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Comparison of subset systems

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Abstract: A subset system, as introduced by [ADJ], is a means for expressing a type of (join-)completeness of posets and (join-)continuity of order-preserving maps. We compare subset systems, and we prove, essentially, that the corresponding types of completeness coincide iff the corresponding types of continuity do. We show how this relates to absolutely free completions of posets (for which a new description is also presented), and as a by-product we exhibit a simplified proof of the result of J. Meseguer that each subset system is equivalent to a union-complete one.

Key words: Subset system, complete poset.

Classification: 06A23

0. Preliminaries. Recall that a subset system \( Z \) is a rule assigning to each poset \( P \) a collection \( Z(P) \) of subsets of \( P \) in such a way that 1. order-preserving maps preserve \( Z \)-sets (i.e., if \( f: P \rightarrow Q \) is order-preserving, then \( M \in Z(P) \) implies \( f(M) \in Z(Q) \)) and 2. if \( \emptyset \in Z(P) \) for some \( P \) then \( \emptyset \in Z(\emptyset) \). A poset \( P \) which has joins of all sets in \( Z(P) \) is said to be \( Z \)-complete. An order-preserving map \( f: P \rightarrow Q \) (not necessarily between \( Z \)-complete posets) is \( Z \)-continuous if it preserves all existing joins of sets in \( Z(P) \).

Examples: 1. \( S_n \) is the subset system of all non-empty subsets of cardinality smaller than \( n \). (\( S_n \)-complete posets are precisely the upper-semilattices, and \( S_n \)-continuous maps are those preserving all existing finite, non-empty joins.)

2. \( \omega \) is the subset system of all increasing \( \omega \)-chains and all finite chains and \( C \) is the subset system of all \( \omega \) chains.

3. \( \Delta \) is the subset system of all directed sets.

4. \( \Delta_{\omega} \) is the subset system of all countably directed sets, i.e., those sets in which every countable subset has an upper bound.

5. Analogously, \( C_{\omega} \) is the subset system of all countably directed chains.
6. For each subset system Z, \( Z^C \) is the subset system of conditional \( Z \)-completeness, i.e. \( X \in Z^C(P) \) iff \( X \subseteq Z(P) \) and \( X \) has an upper bound in \( P \).

A subset system \( Z \) is absolute if for each subposet \( A \) of a poset \( P \), \( A \in Z(A) \) implies \( A \in Z(P) \). For example \( \omega \) is absolute, whereas the subset system \( \omega^C \) of all bounded \( \omega \)-chains and finite chains is non-absolute. Further, a subset system \( Z \) is conditional if \( Z=Z^C \), that is, for every poset \( P \), every \( Z \)-set in \( P \) has an upper bound in \( P \). Finally, a subset system \( Z \) is normal if for every poset \( P \), \( X \in Z(P) \) implies \( X \in Z(x^T) \), where \( x^T \) is \( x \) with a new top element \( T \) added.

For each poset \( P \), we denote by \( J(P) \) the lattice of all ideals (= downsets) in \( P \), ordered by inclusion. Let \( e:P \to J(P) \) be the principal-ideal embedding, i.e. \( e(P)=\{q \in P|q \leq p\} \). We denote by \( Z^e P \) the \( Z \)-closure of \( e(P) \) in \( J(P) \), i.e., the least subposet \( X \) of \( J(P) \) containing \( e(P) \) and such that \( M \subseteq Z(X) \) implies \( UM \in X \). Then \( Z^e P \) is clearly \( Z \)-complete. As proved in [AN], \( Z^e P \) is the absolutely free \( Z \)-completion of \( P \), i.e., the principal-ideal embedding \( P \to Z^e P \) has the universal property that each order-preserving map \( f:P \to Q \) with \( Q \) \( Z \)-complete has a unique \( Z \)-continuous extension \( Z^e f:Z^e P \to Q \).

A subset \( M \) of a poset \( P \) is said to be \( Z \)-closed if for each \( X \in Z(P) \) with \( X \subseteq M \) and with a join \( VX \) in \( P \) we have \( VX \subseteq M \).

