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ON THE BASIC REPRESENTATION OF THE AFFINE
KAC-MOODY LIE ALGEBRAS $D_n^{(1)}$

Thomas N. VOUGIOUKLIS

Abstract: We compute the constants needed for the principal realization given in [3] of the affine Kac-Moody Lie algebras $D_n^{(1)}$, $n \geq 4$.

Key words: Affine Kac-Moody Lie algebras, graded Lie algebras.

Classification: 17B65, 17B70

1. Introduction. Let $n \geq 4$. Let $\{E_{ij}\}_{j=1, \dots, 2n}$ be the standard basis of the space of $2n \times 2n$ complex matrices, so that the matrix E_{ij} is 1 in the ij -entry and 0 in all the other entries. In the case of Kac-Moody Lie algebra $\mathfrak{G}(A)$ of type $D_n^{(1)}$ ($n \geq 4$) we have (see [4], [9])

$$(1) \quad \mathfrak{G} = \mathfrak{o}(2n, \mathbb{C}), \quad (x|y) = \text{tr } xy.$$

Instead of the standard representations [2], we consider all $2n \times 2n$ complex matrices in the form given in [1], for type D_n . We can take the Chevalley generators [3] (cf. [8]) e_i, f_i, h_i given in [10] by the following relations, for $i=1, \dots, n-1$.

$$(2) \quad \left\{ \begin{array}{l} e_0 = E_{2n-1, 1} - E_{2n, 2}, \quad e_i = E_{i, i+1} - E_{2n-i, 2n-i+1}, \\ e_n = E_{n-1, n+1} - E_{n, n+2}, \\ f_0 = E_{1, 2n-1} - E_{2, 2n}, \quad f_i = E_{i+1, i} - E_{2n-i+1, 2n-i}, \\ f_n = E_{n+1, n-1} - E_{n+2, n}, \\ h_0 = E_{2n, 2n} + E_{2n-1, 2n-1} - E_{22} - E_{11}, \\ h_i = E_{2n-i, 2n-i} + E_{ii} - E_{2n-i+1, 2n-i+1} - E_{i+1, i+1}, \\ h_n = E_{nn} + E_{n-1, n-1} - E_{n+2, n+2} - E_{n+1, n+1} \end{array} \right.$$

Let us denote (see [10]) by κ the number defined in the following way:

$\kappa = \kappa$ if $\kappa \leq n$ and $\kappa = \kappa - 1$ if $\kappa > n$ for every κ of Z . Moreover let $h=2(n-1)$ be the Coxeter number [3]. Then we have the following

Proposition 1 (see [10]). A Lie algebra \mathfrak{G} of type D_n is a graded modh where the 1-principal Z/hZ -gradation is given by setting

$$\deg E_{ij} = (j-i) \text{ modh.}$$

So we can write

$$\mathfrak{G} = \bigoplus_{i \in Z/hZ} \mathfrak{G}_i.$$

We take a normalized 1-cyclic element [6] as follows: $E = \beta_i e_i$ where $\beta_0 = \beta_1 = \beta_{n-1} = \beta_n = 1/\sqrt{2}$, $\beta_i = 1$ for $i=2, \dots, n-2$. One obtains the relations

$$(3) \begin{cases} E^{h+1} = (-1)^n E, & E^{2\kappa-1} = (-1)^n t_E^{2(n-\kappa)-1} \text{ for } \kappa \in \mathbb{N}; \kappa \neq \frac{n}{2}, \\ \text{and } \text{tr } E^h = (-1)^n h. \end{cases}$$

The centralizer S of E is a CSA of \mathfrak{G} with dimension n (see [6]). A basis of S is $\{E, E^3, \dots, E^{2n-3}, E_0\}$ where

$$E_0 = E_{1n} - E_{1,n+1} + E_{n1} + E_{n,2n} - E_{n+1,1} - E_{n+1,2n} + E_{2n,n} - E_{2n,n+1},$$

and we have

$$(4) \quad E_0^2 + (-1)^n 4E^h = 4I, \quad \text{tr } E_0^2 = 8, \quad \text{and } E_0 E^\nu = E^\nu E_0 = 0 \text{ for every } \nu.$$

