Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 1, 157--167

Persistent URL: http://dml.cz/dmlcz/106606

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29,1 (1988)

ON CARATHÉODORY'S AND KREIN-MILMAN'S THEOREMS IN FULLY ORDERED GROUPS

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Abstract: In a fully ordered group $(F', \leq, \circ)$ we introduce an algebraic structure by inducing a further binary operation by the order and extending F' about a zero-element $\bar{0}$. Provided with this algebraic structure, we prove in F^n, the n-fold cartesian product of $F := F' \cup \{\bar{0}\}$, the theorems of Carathéodory and Krein-Milman. Here, Carathéodory's theorem is proved not by a reduction step - as be usually done in linear spaces - , but by solving a certain system of equalities which is linear with respect to the operations $\oplus$ and $\circ$. To prove Krein-Milman's theorem, we state some results of separation theory in such algebraic structures.

Key words: Ordered algebraic structure, convexity concept, Theorem of Carathéodory, Theorem of Krein-Milman.

Classification: 06F99, 52A01, 46P05

I. Introduction. Let $(F', \leq, \circ)$ be a fully ordered group with neutral element $\bar{1}$ and let $\bar{0}$ be an element not belonging to F'. Extend $\leq$ and $\circ$ on $F := F' \cup \{\bar{0}\}$ by

$$\bar{0} \circ x = x \circ \bar{0} = \bar{0} \text{ and } \bar{0} \leq x \quad \text{for each } x \in F,$$

and introduce a further binary operation $\oplus$ induced through the fully-order by

$$x \oplus y = y \Longleftrightarrow x \leq y \quad \text{for each } x, y \in F.$$

In this way, F is provided with an algebraic structure. To emphasize this, we denote in the sequel $(F, \leq, \circ)$ by $(F, \oplus, \circ)$. An easy consideration shows (see Helbig [2], Lemma II.1) that $(F, \oplus, \circ)$ is an extremal algebra, a conception introduced by Nedoma [5] and investigated in detail by Zimmermann [6] and Helbig [3]. For that reason we call $(F, \oplus, \circ)$ <u>extremal algebra</u>. If the group F' is complete, i.e. that every non-empty subset of F' which is bounded from above, has a least upper bound, we call $(F, \oplus, \circ)$ a <u>complete extremal algebra</u>. Notice that a subset of F is always bounded from below by $\bar{0}$.

Examples for complete extremal algebras $(F,\oplus,\bullet)$ are $(\mathbf{R}\cup\{-\infty\},\max,+)$, $(\mathbf{R}\cup\{\infty\},\min,+)$, $(\mathbf{R}_0^+,\max,\cdot)$, and $(\mathbf{R}^+\cup\{\infty\},\min,\cdot)$, where $\mathbf{R}$ is the set of the real numbers, $\mathbf{R}_0^+$ is the set of the non-negative and $\mathbf{R}^+$ is the set of the positive real numbers. Exchanging $\mathbf{R}$ by the rational numbers, we obtain examples for extremal algebras.

It is easy to see that (for proofs see Nedoma [5] or Zimmermann [6])

(1) $x \leq y \Rightarrow x \oplus z \leq y \oplus z$ for all $z \in F$;

(2) $x \leq y \Rightarrow x \bullet z \leq y \bullet z$ for all $z \in F$;

(3) $x < y \Rightarrow x \bullet z < y \bullet z$ for all $z \in F$, $z \neq \bar{0}$.

On F^n, the n-fold cartesian product of F, we define a partial order by $x \leq y \Leftrightarrow y_i \leq y_i$ for $i=1,\ldots,n$, where $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$ in F^n, and extend the operations $\oplus$ and $\bullet$ on F^n by defining $x \oplus y$ and $\alpha \bullet x$ componentwise, where x, y in F^n and $\alpha \in F$. Furthermore, we define an <u>extremal inner product</u> on F^n by

$$(x,y) := x_1 \bullet y_1 \oplus \ldots \oplus x_n \bullet y_n := \sum_{i=1}^{n\,\oplus} x_i \bullet y_i \quad \text{for } x, y \text{ in } F^n.$$

By definition of the operation $\oplus$, we have for x, y in F^n

$(x,y) \geq x_i \bullet y_i$ for $i=1,\ldots,n$ and $(x,y)=x_j \bullet y_j$

for at least one $j \in \{1,\ldots,n\}$. For the following, we need some definitions.