Given subset systems \( Z_1 \) and \( Z_2 \), we write \( Z_1 \subseteq Z_2 \) if each \( Z_2 \)-complete poset is \( Z_1 \)-complete, and each \( Z_2 \)-continuous map between \( Z_2 \)-complete posets is \( Z_1 \)-continuous. (Note that this is trivially true if, for each poset \( P \), \( Z_1(P) \subseteq Z_2(P) \)). In other words,

\[
Z_1 \subseteq Z_2 \text{ iff } Z_2^{\text{Pos}} \subseteq Z_1^{\text{Pos}}
\]

where \( Z^{\text{Pos}} \) is the category of \( Z \)-complete posets and \( Z \)-continuous maps. If \( Z_1 \subseteq Z_2 \subseteq Z_1 \), we say that \( Z_1 \) and \( Z_2 \) are equivalent. For example \( \omega^C \subseteq \omega \subseteq S_\omega \), and \( S_\omega \) is equivalent to \( S_1 \). Moreover, as it is well known, \( \Delta \) and \( C \) are equivalent by Iwamura's Lemma [11]. On the other hand, \( \Delta_\omega \) and \( C_\omega \) are not equivalent:

Example: A poset \( P \) which is \( C_\omega \)-complete but not \( \Delta_\omega \)-complete. Let \( F=\prod\omega_n \) with the componentwise order, and let \( P=\{f \in F|f(n)=\omega_n \} \) for at most finitely many \( n \}. \) Further, let \( D=\{f \in F|f(n) \neq \omega_n \} \) for all \( n \geq 1 \}

We will show that \( D \) is a countably-directed subset of \( P \). Since \( D \) has no upper bound at all in \( P \), this will establish that \( P \) is not \( \Delta_\omega \)-complete. For any countable subset \( X \subseteq D \), consider, for each \( n \in \omega \), \( n \neq 0 \), the set
\{f(n) | f \in X\}. The latter is a countable subset of $\omega_n$, and hence has an upper bound, say $x_n$, in $\omega_n$. Define $g(n)=x_n$ for all $n \geq 1$, then $g$ is an upper bound of $X$ in $D$ and hence $D$ is countably-directed.

Next we prove that $P$ is $C_\omega$-complete. Let $\Gamma$ be a chain in $P$ with no countable cofinal subset. Let $g \in P$ be the join of $\Gamma$ in $P$; it is enough to prove $g \in P$. If not, then $g(n)=\omega_n$ for infinitely many $n$. For each such $n$, we have $\omega_n=(\forall \Gamma)(n)=\forall f \in \Gamma$ and hence either there exists $f \in \Gamma$ with $f(n)=\omega_n$ or $\Gamma$ has a cofinal subset of order type $\omega_n$. However, the latter cannot happen for two different natural numbers $n$, and hence there are infinitely many $n \in \omega$ for which there exists $f_n \in \Gamma$ with $f_n(n)=\omega_n$. Let $Y \subseteq \omega$ consist of all such $n$, and for each $n \in Y$ take $f_n \in \Gamma$ with $f_n(n)=\omega_n$. Then the set \{f_n | n \in Y\} is not cofinal in $\Gamma$ (since $\Gamma$ has no countable cofinal subset) and hence has an upper bound, say $h$, in $\Gamma$. But then $h(n)=\omega_n$ for all $n \in Y$, so $h \notin P$, a contradiction. This shows that $P$ is $C_\omega$-complete.

1. Completeness versus continuity. In this section, we consider the relationship of the condition $Z_2 \subseteq Z_1$ with the ostensibly weaker condition that every $Z_1$-complete poset is $Z_2$-complete.

**Theorem 1:** For any subset systems $Z_1$ and $Z_2$, $Z_2 \subseteq Z_1$ iff every $Z_1$ complete poset is $Z_2$-complete.

**Proof.** Assume that every $Z_1$-complete poset is $Z_2$-complete. Let $f: P \rightarrow Q$ be a $Z_1$-continuous map with $P$ $Z_1$-complete. Given $A \subseteq Z_2(P)$ with $p=VA$, we shall prove that $f(p)=Vf(A)$. Assuming the contrary, there exists an upper bound $q \in Q$ of $f(A)$ with $f(p) \notin q$ - we shall derive a contradiction.