In [5] and [3] a construction of the basic representation of the affine Kac-Moody Lie algebras have been introduced which construction is a generalization of the one introduced in [7]. This construction has been called principal realization. For this realization, in the case of $D_n^{(1)}$, one needs n root vectors A_1, \dots, A_n , with respect to S , such that their projections on \mathfrak{G}_0 form a basis of \mathfrak{G}_0 . Moreover if $\{T_s\}_{s=1, \dots, n}$ is a basis of S which is normalized such that $(T_i | T_{n+1-j}) = \delta_{ij}$ for all $i, j=1, \dots, n$, then the constants λ_{rs} defined by the relation $[T_s, A_r] = \lambda_{rs} A_r$ are needed for the principal realization. If we decompose the vectors A_r with respect to the 1-principal gradation

$$A_r = \sum_{\nu} A_{r\nu}, \quad r=1, \dots, n; \quad \nu \in Z/hZ,$$

then the elements $A_{r\nu}, T_s$ where $r, s=1, \dots, n; \nu=0, \dots, n-1$, form a basis of \mathfrak{G} . The aim of this paper is to compute the constants λ_{rs} in the case of $D_n^{(1)}$, $n \geq 4$.

2. A basis of \mathfrak{G} . Let $n=2\kappa$. Then decomposing the S with respect to the 1-principal gradation we have dimension one for the degrees $1, 3, \dots, n-3, n+1, n+3, \dots, 2n-3$, and dimension two for the degree $n-1$. For the dimension $n-1$ we have a basis: $\{E^{n-1}, E_0\}$.

For the one dimensional degrees, in order that the relation $(T_i | T_{n+1-j}) = \delta_{ij}$ should be valid, we can take the vectors

$$(5) \quad \begin{cases} T_s = \frac{1}{\sqrt{n}} E^{2s-1} & \text{for } s=1, \dots, \kappa-1; \\ T_s = t T_{n+1-s} = \frac{1}{\sqrt{n}} E^{2s-3} & \text{for } s=\kappa+2, \dots, n. \end{cases}$$

It remains to find two vectors $T_\kappa, T_{\kappa+1}$ of dimension $n-1$, i.e. from the linear span of E^{n-1}, E , such that $(T_\kappa | T_\kappa) = 0, (T_{\kappa+1} | T_{\kappa+1}) = 0$ and $(T_\kappa | T_{\kappa+1}) = 1$. Using the relations (3) and (4) one obtains the vectors.

$$(6) \quad T_\kappa = \frac{1}{\sqrt{2}} E^{n-1} + \frac{i}{4} E_0, \quad T_{\kappa+1} = \frac{1}{\sqrt{2}} E^{n-1} - \frac{i}{4} E_0, \quad \text{where } i^2 = -1.$$

Therefore a normalized basis in case that $n=2\kappa$ is given by the relations (5) and (6).

Let $n=2\kappa+1$. In this case for S we have only dimension one on the degrees $1, 3, \dots, n-2, n-1, n, \dots, 2n-3$. So in a similar way as above we have the following as a normalized basis of S :

$$(7) \quad \begin{cases} T_s = \frac{1}{\sqrt{n}} E^{2s-1} & \text{for } s=1, \dots, \kappa; \quad T_{\kappa+1} = \frac{1}{2\sqrt{2}} E_0; \\ T_s = -t T_{n+1-s} = \frac{1}{\sqrt{n}} E^{2s-3} & \text{for } s=\kappa+2, \dots, n. \end{cases}$$

Proposition 2. Let $\Theta_r = \frac{r\pi}{n-1}$, and $X = \text{diag}(x_1, \dots, x_n, -x_n, \dots, -x_1)$ be the elements of \mathfrak{G}_n . Then the $(\text{ad}E)^h$ has eigenvalues

$$(8) \quad \lambda_r = (-1)^{n+r} \cdot 2^h \cos^h \frac{\Theta_r}{2} \quad \text{for } r=1, \dots, n-2, \quad \text{and } \lambda_{n-1} = \lambda_n = (-1)^n$$

with corresponding, appropriate, eigenvectors

$$(9) \quad A_{r0} = \text{diag}(0, -\sin \Theta_r, \dots, (-1)^{n-2} \sin(n-2) \Theta_r, 0, \\ 0, -(-1)^{n-2} \sin(n-2) \Theta_r, \dots, \sin \Theta_r, 0),$$

$$(10) \quad \begin{cases} A_{n-1,0} = \text{diag}(1, 0, \dots, \nu_1, -\nu_1, 0, \dots, -1) \\ A_{n0} = \text{diag}(1, 0, \dots, \nu_2, -\nu_2, 0, \dots, -1), \end{cases}$$

where ν_1, ν_2 are the square roots of $(-1)^n$.

