Commentationes Mathematicae Universitatis Carolinae

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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 29 (1988), No. 1, 169--177

Persistent URL: `http://dml.cz/dmlcz/106607`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,1 (1988)

# COMPARISON OF SUBSET SYSTEMS

E. NELSON, J. ADÁMEK, A. JUNG, J. REITERMAN, A. TARLECKI

**Abstract:** A subset system, as introduced by [ADJ], is a means for expressing a type of (join-)completeness of posets and (join-)continuity of order-preserving maps. We compare subset systems, and we prove, essentially, that the corresponding types of completeness coincide iff the corresponding types of continuity do. We show how this relates to absolutely free completions of posets (for which a new description is also presented), and as a by-product we exhibit a simplified proof of the result of J. Meseguer that each subset system is equivalent to a union-complete one.

**Key words:** Subset system, complete poset.

**Classification:** 06A23

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**0. Preliminaries.** Recall that a subset system $\mathbf{Z}$ is a rule assigning to each poset P a collection Z(P) of subsets of P in such a way that 1. order-preserving maps preserve Z-sets (i.e., if $f:P \to Q$ is order-preserving, then $M \in Z(P)$ implies $f(M) \in Z(Q)$) and 2. if $\emptyset \in Z(P)$ for some P then $\emptyset \in Z(\emptyset)$. A poset P which has joins of all sets in Z(P) is said to be Z-complete. An order-preserving map $f:P \to Q$ (not necessarily between Z-complete posets) is Z-continuous if it preserves all existing joins of sets in Z(P).

**Examples:** 1. $S_n$ is the subset system of all non-empty subsets of cardinality smaller than n. ($S_\omega$ complete posets are precisely the upper-semilattices, and $S_\omega$-continuous maps are those preserving all existing finite, non-empty joins.)

2. $\omega$ is the subset system of all increasing $\omega$-chains and all finite chains and **C** is the subset system of all $\omega$ chains.

3. $\Delta$ is the subset system of all directed sets.

4. $\Delta_\omega$ is the subset system of all countably directed sets, i.e., those sets in which every countable subset has an upper bound.

5. Analogously, $C_\omega$ is the subset system of all countably directed chains.

6. For each subset system $Z$, $Z^c$ is the subset system of conditional $Z$-completeness, i.e. $X \in Z^c(P)$ iff $X \in Z(P)$ and $X$ has an upper bound in $P$.

A subset system $Z$ is absolute if for each subposet $A$ of a poset $P$, $A \in Z(P)$ implies $A \in Z(A)$. For example $\omega$ is absolute, whereas the subset system $\omega^c$ of all bounded $\omega$-chains and finite chains is non-absolute. Further, a subset system $Z$ is conditional if $Z = Z^c$, that is, for every poset $P$, every $Z$-set in $P$ has an upper bound in $P$. Finally, a subset system $Z$ is normal if for every poset $P$, $X \in Z(P)$ implies $X \in Z(X^T)$, where $X^T$ is $X$ with a new top element $T$ added.

For each poset $P$, we denote by $J(P)$ the lattice of all ideals (= downsets) in $P$, ordered by inclusion. Let $e: P \to J(P)$ be the principal-ideal embedding, i.e. $e(P) = \{q \in P \mid q \leq p\}$. We denote by $Z^{\#}P$ the Zclosure of $e(P)$ in $J(P)$, i.e., the least subposet $X$ of $J(P)$ containing $e(P)$ and such that $M \in Z(X)$ implies $\bigcup M \in X$. Then $Z^{\#}P$ is clearly $Z$-complete. As proved in [AN], $Z^{\#}P$ is the absolutely free $Z$-completion of $P$, i.e., the principal-ideal embedding $P \to Z^{\#}P$ has the universal property that each order-preserving map $f: P \to Q$ with $Q$ $Z$-complete has a unique $Z$-continuous extension $Z^{\#}f: Z^{\#}P \to Q$.

A subset $M$ of a poset $P$ is said to be Z-closed if for each $X \in Z(P)$ with $X \subseteq M$ and with a join $\bigvee X$ in $P$ we have $\bigvee X \in M$.