Since $f$ is $Z_1$-continuous, the set $M=\{x \in P | x \notin p \text{ and } f(x) \notin q\}$ is $Z_1$-closed in $P$. Define a poset $P^*$ by adding a decreasing $\omega$-chain $d_0 > d_1 > d_2$... to $P$ in such a way that for $x \in P$, $i < \omega$, we have:

\[ x < d_i \text{ in } P^* \text{ iff } x \in M, \]

and

\[ d_i < x \text{ in } P^* \text{ iff } p \notin x. \]

Let us verify that $P^*$ is $Z_1$-complete. The map $h: P^* \rightarrow P$ defined by $h(d_i)=p$ ($i < \omega$) and $h(x)=x$ ($x \in P$) is order-preserving. Thus, given $B \subseteq Z_1(P^*)$, we have $h(B) \subseteq Z_1(P)$. Put $b=Vh(B)$ in $P$,

then $b$ is an upper bound of $B$ in $P^*$ since $h(y) \preceq y$ for all $y \in P^*$. Either $b=Vb$, or $B$ has the upper bound $d_i$ for some $i$. In the latter case there are two
possibilities:

(i) $B \cap \bigcup_{j < \omega} A_j = \emptyset$ and then $d_j=VB$ for a suitable $j < \omega$, or

(ii) $B \subseteq M$, in which case $h(B)=B$ and hence $B \in Z_1(P)$; since $M$ is $Z_1$-closed in $P$, it follows that $b \in M$ and then $b=VB$ in $P^\ast$.

By assumption, it follows that $P^\ast$ is $Z_2$-complete. However, the set $A \in Z_2(P^\ast)$ fails to have a join in $P^\ast$, due to the decreasing chain of upper bounds $d_j$ - a contradiction. □

For subset systems $Z_1$ and $Z_2$, consider the following conditions:

- **COMPL**($Z_1, Z_2$): every $Z_1$-complete poset is $Z_2$-complete
- **CONT**($Z_1, Z_2$): every $Z_1$-continuous map with $Z_1$-complete domain is $Z_2$-continuous
- **CLOSED**($Z_1, Z_2$): every $Z_1$-closed ideal in a $Z_1$-complete poset is $Z_2$-closed.

**Remark.** In the above proof, we actually proved that $COMPL(Z_1, Z_2) \iff CONT((Z_1, Z_2))$. This is stronger than the nontrivial implication above.

**Theorem 2.** For any subset system $Z_1$ and $Z_2$,

$$COMPL(Z_1, Z_2) \iff CONT(Z_1, Z_2) \iff CLOSED(Z_1, Z_2).$$

**Proof.** $COMPL(Z_1, Z_2) \iff CONT(Z_1, Z_2)$ has exactly the same proof as Theorem 1; the set $A$ considered there has an upper bound in $P^\ast$, and hence belongs to $Z_2(P^\ast)$ but fails to have a join in $P^\ast$.

$CONT(Z_1, Z_2) \iff CLOSED(Z_1, Z_2)$: For each $Z_1$-closed ideal $A$ in a $Z_1$-complete poset $P$ define a map

$$f: P \rightarrow \{0, 1\} (0 < 1), \text{ by } f(x) = 0 \text{ iff } x \in A.$$ 

Since $A$ is an ideal, $f$ is order-preserving, and since $A$ is $Z_1$-closed, $f$ is $Z_1$-continuous. Consequently, $f$ is $Z_2$-continuous; in other words, $A$ is $Z_2$-closed.

$CLOSED(Z_1, Z_2) \implies COMPL(Z_1, Z_2)$: Let $P$ be a $Z_1$-complete poset, and suppose $A \in Z_2(P)$ such that $A$ has no join in $P$. Since $A \in Z_2(P)$, we know $A \in Z_2(P)$ and $A$ has an upper bound, $a$, in $P$. Let $\overline{A}$ be the smallest $Z_1$-closed ideal of $P$ containing $A$. Then $a$ is an upper bound of $\overline{A}$ in $P$, but $\overline{A}$ has no largest element (since this would be the join of $A$ in $P$). In fact, $A$ has no upper bound in $\overline{A}$.

We shall verify that $\overline{A}$ is $Z_1$-closed in the (obviously $Z_1$-complete) subset $B=\overline{A} \cup \{a\}$ of $P$: given $x \in Z_2(B)$ with $x \in \overline{A}$ and $x=VX$ in $P$, since $X \in Z_2(P)$ and $\overline{A}$ is $Z_1$-closed, and therefore $Z_2$-closed, in $P$, we conclude that $x \in \overline{A}$ and...
hence $x$ is the join of $X$ in $B$, too. By the hypothesis, $\mathcal{A}$ is $Z_2$-closed in $B$. However, the map $P \rightarrow B$ which maps $B$ identically and maps all other elements of $P$ to $a$, is order-preserving, hence $A \in Z_2(B)$. Now, $a$ is the join of $A$ in $B$, and hence $\mathcal{A}$ is not $Z_2$-closed in $B$, a contradiction.

