Giovanni Rotondaro
An integral formula for closed surfaces and a generalization of $H^p$-theorem

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Abstract: Let $H$, $r$, $p$ be the mean curvature, distance and support functions for an immersion $f:M \to \mathbb{R}^3$ of a closed orientable surface, with area element $dS$. We prove the integral formula \[ \int_M (p^2 - Hpr^2)/r^4 \, dS = 0, \] and deduce that, if $Hp=1$, then $M$ is embedded as a standard sphere.

Key words: Closed surface, support function, mean curvature, sphere.

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Let $f:M \to \mathbb{R}^3$ be a $C^\infty$ immersion of a closed orientable $C^\infty$ surface into Euclidean three-space. Let $n$ denote the unit normal field of $f$, $dS$ the area element, $H$ the mean curvature, $K$ the Gauss curvature, $p=-f \cdot n$ the support function and $r=\|f\|$ the distance function. The well-known $Hp$-theorem ([1],[2]) asserts that if $Hp=1$ and $K>0$ then $M$ is embedded as a standard sphere. In this note we prove an integral formula (Theorem 1) and deduce (Theorem 2) a generalization of $Hp$-theorem which avoids the stringent hypothesis of convexity.

Theorem 1. In the above situation, if the origin of coordinates does not lie in $f(M)$, then
\[ \int_M \frac{p^2 - Hpr^2}{r^4} \, dS = 0. \]

Proof. Consider the conformal diffeomorphism
\[ i: x \in \mathbb{R}^3 \mapsto \frac{c^2}{(r(x))^2} \cdot x \in \mathbb{R}^3 \]
where $c>0$ is a fixed real number. Immerse $M$ in $\mathbb{R}^3$ via $f^* = i \circ f$ and denote by $n^*$, $H^*$, ... the differential-geometric entities associated with $f^*$. Then, by
a routine calculation, we have (*)

\[ df^* \cdot df^* = \frac{c^4}{r^4} df \cdot df \]

\[ dS^* = \frac{c^4}{r^4} ds \]

\[ - df^* \cdot dn^* = \frac{c^2}{r^2} df \cdot dn + \frac{2pc}{r^4} df \cdot df. \]

Hence

\[ K^* dS^* = KdS + 4 \frac{p^2-Hp^2}{r^4} ds. \]

On integration, we have

\[ \int_M \frac{p^2-Hp^2}{r^4} ds = \frac{1}{4} \int_M (KdS-K^*dS^*) = 0 \]

by the Gauss-Bonnet theorem.

**Theorem 2.** If \( H_p=1 \), then \( \mathcal{M} \) is embedded as a standard sphere.

**Proof.** Applying our formula, we have

\[ \int_M \frac{p^2-r^2}{r^4} ds = 0, \]

which implies \( p^2 = r^2 \). Changing orientation, if necessary, this gives \( f = -p_n \).

Then, denoting by subscripts partial derivatives with respect to some local coordinates, \( f_i = -p_i n - p_n i \) \((i=1,2)\), which implies \( p_1 = p_2 = 0 \). Therefore \( p \) is a constant, and so \(|f| = 0\).

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(*) The reader can consult [3,p.110], paying attention to some misprints.

**References**


Università degli Studi di Napoli – Dipartimento di Matematica e Applicazioni "R. Caccioppoli" – Via Mezzocannone, 8 – 80134 Napoli – Italy

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