Definition I.1: Let A be a subset of F^n.

(a) The set A is called <u>extremally convex</u> (for short <u>e-convex</u>), if $x,y \in A$ and α, β in F with $\alpha \oplus \beta = \bar{1}$ imply $\alpha \bullet x \oplus \beta \bullet y \in A$.

(b) The set

$$\text{eco } A := \{\sum_{i\in I}^{\oplus} \alpha_i \bullet a^i \mid \alpha_i \in F, a^i \in F \text{ for } i \in I,\ I \subset \mathbf{N},\ \text{card } I < \infty,\ \sum_{i\in I}^{\oplus} \alpha_j = \bar{1}\}$$

is called the <u>e-convex hull of</u> A.

(c) The set

$$\text{econ } A := \{\sum_{i\in I}^{\oplus} \alpha_i \bullet a^i \mid \alpha_i \in F, a^i \in F \text{ for } i \in I,\ I \subset \mathbf{N},\ \text{card } I < \infty\}$$

is called the <u>e-convex cone of</u> A.

Let x, y be in F^n. The <u>closed segment</u> between x and y is defined by

$$[x,y] := \{\alpha \bullet x \oplus \beta \bullet y \mid \alpha,\beta \in F \text{ with } \alpha \oplus \beta = \bar{1}\},$$

while the <u>open segment</u> between x and y is

$$]x,y[\;:=\; \begin{cases} [x,y]/\{x,y\} & \text{if } x \neq y \\ \{x\} & \text{if } x=y. \end{cases}$$

Definition I.2: Let K be a subset of F^n.

(a) A subset E of K is called extremal subset in K (or extremal in K), if x, y in K, and $]x,y[\cap E\neq\emptyset$ imply $x,y\in E$.

(b) A subset E of K is called weak-extremal subset of K (or w-extremal in K), if $x,y\in K$, and $[x,y]\cap E\neq\emptyset$ imply x or y in E.

(c) An element $x\in K$ is called extreme point of K, if $\{x\}$ is extremal in K.

(d) An element $x\in K$ is called efficient point of K, if $y\leq x$, $y\in K$ implies $x=y$.

We endow F with the so-called open-interval-topology. A basis of neighbourhoods of an element $y\in F$ is given by the e-convex sets

$U_{ab}:=\{x\in F|a<x<b\}$ if $y\neq\bar{0}$ and $U_{\bar{0}b}:=\{x\in F|\bar{0}\leq x<b\}$ if $y=\bar{0}$,

where $a,b\in F$. If F' is complete, then (by a theorem of Hölder (see for instance Kokorin and Kopytov [4], p. 110), $(F',\circ)$ with its order is order-isomorphic to the additive group of the real numbers with the natural order. Thus, a complete extremal algebra $(F,\oplus,\circ)$ is homeomorphic to $(\mathbb{R}\cup\{-\infty\},+)$ with the natural order (for a proof see Helbig [2], Lemma III.2). On account of this conclusion, we characterize the compact sets of F^n, where F^n is endowed with the product topology, and $(F,\oplus,\circ)$ is a complete extremal algebra, as the closed and bounded sets in F^n, and that the extremal inner product is a continuous function.

The aim of this paper is to prove two theorems which are well-known in linear spaces, namely the theorems of Carathéodory and Krein-Milman. Here, we prove Carathéodory's theorem in F^n not by a reduction step - as be usually done in linear spaces -, but by solving a certain system of equalities which is linear with respect to the operations $\oplus$ and $\circ$. From this, we deduce that the number of elements in a set which are needed to describe an element of the e-convex cone of this set, is less or equal to n and that the e-convex hull of a compact set is compact.