The set $\{A_{10}, \dots, A_{n0}\}$ is a basis of \mathfrak{S}_0 .

Proof. The non-degenerate operator $\text{ad}E$ shifts the gradation by 1, see [3], so the vectors

$$(\text{ad}E)^h X = \sum_{s=0}^h (-1)^s \binom{h}{s} E^{h-s} X E^s$$

are of degree zero. Since $E^{h-s} = (-1)^n t_E^s$, $0 < s < h$, we obtain

$$(11) \quad (\text{ad}E)^h X = (-1)^n \sum_{s=1}^{h-1} \binom{h}{s} t_E^s X E^s + E^h X + X E^h,$$

Our problem is to solve the matrix equation $(\text{ad}E)^h X = \lambda X$ or from relation (11) to solve the equation

$$(12) \quad \sum_{s=1}^{h-1} \binom{h}{s} t_E^s X E^s + (-1)^n (E^h X + X E^h) = (-1)^n \lambda X.$$

We have two cases:

Case A. Let $X = \text{diag}(0, x_2, \dots, x_{n-1}, 0, 0, -x_{n-1}, \dots, -x_2, 0)$. In this case depending on the skew symmetry with respect to the second diagonal of x_2, \dots, x_{n-1} in (12) we have to solve the homogeneous system on x_2, \dots, x_{n-1} with coefficient matrix the $(n-2) \times (n-2)$ matrix which has on the ν -th row, $\nu=1, \dots, n-2$, the following entries

$$\begin{aligned} & (-1)^{\nu+1} \left[\binom{h}{\nu-1} - \binom{h}{\nu+1} \right], \quad (-1)^{\nu+2} \left[\binom{h}{\nu-2} - \binom{h}{\nu+2} \right], \dots, \\ & - \left[\binom{h}{1} - \binom{h}{2\nu-1} \right], \quad 2 - \binom{h}{2\nu} - (-1)^n \lambda, \quad - \left[\binom{h}{1} - \binom{h}{2\nu+1} \right], \dots, \\ & (-1)^{n-\nu-3} \left[\binom{h}{n-3-\nu} - \binom{h}{n-3+\nu} \right], \quad (-1)^{n-\nu-2} \left[\binom{h}{n-2-\nu} - \binom{h}{n-2+\nu} \right]. \end{aligned}$$

Solving the above system we finally obtain that the $n-2$ eigenvalues of the $(\text{ad}E)^h$ are given by the relation (8) for $r=1, \dots, n-2$ and the corresponding eigenvectors are given by the relation (9). For the above computation we use the well known identity

$$\cos^h \theta = \frac{1}{2^{h-1}} \sum_{k=0}^{h/2-1} \binom{h}{k} \cos(h-2k)\theta + \frac{1}{2^h} \binom{h}{h/2}$$

Case B. Let $X = \text{diag}(x_1, 0, \dots, 0, x_n, -x_n, 0, \dots, 0, -x_1) = A_{r0}$ where $r=n-1, n$. In this case all matrices $(\text{ad}E)^\nu A_{r0}$ have non-zero entries only on $1, n, n+1, 2n$ rows and columns. Moreover $A_r = \sum_{\nu} A_{r\nu}$ must be an eigenvector of the $\text{ad}E_0$, i.e. $(\text{ad}E_0)A_r = \mu A_r$. Solving the above system we obtain the eigenvalue $(-1)^n$ for $(\text{ad}E)^h$ and a corresponding basis of X 's is the one given by (10). From the cases A and B we obtain that $\{A_{10}, \dots, A_{n0}\}$ is a basis of \mathfrak{S}_0 .