Given subset systems $Z_1$ and $Z_2$, we write $Z_1 \leq Z_2$ if each $Z_2$-complete poset is $Z_1$-complete, and each $Z_2$-continuous map between $Z_2$-complete posets is $Z_1$-continuous. (Note that this is trivially true if, for each poset $P$, $Z_1(P) \subseteq Z_2(P)$.) In other words,

$$Z_1 \leq Z_2 \text{ iff } Z_2\text{-Pos} \subseteq Z_1\text{-Pos}$$

where $Z$-Pos is the category of $Z$-complete posets and $Z$-continuous maps. If $Z_1 \leq Z_2 \leq Z_1$, we say that $Z_1$ and $Z_2$ are equivalent. For example $\omega^c \leq \omega \leq S_{\omega_1}$, and $S_\omega$ is equivalent to $S_3$. Moreover, as it is well known, $\Delta$ and $C$ are equivalent by Iwamura's Lemma [I]. On the other hand, $\Delta_\omega$ and $C_\omega$ are not equivalent:

**Example:** A poset $P$ which is $C_\omega$-complete but not $\Delta_\omega$-complete. Let $\bar{P} = \prod_{0 \neq n \in \omega} (\omega_n + 1)$ with the componentwise order, and let $P = \{f \in \bar{P} \mid f(n) = \omega_n$ for at most finitely many $n\}$. Further, let

$$D = \{f \in \bar{P} \mid f(n) \neq \omega_n \text{ for all } n \geq 1\}.$$

We will show that $D$ is a countably-directed subset of $P$. Since $D$ has no upper bound at all in $P$, this will establish that $P$ is not $\Delta_\omega$-complete. For any countable subset $X \subset D$, consider, for each $n \in \omega$, $n \neq 0$, the set

$\{f(n) | f \in X\}$. The latter is a countable subset of $\omega_n$, and hence has an upper bound, say $x_n$, in $\omega_n$. Define $g(n)=x_n$ for all $n \geq 1$, then g is an upper bound of X in D and hence D is countably-directed.

Next we prove that P is $C_\omega$-complete. Let $\Gamma$ be a chain in P with no countable cofinal subset. Let $g \in \bar{P}$ be the join of $\Gamma$ in $\bar{P}$; it is enough to prove $g \in P$. If not, then $g(n)= \omega_n$ for infinitely many n. For each such n, we have $\omega_n=(V\Gamma)(n)= \bigvee_{f \in \Gamma} f(n)$ and hence either there exists $f \in \Gamma$ with $f(n)= = \omega_n$ or $\Gamma$ has a cofinal subset of order type $\omega_n$. However, the latter cannot happen for two different natural numbers n, and hence there are infinitely many $n \in \omega$ for which there exists $f_n \in \Gamma$ with $f_n(n)= \omega_n$. Let $Y \subseteq \omega$ consist of all such n, and for each $n \in Y$ take $f_n \in \Gamma$ with $f_n(n)= \omega_n$. Then the set $\{f_n | n \in Y\}$ is not cofinal in $\Gamma$ (since $\Gamma$ has no countable cofinal subset) and hence has an upper bound, say h, in $\Gamma$. But then $h(n)= \omega_n$ for all $n \in Y$, so $h \notin P$, a contradiction. This shows that P is $C_\omega$-complete.

**1. Completeness versus continuity.** In this section, we consider the relationship of the condition $Z_2 \leq Z_1$ with the ostensibly weaker condition that every $Z_1$-complete poset is $Z_2$-complete.

**Theorem 1:** <u>For any subset systems $Z_1$ and $Z_2$, $Z_2 \leq Z_1$ iff every $Z_1$ complete poset is $Z_2$-complete</u>.

**Proof.** Assume that every $Z_1$-complete poset is $Z_2$-complete. Let $f:P \to Q$ be a $Z_1$-continuous map with P $Z_1$-complete. Given $A \in Z_2(P)$ with $p=VA$, we shall prove that $f(p)=Vf(A)$. Assuming the contrary, there exists an upper bound $q \in \in Q$ of $f(A)$ with $f(p) \nleq q$ - we shall derive a contradiction.

Since f is $Z_1$-continuous, the set $M= \{ x \in P | x \leq p \text{ and } f(x) \leq q \}$ is $Z_1$-closed in P. Define a poset $P^*$ by adding a decreasing $\omega$-chain $d_0 > d_1 > d_2 \dots$ to P in such a way that for $x \in P$, $i < \omega$, we have:

$$x < d_i \text{ in } P^* \text{ iff } x \in M,$$

and

$$d_i < x \text{ in } P^* \text{ iff } p \leq x.$$

Let us verify that $P^*$ is $Z_1$-complete. The map $h:P^* \to P$ defined by $h(d_i)=p$ $(i < \omega)$ and $h(x)=x$ $(x \in P)$ is order-preserving. Thus, given $B \in Z_1(P^*)$, we have $h(P) \in Z_1(P)$. Put

$$b=V\ h(B) \text{ in } P,$$

then b is an upper bound of B in $P^*$ since $h(y) \geq y$ for all $y \in P^*$. Either $b= =VB$, or B has the upper bound $d_i$ for some i. In the latter case there are two possibilities:

(i) $B\cap\{d_j\}_{j<\omega}\neq\Phi$ and then $d_j=\bigvee B$ for a suitable $j<\omega$, or

(ii) $B\subseteq M$, in which case $h(B)=B$ and hence $B\in Z_1(P)$; since M is $Z_1$-clossed in P, it follows that $b\in M$ and then $b=\bigvee B$ in $P^*$.

By assumption, it follows that $P^*$ is $Z_2$-complete. However, the set $A\in Z_2(P^*)$ fails to have a join in $P^*$, due to the decreasing chain of upper bounds $d_i$ - a contradiction. □

For subset systems $Z_1$ and $Z_2$, consider the following conditions:

COMPL($Z_1,Z_2$) every $Z_1$-complete poset is $Z_2$-complete

CONT($Z_1,Z_2$) every $Z_1$-continuous map with $Z_1$-complete domain is $Z_2$-continuous

CLOSED($Z_1,Z_2$) every $Z_1$-closed ideal in a $Z_1$-complete poset is $Z_2$-closed.

**Remark.** In the above proof, we actually proved that COMPL($Z_1,Z_2$) $\Rightarrow$ CONT(($Z_1,Z_2$).This is stronger than the nontrivial implication above.

**Theorem 2.** For any subset system $Z_1$ and $Z_2$,

$$\mathrm{COMPL}(Z_1,Z_2^c)\Leftrightarrow \mathrm{CONT}(Z_1,Z_2)\Leftrightarrow \mathrm{CLOSED}(Z_1,Z_2).$$

**Proof.** COMPL $(Z_1,Z_2^c)\Rightarrow$ CONT($Z_1,Z_2$) has exactly the same proof as Theorem 1; the set A considered there has an upper bound in $P^*$, and hence belongs to $Z_2^c(P^*)$ but fails to have a join in $P^*$.

CONT($Z_1,Z_2$)$\Rightarrow$CLOSED($Z_1,Z_2$): For each $Z_1$-closed ideal A in a $Z_1$-complete poset P define a map

$$f:P\rightarrow\{0,1\}\ (0<1),\ \text{by } f(x)=0 \text{ iff } x\in A.$$

Since A is an ideal, f is order-preserving, and since A is $Z_1$-closed, f is $Z_1$-continuous. Consequently, f is $Z_2$-continuous; in other words, A is $Z_2$-closed.

CLOSED($Z_1,Z_2$)$\Rightarrow$ COMPL($Z_1,Z_2^c$): Let P be a $Z_1$-complete poset, and suppose $A\in Z_2^c(P)$ such that A has no join in P. Since $A\in Z_2^c(P)$, we know $A\in Z_2(P)$ and A has an upper bound, a, in P. Let $\bar{A}$ be the smallest $Z_1$-closed ideal of P containing A. Then a is an upper bound of $\bar{A}$ in P, but $\bar{A}$ has no largest element (since this would be the join of A in P). In fact, A has no upper bound in $\bar{A}$.

We shall verify that $\bar{A}$ is $Z_1$-closed in the (obviously $Z_1$-complete) subset $B=\bar{A}\cup\{a\}$ of P: given $X\in Z_2(B)$ with $X\subseteq\bar{A}$ and $x=\bigvee X$ in P, since $X\in Z_2(P)$ and $\bar{A}$ is $Z_1$-closed, and therefore $Z_2$-closed, in P, we conclude that $x\in\bar{A}$ and hence x is the join of X in B, too. By the hypothesis, $\bar{A}$ is $Z_2$-closed in B. However, the map $P \longrightarrow B$ which maps B identically and maps all other elements of P to a, is order-preserving, hence $A \in Z_2(B)$. Now, a is the join of A in B, and hence $\bar{A}$ is not $Z_2$-closed in B, a contradiction.