This completes the proof of Theorem 2.

**Remark:** The condition $\text{CONT}(Z_1, Z_2)$ does not, in general, imply $\text{COMPL}(Z_1, Z_2)$: consider $Z_1 = \omega^C$, and $Z_2 = \omega$. Every $\omega^C$-continuous map is $\omega$-continuous, and hence $\text{CONT}(Z_1, Z_2)$ is true, whereas $\text{COMPL}(Z_1, Z_2)$ is false. These conditions are equivalent under additional hypotheses:

**Theorem 3:** For subset systems $Z_1$ and $Z_2$, if either $Z_1$ is absolute or $Z_2$ is conditional then

$$Z_2 \subseteq Z_1 \iff \text{COMPL}(Z_1, Z_2) \iff \text{CONT}(Z_1, Z_2).$$

**Proof.** It is only necessary to prove that $\text{CONT}(Z_1, Z_2) \iff \text{COMPL}(Z_1, Z_2)$. For the case $Z_2$ is conditional, this follows from Theorem 2. So, assume $Z_1$ is absolute, and let $P$ be a $Z_1$-complete poset. Assuming that there is a set $A \subseteq Z_2(P)$ which does not have a join in $P$, we shall derive a contradiction. Let $\mathcal{A}$ denote the least $Z_1$-closed ideal of $P$ containing $A$. Then $\mathcal{A}$ does not have a largest element (since this would clearly be the join of $A$). Let $\mathcal{A}$ be the extension of $\mathcal{A}$ by a largest element $T$: the absoluteness of $Z_1$ guarantees that $\mathcal{A}$ is $Z_1$-complete. Further, let $\mathcal{A}^T$ be an extension of $\mathcal{A}$ by an element $S < T$ which is an upper bound of $\mathcal{A}$. The absoluteness of $Z_1$ guarantees that the inclusion map $e: \mathcal{A}^T \rightarrow \mathcal{A}^T$ is $Z_1$-continuous: if a $Z_1$-set $B$ of $\mathcal{A}^T$ contains $T$, then $T = V B$, and if $T \in B$ then $V B \in \mathcal{A}$ (because $B \in Z_1(\mathcal{A})$), and in both cases, $e(V B) = V e(B)$. Consequently, $e$ is $Z_2$-continuous. Nevertheless, $V A = T$ in $\mathcal{A}^T$, whereas $V e(A) = S$ in $\mathcal{A}^T$ - a contradiction to $A \in Z_2(\mathcal{A}^T)$ [use $f: P \rightarrow \mathcal{A}^T$, $f/A = \text{id}$, $f/P = A \equiv T$].

**Remark.** The following condition strengthens CLOSED($Z_1, Z_2$) above.

CLOSED$^\mathcal{A}(Z_1, Z_2)$: $Z_1$-closed sets are $Z_2$-closed in each $Z_1$-complete poset. These two conditions are equivalent, whenever $Z_2$ is normal. (Recall that every subset system is equivalent to a normal one, [ANR]J.) To see this, assume CLOSED($Z_1, Z_2$), and let us prove CLOSED$^\mathcal{A}(Z_1, Z_2)$. Let $P$ be a $Z_1$-complete poset. For each $Z_1$-closed set $A \subseteq P$ and each $B \in Z_2(P)$ with join $b = V B$ in $P$ we prove that $B \in A$ implies $b \in A$ as follows. Let $B = \{x \in A \mid x \in A\} \cup \{b\}$. Then $B$ is clearly a $Z_1$-complete poset, and $B \cap A$ is a $Z_1$-closed ideal of $B$. Consequently, $B \cap A$ is $Z_2$-closed in $B$. Since $Z_2$ is normal, we have $B \subseteq Z_2(B)$ and $B \subseteq B \cap A$. Thus, $b \in A$. 
On the other hand, if $Z_2$ is not normal, $\text{CLOSED}(Z_1, Z_2)$ need not imply $\text{CLOSED}^*(Z_1, Z_2)$: Consider the subset system $Z_1$ of all subsets having a least element, and $Z_2$ of all subsets having a lower bound. It is obvious that $Z_1$ is equivalent to $Z_2$. However, in the 4-point Boolean algebra $\{0, 1, a, \exists\}$ the $Z_1$-closed set $\{a, \exists\}$ is not $Z_2$-closed.