Remark. Let ϵ_r be a h -primitive root of $(-1)^{n+r}$ for $r=1, \dots, n-2$, and $\epsilon_{n-1} = \epsilon_n = \epsilon$ be a h -primitive root of $(-1)^n$. We set

$$(13) \quad A_r = \sum_{\nu=1}^h \mu_r^{-\nu} (\text{ad} E)^{\nu} A_{r0}$$

where $\mu_r = 2 \epsilon_r \cos \frac{\theta_r}{2}$, and $\mu_{n-1} = \mu_n = \epsilon$. Then I_s for $s=1, \dots, n$ together with the homogeneous components

$$(14) \quad A_{r\nu} = \mu_r^{-\nu} (\text{ad} E)^{\nu} A_{r0}, \quad r=1, \dots, n; \quad \nu=0, \dots, h-1$$

of A_r , form a basis of \mathcal{G} .

In order to write explicitly the matrices A_r for $r=1, \dots, n-2$ which are skew symmetric with respect to the second diagonal lets consider the matrix

$$A_r = \begin{pmatrix} A_{r1} & A_{r2} \\ A_{r3} & A_{r4} \end{pmatrix}$$

where $A_{r1}, A_{r2}, A_{r3}, A_{r4}$ are $n \times n$ matrices.

We set ϵ and $\langle \nu, \mu \rangle$ instead of ϵ_r and $(-1)^{\nu} \sin \mu \frac{\theta_r}{2}$ respectively, and we write simply the upper parts of the skew symmetric, with respect to the second diagonal, matrices A_{r2}, A_{r3} . Then A_{r1}, A_{r2}, A_{r3} are given respectively as follows:

$$\left(\begin{array}{cccccc} 0 & \frac{\langle 1,1 \rangle}{\epsilon \sqrt{2}} & \frac{\langle 2,2 \rangle}{\epsilon^2 \sqrt{2}} & \dots & \frac{\langle n-3, n-3 \rangle}{\epsilon^{n-3} \sqrt{2}} & \frac{\langle n-2, n-2 \rangle}{\epsilon^{n-2} \sqrt{2}} & \frac{\langle n+r, n-1 \rangle}{\epsilon^{n-1} 2} \\ \frac{\langle n+r, 1 \rangle}{\epsilon^{2n-3} \sqrt{2}} & \langle 1,2 \rangle & \frac{\langle 2,3 \rangle}{\epsilon} & \dots & \frac{\langle n-3, n-2 \rangle}{\epsilon^{n-4}} & \frac{\langle n-2, n-1 \rangle}{\epsilon^{n-3}} & \frac{\langle n+r, n-2 \rangle}{\epsilon^{n-2} \sqrt{2}} \\ \frac{\langle n+r, 2 \rangle}{\epsilon^{2n-4} \sqrt{2}} & \frac{\langle n+r-1, 3 \rangle}{\epsilon^{2n-3}} & \langle 2,4 \rangle & \dots & \frac{\langle n-3, n-1 \rangle}{\epsilon^{n-5}} & \frac{\langle n+r-1, n-2 \rangle}{\epsilon^{n-4}} & \frac{\langle n+r, n-3 \rangle}{\epsilon^{n-3} \sqrt{2}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\langle n+r, n-3 \rangle}{\epsilon^{n+1} \sqrt{2}} & \frac{\langle n+r-1, n-2 \rangle}{\epsilon^{n+2}} & \frac{\langle n+r, n-1 \rangle}{\epsilon^{n+3}} & \dots & \langle n+r, 4 \rangle & \frac{\langle n+r-1, 3 \rangle}{\epsilon} & \frac{\langle n+r, 2 \rangle}{\epsilon^2 \sqrt{2}} \\ \frac{\langle n+r, n-2 \rangle}{\epsilon^n \sqrt{2}} & \frac{\langle n+r-1, n-1 \rangle}{\epsilon^{n+1}} & \frac{\langle n-1, n-2 \rangle}{\epsilon^{n+2}} & \dots & \frac{\langle 2,3 \rangle}{\epsilon^{2n-3}} & \langle n+r-1, 2 \rangle & \frac{\langle n+r, 1 \rangle}{\epsilon \sqrt{2}} \\ \frac{\langle n-1, n-1 \rangle}{\epsilon^{n-1} 2} & \frac{\langle n-2, n-2 \rangle}{\epsilon^n \sqrt{2}} & \frac{\langle n-3, n-3 \rangle}{\epsilon^{n+1} \sqrt{2}} & \dots & \frac{\langle 2,2 \rangle}{\epsilon^{2n-4} \sqrt{2}} & \frac{\langle 1,1 \rangle}{\epsilon^{2n-3} \sqrt{2}} & 0 \end{array} \right)$$

