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On the largest generalized joint spectrum


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Abstract. An explicit description of the largest generalized joint spectrum on a Banach algebra is given. It is proved that this spectrum coincides with the rationally convex joint spectrum introduced by Waelbroeck. This answers questions posed in \[14\].

Key words: Banach algebra, generalized joint spectrum.

Classification: 46H05

Let $A$ be a complex Banach algebra with the unit $1$. By $\sigma^A(a)$, or simply $\sigma(a)$ if there is no confusion, we shall denote the usual spectrum of an element $a \in A$. A generalized joint spectrum on $A$ is a function $\mathcal{G}$ which assigns to each finite collection $\{a_1, \ldots, a_n\}$ of elements in $A$ a compact subset of $\mathbb{C}^n$ (possibly empty) in such a way that the following three conditions are satisfied:

(I) $\mathcal{G}(a_1, \ldots, a_n) \subseteq \prod_{k=1}^n \sigma(a_k)$

(For simplicity we write $\mathcal{G}(a_1, \ldots, a_n)$ instead of $\mathcal{G}(\{a_1, \ldots, a_n\})$;

(II) $p(\mathcal{G}(a_1, \ldots, a_n)) \subseteq \mathcal{G}(p(a_1, \ldots, a_n))$

where $p$ is an arbitrary $n$-tuple of polynomials over $\mathbb{C}$ in $n$ non-commutative indeterminates;

(III) $\mathcal{G}(a_1, \ldots, a_n) \not= \emptyset$ whenever elements $a_1, \ldots, a_n$ are pairwise commuting,

The above definition was given in \[14\]. It was shown that there exists the largest generalized joint spectrum (with respect to the following obvious partial order: $\mathcal{G}_1 \preceq \mathcal{G}_2$ if and only if $\mathcal{G}_1(a_1, \ldots, a_n) \subseteq \mathcal{G}_2(a_1, \ldots, a_n)$ for all finite subsets $\{a_1, \ldots, a_n\}$ of $A$).

It was asked if one can give a simple characterization of this spectrum.
The bicommutant joint spectrum was given as a candidate for the largest generalized joint spectrum.

The purpose of the present paper is to give a description of this largest spectrum. We prove that it coincides with the rationally convex joint spectrum introduced by Waelbroeck. We also show (see Example 2 below) that it differs from the bicommutant joint spectrum in general.

Following L. Waelbroeck (see [61 or 171]) we shall give

**Definition.** Let $a_1,\ldots,a_n \in A$ (we do not assume them to commute). The rationally convex joint spectrum of the $n$-tuple $(a_1,\ldots,a_n)$ is the set

$$\mathcal{J}(a_1,\ldots,a_n) = \{(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n : p(\lambda_1,\ldots,\lambda_n) \in \mathcal{J}(p(a_1,\ldots,a_n))$$

for every $p \in P_n\}$$

where $P_n$ denotes the set of all polynomials over $\mathbb{C}$ in $n$ non-commutative indeterminates.

**Theorem.** The largest generalized joint spectrum and the rationally convex joint spectrum coincide.

**Proof.** First we show that the rationally convex joint spectrum is a generalized joint spectrum, i.e. it satisfies conditions (I) - (III).

To see that (I) is fulfilled, take $p_j(x_1,\ldots,x_n) = x_j \ (j=1,\ldots,n)$. Then

$$(\lambda_1,\ldots,\lambda_n) \in \mathcal{J}(a_1,\ldots,a_n) \quad \text{implies} \quad \lambda_j = p_j(\lambda_1,\ldots,\lambda_n) \in \mathcal{J}(p_j(a_1,\ldots,a_n)) = \mathcal{J}(a_j)$$

which gives (I).

It is also clear that (II) holds true. If $(\alpha_1,\ldots,\alpha_m) \in \mathcal{J}(a_1,\ldots,a_n)$ then $(\lambda_1,\ldots,\lambda_n) = (\alpha_1,\ldots,\alpha_m) \in \mathcal{J}(a_1,\ldots,a_n)$ for some $(\lambda_1,\ldots,\lambda_n) \in \mathcal{J}(a_1,\ldots,a_n)$. Taking an arbitrary $q \in P_m$ we get $q \cdot p \in P_n$ and $(q \cdot p)(\lambda_1,\ldots,\lambda_n) = (q \cdot (p(a_1,\ldots,a_n)))$ which means that $(\alpha_1,\ldots,\alpha_m) \in \mathcal{J}(p(a_1,\ldots,a_n))$.

Finally (III) is trivially satisfied since we always have $\mathcal{J}(a_1,\ldots,a_n) \subseteq \mathcal{J}(a_1,\ldots,a_n)$ where $\mathcal{J}(a_1,\ldots,a_n)$ denotes the Harte's spectrum (= the union of the left and the right joint spectra) of the $n$-tuple $(a_1,\ldots,a_n)$.

Moreover we have $\mathcal{J}(a_1,\ldots,a_n) \subseteq \mathcal{J}(a_1,\ldots,a_n)$ for every generalized joint spectrum $\mathcal{J}$ on $A$. Indeed, if $(\lambda_1,\ldots,\lambda_n) \in \mathcal{J}(a_1,\ldots,a_n)$, then by (II) and (I) $p(\lambda_1,\ldots,\lambda_n) \in \mathcal{J}(p(a_1,\ldots,a_n)) \subseteq \mathcal{J}(p(a_1,\ldots,a_n))$ for every $p \in P_n$.