Furthermore, we prove that a non-empty compact subset of F^n has extreme points, and is the closed, e-convex hull of its extreme points, if it is additionally e-convex. For this, we need a separation theorem in a complete extremal algebra , which we obtain as a conclusion of a theorem of Zimmermann [6].

II. The theorem of Carathéodory. Let $(F,\oplus,\circ)$ be an extremal algebra, whose group operation $\circ$ is not necessarily commutative.

Theorem II.1: Let A be a subset of F^n. If $b \in \mathrm{eco}\, A$, then there exist $k \leq n+1$ elements $a^j \in A$, $j=1,\dots,k$, such that $b \in \mathrm{eco}\,(\{a^1,\dots,a^k\})$.

Proof: Since $b \in \mathrm{eco}\, A$, there exist m elements $a^j \in A$, $j=1,\dots,m$, such that

$$b = \sum_{j=1}^{m}{}^{\oplus} \alpha_j \circ a^j,$$

where $\alpha_j \in F$ for $j=1,\dots,m$ with $\sum_{j=1}^{m}{}^{\oplus} \alpha_j = \bar{1}$. By definition of the operation $\oplus$, for all indices $i \in \{1,\dots,n\}$ there exists an index $j \in \{1,\dots,m\}$ such that

(2.1) $\quad b_i \geq \alpha_l \circ a_i^l$ for all $l \in \{1,\dots,m\}$

and

(2.2) $\quad b_i = \alpha_j \circ a_i^j$.

Define a function $f: \{1,\dots,n\} \to \{1,\dots,m\}$ by

$$f(i) := \min_{\substack{j \in \{1,\dots,m\} \\ b_i = \alpha_j \circ a_i^j}} j$$

and a set N by

$$N := \{j \in \{1,\dots,m\} \mid \exists\, i \in \{1,\dots,n\}: f(i)=j\}.$$

Since $b \in F^n$ we have card $N \leq n$. By (2.1) and (2.2), it follows

$$b = \sum_{j \in N}{}^{\oplus} \alpha_j \circ a^j,$$

If $\alpha_j = \bar{1}$ for some $j \in N$, $b \in \mathrm{eco}(\{a^j \mid j \in N\})$ and $k = \mathrm{card}\, N < n+1$. Thus the proof is finished in this case. Otherwise, there exists $l \in \{1,\dots,m\}$, $l \notin N$, with $\alpha_l = \bar{1}$. With (2.1) we obtain

$$b = \sum_{j \in N}{}^{\oplus} \alpha_j \circ a^j \oplus \alpha_l \circ a^l.$$

Thus, $b \in \mathrm{eco}\,(\{a^j \mid j \in N\} \cup \{a^l\})$ and $k = \mathrm{card}\, N + 1 \leq n+1$. $\square$

In the same manner, we deduce

Theorem II.2: Let A be a subset of F^n. If $b \in \mathrm{econ}\, A$, then there exist $k \leq n$ elements $a^j \in A$, $j=1,\dots,k$, such that $b \in \mathrm{econ}\,(\{a^1,\dots,a^k\})$. $\square$

Corollary II.3: Let $(F, \oplus, \circ)$ be a complete extremal algebra and let A be a non-empty compact subset of F^n. Then eco A is a compact set.