Example 1. The assumption that the domain be $Z_1$-complete in the above condition $\text{CONT}(Z_1, Z_2)$ is essential. Consider the absolute, equivalent subset systems $S_3$ and $S_\omega$. There is an $S_3$-continuous map which is not $S_\omega$-continuous: consider the following poset $P$.

The map $f: P \to P$ defined by $f(x) = x$ for all $x \in d$, $f(d) = T$, is not $S_\omega$-continuous because it does not preserve the join $V\{a, b, c\} = d$; however, $f$ is $S_3$-continuous (by default).

This shows that the above $\text{CONT}(Z_1, Z_2)$ is not equivalent to the following (more natural) condition:

$\text{CONT}^*(Z_1, Z_2)$: every $Z_1$-continuous map between arbitrary posets is $Z_2$-continuous

Consider furthermore the following conditions:

$\text{CLOSED}(Z_1, Z_2)$: every $Z_1$-closed ideal in any poset is $Z_2$-closed,

$\text{CLOSED}^*(Z_1, Z_2)$: every $Z_1$-closed subset of any poset is $Z_2$-closed.

For all $Z_1, Z_2$, $\text{CONT}(Z_1, Z_2) \iff \text{CLOSED}(Z_1, Z_2)$, and for $Z_2$ normal, these are equivalent to $\text{CLOSED}^*(Z_1, Z_2)$; the proof is like that of Theorem 1.

2. Saturated subset systems

Definition. The saturation of a subset system $Z$ is the following subset system $\hat{Z}$: For each poset $P$,

$M \in \hat{Z}(P)$ iff for each order-preserving map $h: P \to Q$, if $Q$ is $Z$-complete then $Vh(M)$ exists.

A subset system $Z$ is saturated if $Z = \hat{Z}$.

Corollary 1. Each subset system is equivalent to its saturation.
In fact, $Z$-completeness and $\hat{Z}$-completeness are clearly equivalent, and hence, Theorem 1 can be applied.

Observe that $\hat{Z}$ is the largest subset system, under inclusion, equivalent to $Z$ (where the inclusion $Z_1 \subseteq Z_2$ means that $Z_1(P) \subseteq Z_2(P)$ for each poset $P$). Also, for subset systems $Z_1$ and $Z_2$,

$$Z_1 \subseteq Z_2 \iff \hat{Z}_1 \subseteq \hat{Z}_2.$$ 

Examples: 1. $\Delta$ is saturated, and moreover, is the saturation of $C$.

2. $S_\omega$ is not saturated; $\hat{S}_\omega$ consists of those sets that have a finite cofinal subset.

Proposition 1. For a saturated subset system $Z$,

$$Z^P = J(P) \cap Z(P),$$

i.e., the $Z$-closure in $J(P)$ of the set of principal ideals consists of all ideal $Z$-sets in $P$.

Proof. Since $Z$ is saturated, all principal ideals are clearly $Z$-sets, and hence, the set $X = J(P) \cap Z(P)$ contains $e(P)$. We shall prove that $M \in Z(X)$ implies $\bigcup M \in Z(P)$. (It is then clear that $X$ is the least subset of $J(P)$ with the above properties.) Thus, we are to show that $\bigvee h(UM)$ exists in each $Z$-complete poset $Q$ for each order-preserving map $h: P \rightarrow Q$. Define a map

$$h': J(P) \cap Z(P) \rightarrow Q, \text{ by } h'(I) = \bigvee h(I) \text{ for } I \in J(P) \cap Z(P).$$

Since $h'$ is clearly order-preserving, we have $h'(M) \in Z(Q)$. Thus, the set $h'(M)$ has a join. Obviously,

$$\bigvee_{I \in M} h(I) = \bigvee_{I \in M} h_{h'}(I) = \bigvee_{I \in M} h(I). \quad \square$$

Remark. Recall [ADJ] that a subset system is called union complete iff for $J_Z(P)$ = all $Z$-generated ideals in $P$, if $M \in Z(J_Z(P))$ then $\bigcup M \in J_Z(P)$. If $Z$ is saturated then $J_Z(P) = J(P) \cap Z(P)$, and the above proof actually verifies that every saturated subset system is union-complete. Together with Corollary 1 this yields the following result, proved (much more technically) by J. Mese-

guer [M]: Each subset system is equivalent to a union-complete subset system.

Note that union complete does not imply saturated: $Z = S_\omega$ is a counterexample. It does imply if $Z$ fulfills $(M \leq P, N \leq ZP$ cofinal in $\downarrow M \rightarrow M \leq ZP$.