This completes the proof of Theorem 2. □

**Remark:** The condition $\mathrm{CONT}(Z_1, Z_2)$ does not, in general, imply $\mathrm{COMPL}(Z_1, Z_2)$: consider $Z_1 = \omega^c$, and $Z_2 = \omega$. Every $\omega^c$-continuous map is $\omega$-continuous, and hence $\mathrm{CONT}(Z_1, Z_2)$ is true, whereas $\mathrm{COMPL}(Z_1, Z_2)$ is false. These conditions are equivalent under additional hypotheses:

**Theorem 3:** <u>For subset systems $Z_1$ and $Z_2$, if either $Z_1$ is absolute or $Z_2$ is conditional then</u>

$$Z_2 \leq Z_1 \Longleftrightarrow \mathrm{COMPL}(Z_1, Z_2) \Longleftrightarrow \mathrm{CONT}(Z_1, Z_2).$$

**Proof.** It is only necessary to prove that $\mathrm{CONT}(Z_1, Z_2) \Longrightarrow \mathrm{COMPL}(Z_1, Z_2)$. For the case $Z_2$ is conditional, this follows from Theorem 2. So, assume $Z_1$ is absolute, and let P be a $Z_1$-complete poset. Assuming that there is a set $A \in Z_2(P)$ which does not have a join in P, we shall derive a contradiction. Let $\bar{A}$ denote the least $Z_1$-closed ideal of P containing A. Then $\bar{A}$ does not have a largest element (since this would clearly be the join of A). Let $\bar{A}^T$ be the extension of $\bar{A}$ by a largest element T: the absoluteness of $Z_1$ guarantees that $\bar{A}^T$ is $Z_1$-complete. Further, let $\bar{A}^{TS}$ be an extension of $\bar{A}^T$ by an element $S < T$ which is an upper bound of $\bar{A}$. The absoluteness of $Z_1$ guarantees that the inclusion map $e: \bar{A}^T \longrightarrow \bar{A}^{TS}$ is $Z_1$-continuous: if a $Z_1$-set B of $\bar{A}^T$ contains T, then $T = \vee B$, and if $T \notin B$ then $\vee B \in \bar{A}$ (because $B \in Z_1(\bar{A})$), and in both cases, $e(\vee B) = \vee e(B)$. Consequently, e is $Z_2$-continuous. Nevertheless, $\vee A = T$ in $\bar{A}^T$, whereas $\vee e(A) = S$ in $\bar{A}^{TS}$ - a contradiction to $A \in Z_2(\bar{A}^T)$ [use $f: P \longrightarrow \bar{A}^T$, $f/A = id$, $f/P-A \equiv T$].

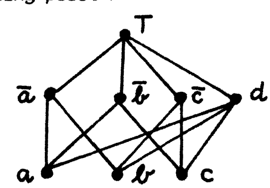
**Remark.** The following condition strengthens $\mathrm{CLOSED}(Z_1, Z_2)$ above.

$\mathrm{CLOSED}^*(Z_1, Z_2)$: $Z_1$-closed sets are $Z_2$-closed (in each $Z_1$-complete poset).

These two conditions are equivalent, whenever $Z_2$ is normal. (Recall that every subset system is equivalent to a normal one, [ANR].) To see this, assume $\mathrm{CLOSED}(Z_1, Z_2)$, and let us prove $\mathrm{CLOSED}^*(Z_1, Z_2)$. Let P be a $Z_1$-complete poset. For each $Z_1$-closed set $A \subseteq P$ and each $B \in Z_2(P)$ with join $b = \vee B$ in P we prove that $B \subseteq A$ implies $b \in A$ as follows. Let $\bar{B} = \{x \in A \mid x \not\leq b\} \cup \{b\}$. Then $\bar{B}$ is clearly a $Z_1$-complete poset, and $\bar{B} \cap A$ is a $Z_1$-closed ideal of $\bar{B}$. Consequently, $\bar{B} \cap A$ is $Z_2$-closed in $\bar{B}$. Since $Z_2$ is normal, we have $B \in Z_2(\bar{B})$ and $B \subseteq \bar{B} \cap A$. Thus, $b \in A$.

On the other hand, if $Z_2$ is not normal, CLOSED$(Z_1,Z_2)$ need not imply CLOSED$^*(Z_1,Z_2)$: Consider the subset system $Z_1$ of all subsets having a least element, and $Z_2$ of all subsets having a lower bound. It is obvious that $Z_1$ is equivalent to $Z_2$. However, in the 4-point Boolean algebra $\{0,1,a,\bar{a}\}$ the $Z_1$-closed set $\{a,\bar{a}\}$ is not $Z_2$-closed.