Proof: By Theorem II.1, it follows

$$\mathrm{eco}\, A = \{\sum_{j=1}^{n+1}{}^{\oplus} \alpha_j \circ a^j \in F^n \mid \alpha_j \in F, a^j \in F^n \text{ for } j=1,\dots,n+1, \sum_{j=1}^{n+1}{}^{\oplus} \alpha_j = \bar{1}\}.$$

Let $T := \{\alpha \in F^{n+1} \mid \sum_{j=1}^{n+1}{}^{\oplus} \alpha_j = 1\}$. By the considerations of Section I, T is compact. Hence $T\times A^{n+1}$ is compact. The mapping $f:T\times A^{n+1}\longrightarrow \text{eco } A$, which is defined by

$$(\alpha,a^1\dots,a^{n+1})\mapsto \sum_{j=1}^{n+1}{}^{\oplus}\ \alpha_j\circ a^j,$$

is continuous. Thus, the set $f(T\times A^{n+1})=\text{eco } A$ is compact. □

III. A separation theorem. Although the separation theory in extremal algebras is of own interest, we only make available one separation theorem which we will need to show the Krein-Milman-Theorem. Let $(F,\oplus,\circ)$ be an extremal algebra. For $h\in F^n$ and $N\subset\{1,\dots,n\}$, $N\neq\emptyset$, define

$$H(h,N):=\{x\in F^n \mid \sum_{i\in N}^{\oplus} h_i\circ x_i=\sum_{i\in CN}^{\oplus} h_i\circ x_i\oplus \bar{1}\},$$

$$H^+(h,N):=\{x\in F^n \mid \sum_{i\in N}^{\oplus} h_i\circ x_i\geq\sum_{i\in CN}^{\oplus} h_i\circ x_i\oplus \bar{1}\}, \text{ and}$$

$$H^-(h,N):=\{x\in F^n \mid \sum_{i\in N}^{\oplus} h_i\circ x_i\leq\sum_{i\in CN}^{\oplus} h_i\circ x_i\oplus \bar{1}\},$$

where $CN:=\{1,\dots,n\}\setminus N$. Obviously, we have

(1) $H(h,N)=H^+(h,N)\cap H^-(h,N)$;

(2) $H^+(h,N)\cup H^-(h,N)=F^n$;

(3) $H^+(h,N)$ and $H^-(h,N)$, and hence $H(h,N)$ are e-convex.

If $(F,\oplus,\circ)$ is a complete extremal algebra,

(4) $H^+(h,N)$ and $H^-(h,N)$, and hence $H(h,N)$ are closed,

since the extremal inner product is a continuous function. The sets $H^+(h,N)$ and $H^-(h,N)$ are called the <u>halfspaces belonging to</u> $H(h,N)$.

Lemma III.1: Let $(F,\oplus,\circ)$ be an extremal algebra with the following properties:

(1) Let x, y, z be in F such that $x<y\leq z$. Then there exists $\alpha\in F$ with $\bar{0}<\alpha<\bar{1}$ such that $x<\alpha\circ z<y$.

(2) There exists a metric $d:F\times F\longrightarrow \mathbf{R}$ with

(a) Let x, y, z be in F such that $x<y<z$. Then $d(y,z)<d(x,z)$ and $d(x,y)<d(x,z)$.

(b) Let x be in F and $\beta\in\mathbf{R}$. Then the set $\{y\in F\mid y>x \text{ and } d(x,y)<\beta\}$ is non-empty.

(c) Let x be in F such that $x\neq\bar{0}$ and $\beta\in\mathbf{R}$. Then the set $\{y\in F\mid y<x \text{ and } d(x\ y)<\beta\}$ is non-empty.

Suppose a closed e-convex subset A of F and $p\in F^n\setminus A$. Then there exist $h\in F^n$

with $h_i \neq \bar{0}$ for $i=1,\ldots,n$, and a non-empty subset N of $\{1,\ldots,n\}$ such that

$$A \subset H^+(h,N) \setminus H(h,N) \text{ and } p \in H^-(h,N) \setminus H(h,N),$$

or conversely.