In our opinion, the role that union-completeness was intended to play, that is, to obtain a description of free $Z$-completions via ideals, can be accomplished more naturally using the concept of saturation.

The equivalence of the first two conditions in the following result is
essentially due to Meseguer [M, Prop. 3.13], where the proof relies on the fact that every subset system is equivalent to a union complete one.

**Proposition 2.** For arbitrary subset systems $Z_1$ and $Z_2$, the following are equivalent:

1. $Z_1 \triangleleft Z_2$
2. $\text{INCL}(Z_1, Z_2)$: $Z_1^P \triangleleft Z_2^P$ (for each poset $P$); i.e., the $Z_2$-closure of $e(P)$ in $J(P)$ contains its $Z_1$-closure.
3. $\text{FREE}(Z_1, Z_2)$: For each poset $P$ there is a $Z_1$-continuous map $\lambda_P : Z_1^P \rightarrow Z_2^P$ such that the following triangle commutes:

\[
\begin{array}{ccc}
P & \xrightarrow{e_1} & Z_1^P \\
& \searrow & \downarrow \lambda_P \\
& e_2 & \swarrow & Z_2^P \\
\end{array}
\]

**Proof.** By Proposition 1 we have $Z_1^P = Z_2^P \wedge Z_1^P$, and, as remarked earlier, $Z_1^P$ is the absolutely free $Z_1$-completion of $P$, and analogously with $Z_2$. Since $Z_1 \leq Z_2$ is equivalent to $Z_1 \triangleleft Z_2$, the implications

$Z_1 \triangleleft Z_2 \Rightarrow \text{INCL}(Z_1, Z_2) \Rightarrow \text{FREE}(Z_1, Z_2)$

follow immediately.

To prove $\text{FREE}(Z_1, Z_2) \Rightarrow Z_1 \triangleleft Z_2$, let $P$ be a $Z_2$-complete poset. It is sufficient to prove that $P$ is $Z_1$-complete - this implies $Z_1 \triangleleft Z_2$ by Theorem 1. For any $A \in Z_1^P$, put $a = \ve_1(A)$ in $Z_1^P$; by (FREE) we have $\lambda_P(a) = \ve_1(A)$ in $Z_2^P$. Since $P$ is $Z_2$-complete, there is a unique $Z_2$-continuous map $f : Z_2^P \rightarrow P$ with $f \circ e_2 = \text{id}_P$. We claim that $VA = f(\lambda_P(a))$ in $P$:

(i) $x \in A$ implies $x = f(e_2(x)) = f(\lambda_P(e_1(x))) = f(\lambda_P(a)),$

and

(ii) each upper bound $b$ of $A$ in $P$ fulfills $e_1(a) \leq e_1(b)$ in $Z_1^P$ and hence, $f(\lambda_P(a)) \leq f(\lambda_P(e_1(b))) = f(e_2(b)) = b$. $\square$

**Remark.** Analogous considerations concerning colimits in categories are presented by M.H. Albert and G.M. Kelly [AKJ]. Given a collection $\mathcal{A}$ of small categories and a small category $J$, they investigate conditions under which the existence of $\mathcal{A}$-colimits always implies the existence of $\mathcal{A} \cup \{J\}$-colimits. They obtain a characterization theorem analogous to the equivalences.
\( Z_1 
\iff \text{COMPL}(Z_2, Z_1) \iff \text{INCL}(Z_1, Z_2) \) above. The role of \( Z^P \) is, in the categorical context, played by the \( \lambda \)-colimit closure of a category \( P \) in its Yoneda embedding into \( \text{Set}^{P^{op}} \). Note, however, that although an absolute subset system \( Z \) can be viewed as a special collection of categories (viz., of all posets \( P \) with \( P \in Z(P) \)) the categorical result does not imply the order-theoretic one, not even for absolute subset systems: if \( Z_1 \nsubseteq Z_2 \), then \( Z_2 \)-cocompleteness of categories need not imply \( Z_1 \)-cocompleteness. For example, let \( Z_2 = S_3 \) and let \( Z_1 \) be the subset system consisting of all \( Z_2 \)-sets, plus subsets of the form \( \omega_{\omega} \); then clearly \( Z_1 \) and \( Z_2 \) are equivalent. However, a category is \( Z_2 \)-complete if it has binary coproducts, whereas \( Z_1 \)-cocompleteness entails the existence of pushouts. For example, the dual of the category of compact topological spaces has coproducts but not pushouts [A; p. 38].

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