**Example 1.** The assumption that the domain be $Z_1$-complete in the above condition CONT$(Z_1,Z_2)$ is essential. Consider the absolute, equivalent subset systems $S_3$ and $S_\omega$. There is an $S_3$-continuous map which is not $S_\omega$-continuous: consider the following poset P




The map $f:P \longrightarrow P$ defined by $f(x)=x$ for all $x \neq d$, $f(d)=T$, is not $S_\omega$-continuous because it does not preserve the join $\bigvee\{a,b,c\}=d$; however, f is $S_3$-continuous (by default).

This shows that the above CONT$(Z_1,Z_2)$ is not equivalent to the following (more natural) condition:

CONT$(Z_1,Z_2)^*$ : every $Z_1$-continuous map between arbitrary posets is $Z_2$-continuous

Consider furthermore the following conditions:

CLOSED$(Z_1,Z_2)^*$ : every $Z_1$-closed ideal in any poset is $Z_2$-closed,

CLOSED$^*(Z_1,Z_2)^*$ : every $Z_1$-closed subset of any poset is $Z_2$-closed.

For all $Z_1,Z_2$, CONT$(Z_1,Z_2)^* \iff$ CLOSED$(Z_1,Z_2)^*$, and for $Z_2$ normal, these are equivalent to CLOSED$^*(Z_1,Z_2)^*$; the proof is like that of Theorem 1.

**2. Saturated subset systems**

**Definition.** The <u>saturation</u> of a subset system Z is the following subset system $\hat{Z}$: For each poset P,

$M \in \hat{Z}(P)$ iff for each order-preserving map $h:P \longrightarrow Q$, if Q is Z-complete then $\bigvee h(M)$ exists.

A subset system Z is <u>saturated</u> if $Z=\hat{Z}$.

**Corollary 1.** <u>Each subset system is equivalent to its saturation.</u>

In fact, $Z$-completeness and $\hat{Z}$-completeness are clearly equivalent, and hence, Theorem 1 can be applied.

Observe that $\hat{Z}$ is the largest subset system, under inclusion, equivalent to $Z$ (where the inclusion $Z_1 \subseteq Z_2$ means that $Z_1(P) \subseteq Z_2(P)$ for each poset $P$). Also, for subset systems $Z_1$ and $Z_2$,

$$Z_1 \leq Z_2 \text{ iff } \hat{Z}_1 \subseteq \hat{Z}_2.$$

**Examples:** 1. $\Delta$ is saturated, and moreover, is the saturation of **C**.

2. $S_\omega$ is not saturated; $\hat{S}_\omega$ consists of those sets that have a finite cofinal subset.

**Proposition 1.** For a saturated subset system $Z$,

$$Z^* P = J(P) \cap Z(P),$$

i.e. , the $Z$-closure in $J(P)$ of the set of principal ideals consists of all ideal $Z$-sets in $P$.

**Proof.** Since $Z$ is saturated, all principal ideals are clearly $Z$-sets, and hence, the set $X=J(P)\cap Z(P)$ contains $e(P)$. We shall prove that $M \in Z(X)$ implies $\bigcup M \in Z(P)$. (It is then clear that $X$ is the least subset of $J(P)$ with the above properties.) Thus, we are to show that $\bigvee h(\bigcup M)$ exists in each $Z$-complete poset $Q$ for each order-preserving map $h:P \longrightarrow Q$. Define a map

$$h':J(P)\cap Z(P) \longrightarrow Q, \text{ by } h'(I)=\bigvee h(I) \text{ for } I \in J(P)\cap Z(P).$$

Since $h'$ is clearly order-preserving, we have $h'(M) \in Z(Q)$. Thus, the set $h'(M)$ has a join. Obviously,

$$\bigvee h'(M) = \bigvee_{I \in M} \bigvee h(I) = \bigvee h(\bigcup M). \qquad \square$$

**Remark.** Recall [ADJ] that a subset system is called union complete iff for $J_Z(P)$= all $Z$-generated ideals in $P$, if $M \in Z(J_Z(P))$ then $\bigcup M \in J_Z(P)$. If $Z$ is saturated then $J_Z(P)=J(P)\cap Z(P)$, and the above proof actually verifies that every saturated subset system is union-complete. Together with Corollary 1 this yields the following result, proved (much more technically) by J. Meseguer [M]: Each subset system is equivalent to a union-complete subset system.

Note that union complete does not imply saturated: $Z=S_\omega$ is a counterexample. It does imply if $Z$ fulfils ($M \subseteq P$, $N \in ZP$ cofinal in $\downarrow M$) $\Longrightarrow M \in ZP$.