Proof: See Zimmermann [6], Theorem 4. □

Theorem III.2: Let $(F, \oplus, \circ)$ be a complete extremal algebra. Furthermore, let A be a closed, e-convex subset of F^n and let p be in $F^n \setminus A$. Then there exist $h \in F^n$ with $h_i \neq \bar{0}$ for $i=1,\ldots,n$, and a non-empty subset N of $\{1,\ldots,n\}$ such that the set $H(h,N)$ separates A and p strictly, i.e.

$$A \subset H^+(h,N) \setminus H(h,N) \text{ and } p \in H^-(h,N) \setminus H(h,N),$$

or conversely.

Proof: We show that a complete extremal algebra $(F, \oplus, \circ)$ satisfies the properties of Lemma III.1. Let φ be the homeomorphism between $(F, \circ)$ and $(\mathbb{R}_0^+, \cdot)$, which exists by the result mentioned in Section I. Furthermore, let x, y, z be in F such that $x < y \not\leq z$. Since φ preserves the order,

$$\varphi(x) < \varphi(y) \not\leq \varphi(z).$$

Then there exists $s \in \mathbb{R}_0^+$ with $0 < s < 1$ such that $\varphi(x) < s\varphi(z) < \varphi(y)$. This implies

$$x < \varphi^{-1}(s) \circ z < y$$

with $\bar{0} < \varphi^{-1} < \bar{1}$. Hence, the property (1) of the above lemma is fulfilled. Define a metric d on F by

$$d(x,y) := |\varphi(x) - \varphi(y)| \quad \text{for } x,y \in F.$$

Obviously, the properties (2)(a) - (c) are fulfilled. Then the assertion follows by Lemma III.1. □

For a more detailed discussion of the sets $H(h,N)$, $H^+(h,N)$, and $H^-(h,N)$ see Zimmermann [6], and Helbig [3], Chapter I.4.

IV. The theorem of Krein-Milman. Let $(F, \oplus, \circ)$ be a complete extremal algebra and let K be a subset of F^n. Denote the set of all extreme points of K by ext K, the set of all efficient points of K by eff K, and the closed, e-convex hull of K by $\overline{\text{eco}}\, K$. Of course, an extremal subset of K is w-extremal in K. In general, the converse is not true. Nevertheless,

Lemma IV.1: Let K be a subset of F^n and $v \in K$.

(a) The set $\{v\}$ is w-extremal in K iff v is an extreme point.

(b) If v is efficient in K, then v is an extreme point of K.

Proof: (a) The "if"-part follows immediately. To show the "only if"-part let x, y be in K with $v \in]x,y[$. It follows $v \in]x,y[\subset [x,y]$. Since $\{v\}$ is w-extremal in K, we have v=x or v=y. W.l.o.g. let v=x. If $x \neq y$, then $x=v \notin]x,y[$, which is a contradiction to the assumption. For this, v=x=y. Thus, the assertion is proved.

(b) Let x, y be in K such that $v \in [x,y]$, i.e. $v= \alpha \circ x \oplus \beta \circ y$ for suitable $\alpha, \beta \in F$ with $\alpha \oplus \beta = \bar{1}$. W. l.o.g. let $\alpha = \bar{1}$. By definition of the operation $\oplus$, the equalities $v_i=x_i \oplus \beta \circ y_i$ for i=1,...,n imply $x_i \leq v_i$ for i=1,... ...,n, i.e. $x \leq v$. Since v is efficient in K, we have v=x. Thus, the set $\{v\}$ is w-extremal in K. Part (a) finishes the proof. □

The next lemma shows that sets which are described by extremally linear functionals, this means linear with respect to the operations $\oplus$ and $\circ$,such as halfspaces, are w-extremal sets.

Lemma IV.2: (a) Let K be a non-empty, compact subset of F^n and let p be in F^n.

Then the set $G:=\{x \in K \mid \max_{y \in K} (p,y)=(p,x)\}$ is a non-empty, compact, w-extremal subset of K.

(b) Let p, q be in F^n and $c \in F$. Then the sets $A^{\geq} := \{x \in F^n \mid (p,x) \geq (q,x) \oplus c\}$ and $A^{\leq} := \{x \in F^n \mid (p,x) \leq (q,x) \oplus c\}$ are w-extremal in F^n.