$$\begin{pmatrix}
\frac{\langle n+r, n-1 \rangle}{\epsilon^{n-1} 2} & \frac{\langle n+r, n-2 \rangle}{\epsilon^n \sqrt{2}} & \frac{\langle n+r, n-3 \rangle}{\epsilon^{n+1} \sqrt{2}} & \dots & \frac{\langle n+r, 2 \rangle}{\epsilon^{2n-4} \sqrt{2}} & \frac{\langle n+r, 1 \rangle}{\epsilon^{2n-3} \sqrt{2}} & 0 \\
\frac{\langle n+r, n-2 \rangle}{\epsilon^{n-2} \sqrt{2}} & \frac{\langle n+r, n-3 \rangle}{\epsilon^{n-1}} & \frac{\langle n+r, n-4 \rangle}{\epsilon^n} & \dots & \frac{\langle n+r, 1 \rangle}{\epsilon^{2n-5}} & 0 & \\
\frac{\langle n+r, n-3 \rangle}{\epsilon^{n-3} \sqrt{2}} & \frac{\langle n+r, n-4 \rangle}{\epsilon^{n-2}} & \frac{\langle n+r, n-5 \rangle}{\epsilon^{n-1}} & \dots & 0 & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \\
\frac{\langle n+r, 2 \rangle}{\epsilon^2 \sqrt{2}} & \frac{\langle n+1, 1 \rangle}{\epsilon^3} & 0 & & & & \\
\frac{\langle n+r, 1 \rangle}{\epsilon \sqrt{2}} & 0 & & & & & \\
0 & & & & & &
\end{pmatrix}$$

$$\begin{pmatrix}
\frac{\langle n-1, n-1 \rangle}{\epsilon^{n-1} 2} & \frac{\langle n-2, n-2 \rangle}{\epsilon^n \sqrt{2}} & \frac{\langle n-3, n-3 \rangle}{\epsilon^{n+1} \sqrt{2}} & \dots & \frac{\langle 2, 2 \rangle}{\epsilon^{2n-4} \sqrt{2}} & \frac{\langle 1, 1 \rangle}{\epsilon^{2n-3} \sqrt{2}} & 0 \\
\frac{\langle n-2, n-2 \rangle}{\epsilon^{n-2} \sqrt{2}} & \frac{\langle n-3, n-3 \rangle}{\epsilon^{n-1}} & \frac{\langle n-4, n-4 \rangle}{\epsilon^n} & \dots & \frac{\langle 1, 1 \rangle}{\epsilon^{2n-5}} & 0 & \\
\frac{\langle n-3, n-3 \rangle}{\epsilon^{n-3} \sqrt{2}} & \frac{\langle n-4, n-4 \rangle}{\epsilon^{n-2}} & \frac{\langle n-5, n-5 \rangle}{\epsilon^{n-1}} & \dots & 0 & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \\
\frac{\langle 2, 2 \rangle}{\epsilon^2 \sqrt{2}} & \frac{\langle 1, 1 \rangle}{\epsilon^3} & 0 & & & & \\
\frac{\langle 1, 1 \rangle}{\epsilon \sqrt{2}} & 0 & & & & & \\
0 & & & & & &
\end{pmatrix}$$

For the skew symmetric with respect to the second diagonal matrices A_{n-1}, A_n we have, analogously for $v = v_1, v_2$ the following upper part:

$$\left(\begin{array}{cccccccc}
 1 & \frac{-1}{\epsilon\sqrt{2}} & \dots & \frac{(-1)^{n-2}}{\epsilon^{n-2}\sqrt{2}} & \frac{\nu(-1)^n}{\epsilon^{n-1}2} & \frac{-\nu(-1)^n}{\epsilon^{n-1}2} & \frac{-(-1)^n}{\epsilon^n\sqrt{2}} & \dots & \frac{-(-1)^n}{\epsilon^{2n-3}\sqrt{2}} \\
 \frac{(-1)^n}{\epsilon^{2n-3}\sqrt{2}} & 0 & \dots & 0 & \frac{\nu}{\epsilon^{n-2}\sqrt{2}} & \frac{-\nu}{\epsilon^{n-2}\sqrt{2}} & 0 & \dots & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \frac{(-1)^n}{\epsilon^n\sqrt{2}} & 0 & \dots & 0 & \frac{\nu}{\epsilon^2\sqrt{2}} & \frac{-\nu}{\epsilon^2\sqrt{2}} & 0 & \dots & 0 \\
 \frac{-\nu+(-1)^n}{\epsilon^{n-1}2} & \frac{\nu}{\epsilon^n\sqrt{2}} & \dots & \frac{-(-1)^n\nu}{\epsilon^{2n-3}\sqrt{2}} & \nu & 0 & \dots & \dots & \dots \\
 \frac{\nu+(-1)^n}{\epsilon^{n-1}2} & \frac{-\nu}{\epsilon^n\sqrt{2}} & \dots & \frac{(-1)^n\nu}{\epsilon^{2n-3}\sqrt{2}} & 0 & \dots & \dots & \dots & \dots \\
 \frac{-(-1)^n}{\epsilon^{n-2}\sqrt{2}} & 0 & \dots & 0 & \dots & \dots & \dots & \dots & \dots \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \frac{1}{\epsilon\sqrt{2}} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
 \end{array} \right)$$

Now using the formulas (13) or (14) it is a simple calculation to obtain all Lie brackets of A_r with the powers of the element E as well as with E_0 . Note that only the odd powers of E are elements of \mathfrak{S} . So the following proposition is obtained.

Proposition 3. If $j \in \{1, 3, \dots, n-1\}$, then we have

$$(\text{ad} E^j)A_r = 2 \epsilon^j \cos \frac{j\theta_r}{2} A_r \text{ for } r=1, \dots, n-2$$

and

$$(\text{ad} E^j)A_r = \epsilon^j A_r \text{ for } r=n-1, n.$$

Moreover

$$(\text{ad} E_0)A_r = 0 \text{ for } r=1, \dots, n-2, \text{ and}$$

$$(\text{ad} E_0)A_r = \frac{-2\nu}{\epsilon^{n-1}} A_r \text{ for } r=n-1, n \text{ where } \nu = \nu_1, \nu_2.$$

3. The results. Using the above notation we can compute the constants λ_{rs} from the relation $[T_s, A_r] = \lambda_{rs} A_r$. From Proposition 3 we obtain that all constants λ_{rs} , where $r, s=1, \dots, n$, are the following:

3.1. Let $r=1, \dots, n-2$ then

$$\lambda_{rs} = \frac{2}{\sqrt{h}} \epsilon_r^{2s-1} \cos \frac{(2s-1)\theta_r}{2} \text{ for } \begin{cases} s=1, \dots, k-1 & \text{if } n=2k \text{ and} \\ s=1, \dots, k & \text{if } n=2k+1. \end{cases}$$

$$\lambda_{rs} = \frac{2}{\sqrt{h}} \epsilon_r^{2s-3} \cos \frac{(2s-3)\theta_r}{2} \text{ for } s=k+2, \dots, n \text{ if } n=2k \text{ or } n=2k+1.$$

Moreover

$$\lambda_{rk} = \lambda_{r,k+1} = \frac{1}{\sqrt{2h}} \epsilon_r^{n-1} \cos \frac{(n-1)\theta_r}{2} \text{ if } n=2k \text{ and}$$

$$\lambda_{r,k+1} = 0 \text{ if } n=2k+1.$$

3.2. Let $r=n-1, n$ then for $\nu=\nu_1, \nu_2$ respectively we have

$$\lambda_{rs} = \frac{\epsilon_r^{2s-1}}{\sqrt{h}} \text{ for } s=1, \dots, k-1 \text{ if } n=2k, \text{ and } s=1, \dots, k \text{ if } n=2k+1.$$

$$\lambda_{rs} = \frac{\epsilon_r^{2s-3}}{\sqrt{h}} \text{ for } s=k+2, \dots, n \text{ if } n=2k \text{ or } n=2k+1.$$

Moreover

$$\lambda_{rk} = \left(\frac{1}{\sqrt{2h}} - \frac{\nu i}{2} \right) \epsilon^{n-1}, \quad \lambda_{r,k+1} = \left(\frac{1}{\sqrt{2h}} - \frac{\nu i}{2} \right) \epsilon^{n-1} \text{ if } n=2k$$

where $i^2 = -1$,

and

$$\lambda_{r,k+1} = \frac{\nu \epsilon^{n-1}}{\sqrt{2}} \text{ if } n=2k+1.$$

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