In our opinion, the role that union-completeness was intended to play, that is, to obtain a description of free $Z$-completions via ideals, can be accomplished more naturally using the concept of saturation.

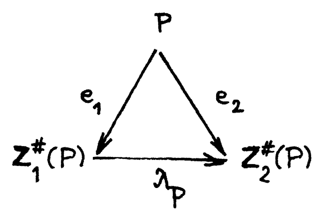
The equivalence of the first two conditions in the following result is essentially due to Meseguer [M, Prop. 3.13], where the proof relies on the fact that every subset system is equivalent to a union complete one.

**Proposition 2.** For arbitrary subset systems $Z_1$ and $Z_2$, the following are equivalent:

$Z_1 \leq Z_2$

$\mathrm{INCL}(Z_1,Z_2)$: $Z_1^{\#}P \subseteq Z_2^{\#}P$ (for each poset P); i.e., the $Z_2$-closure of e(P) in J(P) contains its $Z_1$-closure.

$\mathrm{FREE}(Z_1,Z_2)$: For each poset P there is a $Z_1$-continuous map $\lambda_P : Z_1^{\#}(P) \longrightarrow$ $\longrightarrow Z_2^{\#}(P)$ such that the following triangle commutes:




**Proof.** By Proposition 1 we have $Z_1^{\#}P = J(P) \cap \hat{Z}_1(P)$, and, as remarked earlier, $Z_1^{\#}P$ is the absolutely free $Z_1$-completion of P, and analogously with $Z_2$. Since $Z_1 \leq Z_2$ is equivalent to $\hat{Z}_1 \subseteq \hat{Z}_2$, the implications

$$Z_1 \leq Z_2 \Longrightarrow \mathrm{INCL}(Z_1,Z_2) \Longrightarrow \mathrm{FREE}(Z_1,Z_2)$$

follow immediately.

To prove $\mathrm{FREE}(Z_1,Z_2) \Longrightarrow Z_1 \leq Z_2$, let P be a $Z_2$-complete poset. It is sufficient to prove that P is $Z_1$-complete - this implies $Z_1 \leq Z_2$ by Theorem 1. For any $A \in Z_1(P)$, put $a = \vee e_1(A)$ in $Z_1^{\#}P$; by (FREE) we have $\lambda_P(a) = \vee \lambda_P e_1(A)$ in $Z_2^{\#}P$. Since P is $Z_2$-complete, there is a unique $Z_2$-continuous map $f: Z_2^{\#}P \longrightarrow P$ with $f \cdot e_2 = id_P$. We claim that $\vee A = f(\lambda_P(a))$ in P:

(i) $x \in A$ implies $x = f(e_2(x)) = f(\lambda_P(e_1(x)) \leq f(\lambda_P(a))$,

and

(ii) each upper bound b of A in P fulfils $e_1(a) \leq e_1(b)$ in $Z_1^{\#}P$ and hence, $f(\lambda_P(a)) \leq f(\lambda_P e_1(b)) = f(e_2(b)) = b$. □

**Remark.** Analogous considerations concerning colimits in categories are presented by M.H. Albert and G.M. Kelly [AK]. Given a collection $\Lambda$ of small categories and a small category J, they investigate conditions under which the existence of $\Lambda$-colimits always implies the existence of $\Lambda \cup \{J\}$-colimits. They obtain a characterization theorem analogous to the equivalences

$Z_1 \leq Z_2 \Longleftrightarrow$ COMPL$(Z_2,Z_1) \Longleftrightarrow$ INCL$(Z_1,Z_2)$ above. The role of $Z^{*}P$ is, in the categorical context, played by the $\Lambda$-colimit closure of a category P in its Yoneda embedding into $\mathrm{Set}^{P^{op}}$. Note, however, that although an absolute subset system Z can be viewed as a special collection of categories (viz., of all posets P with $P \in Z(P)$) the categorical result does not imply the order-theoretic one, not even for absolute subset systems: if $Z_1 \leq Z_2$, then $Z_2$-cocompleteness of categories need not imply $Z_1$-cocompleteness. For example, let $Z_2 = S_3$ and let $Z_1$ be the subset system consisting of all $Z_2$-sets, plus subsets of the form $\circ \searrow \circ \nearrow \circ$; then clearly $Z_1$ and $Z_2$ are equivalent. However, a category is $Z_2$-complete if it has binary coproducts, whereas $Z_1$-cocompleteness entails the existence of pushouts. For example, the dual of the category of compact topological spaces has coproducts but not pushouts [A, p. 38].

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(Oblatum 1.9. 1987)