Proof: (a) Since K is compact and the extremal inner product is continuous, the set G is non-empty, closed, and, as a subset of K, compact. To show the third property of G let x, y be in K and $\alpha, \beta \in F$ with $\alpha \oplus \beta = \bar{1}$ such that $v:= \alpha \circ x \oplus \beta \circ y$ is an element of G, i.e. $v \in [x,y] \cap G$. If x and y both are not in G, then $(p,x) \oplus (p,y) < (p,v)$. This implies

$$(p,v)=(p, \alpha \circ x \oplus \beta \circ y)= \alpha \circ (p,x) \oplus \beta \circ (p,y) < (p,v).$$

Because of this contradiction $x \in G$ or $y \in G$, i.e. G is w-extremal in K.

(b) The proof is similar to the proof of (a). □

With these preliminaries we state the main theorems.

Theorem IV.3: Let K be a non-empty, compact subset of F^n. Then there exists an efficient point of K, and therefore an extreme point of K, i.e. $\emptyset \subset \text{eff } K \subset \text{ext } K$.

Proof: Set $K^1 := K$. Then there exists $v^1 \in K^1$ such that $v_1^1 := \min_{x \in K^1} x_1$. Define recursively for i=2,...,n

$K^i := \{x \in K^{i-1} | x_j = v_j^j \text{ for } j < i\}$ and $v^i \in K^i$ such that $v_i^i = \min_{x \in K^i} x_i$.

Since $v^{i-1} \in K^i$ for i=2,...,n, the sets K^i, i=2,...,n, are non-empty. Because of the compactness of K^i, i=1,...,n, the elements v^i, i=1,...,n, exist. We claim that $v := v^n$ is an efficient point of K. To show this, let x be in F^n with $x \leq v$. If $x \neq v$, then there exists $k \in \{1,\dots,n\}$ such that $x_k < v_k$. W.l.o.g. let k be the least index with this property. Because $x_j = v_j = v_j^j$ for $j < k$ if $k > 1$ and $x \in K^1 = K$ if k=1, we have $x \in K^k$. Then the inequality $x_k < v_k = v_k^k$ is a contradiction to the choice of v^k. Thus, $v \in \text{eff } K \subset \text{ext } K$. □

As a corollary from this theorem we deduce a theorem of Butkovic [1], Theorem 1.

Corollary IV.4: A non-empty, closed subset K of F^n has extreme points.

Proof: For arbitrary $w \in K$ define $C := \{x \in K | x \leq w\}$. This set is closed and bounded, and hence compact. With Theorem IV.3, we have eff $C \neq \emptyset$. We claim that v in eff C is an efficient point of K. For this, let y be in K such that $y \leq v$. Since $v \leq w$, we obtain $y \leq w$, i.e. $y \in C$. The efficiency of v implies y=v. Thus, eff K is non-empty, and therefore ext $K \neq \emptyset$ by Lemma IV.1(b).

It follows the theorem of Krein-Milman in complete extremal algebras.

Theorem IV.5: Let K be a non-empty, e-convex, and compact subset of F^n. Then the set K is the closed, e-convex hull of its extreme points, i.e. $K = \overline{\text{eco}}$ ext K.

Proof: Set $B := \overline{\text{eco}}$ ext K. Since the inclusion $B \subset K$ follows immediately, it suffices to show $K \subset B$. Assume that there exists $z \in K$ with $z \notin B$. By Theorem III.2, there exist h in F^n and a non-empty set $N \subset \{1,\dots,n\}$ such that the set $H := H(h,N)$ separates the point z and the closed, e-convex set B strictly. Let $H^+ := H^+(h,N)$ and $H^- := H^-(h,N)$ be the halfspaces belonging to H. To simplify matters define

$$p_i := \begin{cases} h_i & \text{if } i \in N \\ \bar{0} & \text{if } i \in \{1,\dots,n\} \setminus N \end{cases}$$

and

$$q_i := \begin{cases} \bar{0} & \text{if } i \in N \\ h_i & \text{if } i \in \{1,\dots,n\} \setminus N, \end{cases}$$

