Miroslav Engliš

A note on Toeplitz operators on Bergman spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 2, 217--219

Persistent URL: http://dml.cz/dmlcz/106628

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,2 (1988)

A NOTE ON TOEPLITZ OPERATORS ON BERGMAN SPACES

Miroslav ENGLIŠ

Abstract: Toeplitz operators on the Hardy space H^2 of the unit circle are characterized by the intertwining relation

$$S^*TS=T.$$

In this paper it is shown that no such characterization exists for Toeplitz operators on the Bergman space of the unit disc.

Key words: Toeplitz operators, Bergman space, intertwining relations.

Classification: 47B35

Let H^2 be the Hardy space on the unit circle $\mathbf{T}$ and let $f\in L^\infty(\mathbf{T})$. The Toeplitz operator with the symbol f is the operator on H^2 sending $x\in H^2$ to P_+fx, where P_+ is the orthogonal projection of $L^2(\mathbf{T})$ onto H^2. It is easily seen that

$$T_z^* T_f T_z = T_f \text{ for any } f\in L^\infty(\mathbf{T}).$$

According to a classical result, the converse also holds: if an operator T on H^2 satisfies $T_z^*TT_z=T$, then $T=T_f$ for some $f\in L^\infty(\mathbf{T})$. This result serves as a starting point for the theory of symbols of operators (cf. [1],[2]).

Consider now the Bergman space $H^2(\mathbf{D})$, the (closed) subspace of $L^2(\mathbf{D})$, consisting of functions analytic in the unit disc $\mathbf{D}$. For $f\in L^\infty(\mathbf{D})$, we can define the Toeplitz operator T_f in the same way as above. It is natural to ask if there is a similar intertwining relation characterizing these Toeplitz operators.

The following theorem shows that the answer is negative.

Theorem. Let A, B be operators on $H^2(\mathbf{D})$ such that

$$AT_fB=T_f \text{ for all } f\in L^\infty(\mathbf{D}).$$

Then both A and B are scalar multiples of the identity.

Proof. For any $f\in L^\infty(\mathbf{D})$ and $x\in H^2(\mathbf{D})$, we have

$$T_fT_zx=P_+fP_+zx=P_+fzx=T_{fz}x,$$

i.e. $T_fT_z=T_{fz}$, and so

$$AT_fBT_z=T_fT_z=T_{fz}=AT_{fz}B=AT_fT_zB,$$

consequently

$$AT_f(BT_z-T_zB)=0.$$

We are going to prove $BT_z-T_zB=0$. Suppose on the contrary that there is some $x\neq 0$ in $\mathrm{Ran}(BT_z-T_zB)$. Then, by the last relation,

$$AT_fx=0 \text{ for all } f\in L^{\infty}(D),$$

so the kernel of A contains the set $\{T_fx; f\in L^{\infty}(D)\}$. Consider some $y\in H^2(D)$ orthogonal to this set. Then (dz is the planar Lebesgue measure on D)

$$0=\langle y,T_fx\rangle=\langle y,P_+fx\rangle=\langle y,fx\rangle=\int_D y(z)\overline{f(z)}\overline{x(z)}\,dz$$

for all $f\in L^{\infty}(D)$; because $\bar{x}y\in L^1(D)$, we conclude that $\bar{x}y=0$, and this is only possible if at least one of the analytic functions x, y is identically zero. But $x\neq 0$ by assumption, so y must be zero, which means that our set is dense in $H^2(D)$. Because this set is contained in Ker A, we have A=0, so $T_f=AT_fB=0$ for all f - a contradiction. This proves that $BT_z-T_zB=0$.

Denote $B_1=g\in H^2(D)$. Then

$$Bz^n=BT_z^n1=T_z^nB1=z^ng \text{ for all } n\geq 0,$$

and, consequently,

$$Bp=g\cdot p$$

for all polynomials p(Z). For $x\in H^2(D)$, take a sequence $\{p_n\}$ of polynomials, converging to x in the $H^2(D)$ norm. Then also $Bp_n\rightarrow Bx$ in norm. Because point evaluations are continuous functionals, we have

$$p_n(z)\rightarrow x(z) \text{ and } (Bp_n)(z)\rightarrow (Bx)(z)$$

for any $z\in D$. On the other hand,

$$(Bp_n)(z)=(p_ng)(z)=p_n(z)g(z)\rightarrow x(z)g(z), \text{ for all } z\in D.$$

Consequently, Bx=gx for all $x\in H^2(D)$, i.e. B is the operator of multiplication by $g\in H^2(D)$.

Now $AT_fB=T_f$ for all $f\in L^{\infty}(D)$ implies $B^*T_fA^*=T_f$ for all $f\in L^{\infty}(D)$; thus, we can deduce in the same way that A^* is the operator of multiplication by some $h\in H^2(D)$. Hence $A=P_+\bar{h}=T_{\bar{h}}$.

Summing up, we see that our original relation has the form

$$T_{\bar{h}}T_fT_g=T_f \text{ for all } f\in L^{\infty}(\mathbf{D}).$$

Take f=1 and note that $T_1=T$ and

$$T_{\bar{h}}T_g x=P_+\bar{h}P_+gx=P_+\bar{h}gx=T_{\bar{h}g}x \text{ for all } x\in H^2(\mathbf{D}),$$

because g is analytic in $\mathbf{D}$; so

$$T_{\bar{h}g}=I.$$

For m, n nonnegative integers, z^m and z^n belong to $H^2(\mathbf{D})$, and the last formula gives

$$\langle \bar{h}gz^m,z^n\rangle = \langle z^m,z^n\rangle ,$$

i.e.

$$\int_{\mathbf{D}} z^m\bar{z}^n\overline{h(z)}g(z)\, dz= \int_m z^m\bar{z}^n\, dz.$$

This means that the finite complex measure $(\overline{h(z)}g(z)-1)dz$ on $\mathbf{D}$ is annihilated by all monomials $z^m\bar{z}^n$, $m,n\geq 0$; by linearity and the Stone-Weierstrass theorem, it is annihilated by all functions continuous on $\bar{\mathbf{D}}$, and so is the zero measure and necessarily

$$\bar{h}g=1 \text{ on } \mathbf{D}.$$

But this means that the function $\bar{h}=1/g$ is both analytic and antianalytic, and so must be constant. Q.E.D.

References

[1] V. PTÁK, P. VRBOVÁ: Operators of Toeplitz and Hankel type, Acta Sci. Math. Szeged, in print.

[2] V. PTÁK, P. VRBOVÁ: Lifting intertwining relations, Int. Eq. Oper. Theory, in print.

[3] B. SZÖKEFÁLVY-NAGY, C. FOIAS: Toeplitz type operators and hyponormality, in Dilation theory, Toeplitz operators and other topics, Operator Theory 11, Birkhäuser Verlag 1983, pp. 371-378.

Mathematical Institute of ČSAV, Žitná 25, 115 67 Prague, Czechoslovakia

(Oblatum 18.11. 1987)