorem, Contemporary Math., Proceedings Symposia AMS "Logic and Combinatorics", ed. S. Simpson, Amer. Math. Soc. (1987), 275-280.

[3] J. NEŠETŘIL: Some non standard Ramsey-like applications, Theoret. Comp. Science 34(1984), 3-15.

[4] S. FEFERMAN: Classifications of recursive functions by means of hierarchies, Transactions of the American Mathematical Society 104 (1962), 101-122.

Charles University, KAM MFF, Malostranské nám. 25, 118 00 Praha 1, Czechoslovakia

(Oblatum 19.10. 1987)