Then

$$H^{+}=\{x\in F^{n} \mid (p,x)\geq (q,x)\oplus \bar{1} \quad \text{and} \quad H^{-}=\{x\in F^{n} \mid (p,x)\leq (q,x)\oplus \bar{1}\}.$$

where $(\cdot,\cdot)$ denotes the extremal inner product. Distinguish two cases:

Case 1: $z\in H^{+}\setminus H$ and $B\subset H^{-}\setminus H$

Let $G:=\{x\in H^{+}\cap K \mid \max_{y\in H^{+}\cap K}(p,y)=(p,x)\}$. Since $z\in H^{+}\cap K$, the set $H^{+}\cap K$ is non-empty. It is compact as a closed subset of K. Hence, by Lemma IV.2(a), the set G is non-empty, compact, and w-extremal in $H^{+}\cap K$. Applying Theorem IV.3, there exist an efficient point v in G. We claim that $v\in \text{ext}\, K$, or equivalently, by Lemma IV.1(a), $\{v\}$ is w-extremal in K. To show this, let x, y be in K such that $v\in[x,y]$. Then there exist $\alpha,\beta\in F$ with $\alpha\oplus\beta=\bar{1}$ such that $v=\alpha\circ x\oplus\beta\circ y$. Since the set H^{+} is w-extremal in F^{n} by Lemma IV.2(b), $x\in H^{+}\cap K$ or $y\in H^{+}\cap K$.

Subcase 1a: $x\in H^{+}\cap K$ and $y\in H^{+}\cap K$

Since G is w-extremal in $H^{+}\cap K$, w.l.o.g. the point x is in G. If $\alpha=\bar{1}$, then $x\leq v$. By efficiency of v, this implies x=v. Thus, $\{v\}$ is w-extremal in K, and, hence, v is an extreme point of K. If $\alpha<\bar{1}$, then $\beta=\bar{1}$ and $y\leq v$. Because of $x\in H^{+}\cap K$, it follows $(p,x)\geq\bar{1}>\bar{0}$. Therefore, $(p,v)=(p,x)>\alpha\circ(p,x)$. This implies

$$(p,v)=(p,\alpha\circ x\oplus\beta\circ y)=\beta\circ(p,y)=(p,y),$$

i.e. $y\in G$. By efficiency of v, the inequality $y\leq v$ leads to y=v. Thus, $\{v\}$ is w-extremal in K. Hence, v is an extreme point of K.

Subcase 1b: $x\in H^{+}\cap K$ and $y\in (H^{-}\setminus H)\cap K$

Then

$$(6.1) \qquad (p,y)<(q,y)\oplus\bar{1}.$$

Assume that $x\notin G$ or $\alpha<\bar{1}$. Then we obtain because of $(p,v)\geq\bar{1}>\bar{0}$ if $\alpha<\bar{1}$ and directly if $x\notin G$ that $\alpha\circ(p,x)<(p,v)$. This implies $(p,v)=(p,\alpha\circ x\oplus\beta\circ y)=\beta\circ(p,y)$. Since $(p,v)>\bar{0}$, we have $\beta\neq\bar{0}$. With (6.1) the following inequality holds

$$(6.2) \quad (q,v)\oplus\bar{1}\leq(p,v)=\beta\circ(p,y)<\beta\circ(q,y)\oplus\beta\leq$$
$$\leq\beta\circ(q,y)\oplus\bar{1}\leq\alpha\circ(q,x)\oplus\beta\circ(q,y)\oplus\bar{1}=(q,v)\oplus\bar{1}.$$

This is a contradiction. Thus, $x\in G$ and $\alpha=\bar{1}$. Now, we deduce like in subcase 1a that $v\in \text{ext}\, K$.

As well in subcase 1a as in subcase 1b, the element $v\in H^{+}\cap K$ is an extreme point of K. Since $B\subset H^{-}\setminus H$, we have $v\notin B$, which is a contradiction to the definition of B.

Case 2: $z \in H^- \setminus H$ and $B \subset H^+ \setminus H$

Subcase 2a: $(p,z) \leq \bar{1}$

Define a compact set A by $A := \{x \in K | x \leq z\}$. Since $B \subset H^+ \setminus H$, we have

$$(p,x) > (q,x) \oplus \bar{1} \quad \text{for } x \in B.$$

Then there exists $i \in \{1,\ldots,n\}$ with $p_i \bullet x_i > \bar{1} \geq p_i \bullet z_i$. This implies $x_i > z_i$. Thus, $x \notin A$. Since x was arbitrary, $A \cap B = \emptyset$.

By Theorem IV.3, the set A has an efficient point v. To show that $v \in \text{ext } K$, let x, y be in K such that $v \in [x,y]$, i.e. $v = \alpha \bullet x \oplus \beta \bullet y$ for suitable $\alpha, \beta \in F$ with $\alpha \oplus \beta = \bar{1}$. W.l.o.g. let $\alpha = \bar{1}$. Then $x \leq v$ and, by efficiency of v, v=x. This implies that $\{v\}$ is w-extremal in K, and hence v is an extreme point of K by Lemma IV.1(a). Notice that $v \notin B$ as $v \in A$.

Subcase 2b: $(p,z) > \bar{1}$

In this case, the set $\hat{H} := \{x \in F^n | (p,x) = (q,x)\}$ separates the set B and the point z strictly, since $z \in H^- \setminus H$ and $(p,z) > \bar{1}$ implies $\bar{1} < (p,z) < (q,z) \oplus \bar{1}$. Thus,

$$\bar{1} < (p,z) < (q,z) \oplus \bar{1} = (q,z)$$

and

$$(p,x) > (q,x) \oplus \bar{1} \geq (q,x) \quad \text{for all } x \in B.$$

Let $\hat{H}^+ := \{x \in F^n | (p,x) \geq (q,x)\}$ and let $\hat{H}^-$ be analogously defined. Moreover, let

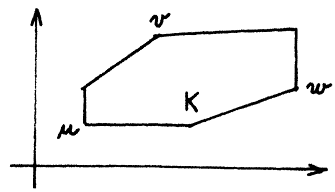
$$G := \{x \in \hat{H}^- \cap K | \max_{y \in \hat{H}^- \cap K} (q,y) = (q,x)\}.$$

The set $\hat{H}^- \cap K$ is compact and, since $z \in \hat{H}^- \cap K$, non-empty. By Lemma IV.2(a) and Theorem IV,3, the set G is non-empty, compact, and w-extremal in $\hat{H}^- \cap K$, and has an efficient point v. Exchanging the roles of p and q and cancelling the term "$\oplus \bar{1}$" in (6.1) and (6.2), we obtain like in subcases 1a and 1b that $v \in \text{ext } K$, but $v \notin B$ since $v \in \hat{H}^-$.

In both subcases 2a and 2b, the element v is an extreme point of K, but $v \notin B$. This is a contradiction to the definition of B.

Combining the results of cases 1 and 2, we have $z \in B$. Therefore, $K \subset B$. This completes the proof. □

Example IV.6: Consider the following compact set K in F^2, whereby $(F, \oplus, \bullet) = (\mathbb{R}_n^+, \max, \cdot)$:



We have eff K={u}, ext K={u,v,w}, and K=eco ({u,v,w}). □

Acknowledgement. The author wishes to thank Prof. Dr. B. Brosowski, who gives the idea of a shorter version of the proof of Carathéodory's Theorem than in [3], and the referee for some helpful remarks.

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(Oblatum 2.11. 1987)