Hence $(\lambda_1,\ldots,\lambda_n) \in \mathcal{J}(a_1,\ldots,a_n)$ and we are done. So, $\mathcal{J}$ is the largest generalized joint spectrum.
Let $K$ be a compact subset of $\mathbb{C}^n$, $(1 \leq n < \infty)$. Then the rationally convex hull $\overline{K}$ of $K$ is defined (see [11] or [17]) as the set of all $n$-tuples $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $|f(\lambda_1, \ldots, \lambda_n)| \leq \sup \{ |f(z_1, \ldots, z_n)| : (z_1, \ldots, z_n) \in K \}$ for every rational function $f$ analytic on the set $K$. Equivalently, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ belongs to $\overline{K}$ if and only if $p(\lambda_1, \ldots, \lambda_n) \in p(K)$ for every polynomial $p \in \mathbb{P}_n$. Next corollary shows that if $a_1, \ldots, a_n$ are pairwise commuting elements of a Banach algebra $A$ then $\sigma(a_1, \ldots, a_n)$ is equal to the rationally convex hull of $\sigma(a_1, \ldots, a_n)$. Example 1 below will show that this is not the case when $a_1, \ldots, a_n$ do not commute.

**Corollary 1.** Let $a_1, \ldots, a_n$ be pairwise commuting elements of a Banach algebra $A$. Then $\sigma(a_1, \ldots, a_n)$ is the rationally convex hull of the Harte's spectrum $\sigma(a_1, \ldots, a_n)$.

**Proof.** If elements $a_1, \ldots, a_n$ are pairwise commuting then the Harte's spectrum has the spectral mapping property. In particular, $\sigma(p(a_1, \ldots, a_n)) = p(\sigma(a_1, \ldots, a_n))$ for all $p \in \mathbb{P}_n$ (see [2]). This implies immediately that $\sigma(a_1, \ldots, a_n)$ is the rationally convex hull of the Harte's spectrum $\sigma(a_1, \ldots, a_n)$.

**Corollary 2.** Let $a_1, \ldots, a_n$ be elements of a Banach algebra $A$. Then $\sigma(a_1, \ldots, a_n) \subseteq \sigma(p(a_1, \ldots, a_n))$ where $\sigma([a_1, \ldots, a_n])$ denotes the Harte's spectrum of the n-tuple $(a_1, \ldots, a_n)$ in the algebra $[a_1, \ldots, a_n]$ generated by $a_1, \ldots, a_n$ and the unit.

**Proof.** Let $(\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n)$. Then $p(\lambda_1, \ldots, \lambda_n) \in \sigma(p(a_1, \ldots, a_n))$ for every $p \in \mathbb{P}_n$ (see [2]). Hence $(\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n)$ and the rationally convex hull of $\sigma(a_1, \ldots, a_n)$ is contained in $\sigma(a_1, \ldots, a_n)$.

Property (II) implies that $\sigma$ is translation invariant, i.e.

$(\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n)$ if and only if $(0, \ldots, 0) \in \sigma(a_1 - \lambda_1, \ldots, a_n - \lambda_n)$.

Therefore to prove the second inclusion it is sufficient to show that $(0, \ldots, 0) \in \sigma(a_1, \ldots, a_n)$ implies $(0, \ldots, 0) \in \sigma([a_1, \ldots, a_n])$.

Suppose $(0, \ldots, 0) \in \sigma(a_1, \ldots, a_n)$. Then $M = \{ p(a_1, \ldots, a_n) : p \in \mathbb{P}_n, p(0, \ldots, 0) \}$ is a linear subspace of codimension 1 in the algebra $[a_1, \ldots, a_n]$.
consisting of singular elements in $A$ (and thus singular in $[a_1, \ldots, a_n]$). By
the Gleason-Kahane-Żelazko theorem (see [8], p. 87) $M$ is a maximal two-sided
ideal in $[a_1, \ldots, a_n]$ and $(0, \ldots, 0) \in [a_1, \ldots, a_n]$.

Now we proceed to the previously mentioned examples.

Example 1 (cf. [5], Example 1). Let $A$ be the algebra $M_5(\mathbb{C})$ of all $5 \times 5$
matrixes with complex entries. Take the following two elements of $A$:

$$a_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } a_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we have $a_1^3 = a_2^3 = 0$. Hence $\mathfrak{C}^A(a_1) = \mathfrak{C}^A(a_2) = \{0\}$. This implies $\mathfrak{C}^A(a_1, a_2) \subset \mathfrak{C} \{0, 0\}$. Further $a_1 a_3 + a_2 a_4 = 1$ and $a_2 a_1 + a_4 a_2 = 1$ where

$$a_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } a_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $\mathfrak{C}^A(a_1, a_2) \neq \emptyset$. Let $B = [a_1, a_2]$. If we assign to each element $b \in B$ the entry of $b$ which is placed in the
third row and the third column, then we shall get a linear functional $\varphi$ on
$B$. We prove that $\varphi$ is multiplicative on $B$. By the Gleason-Kahane-Żelazko theo-
rem it is sufficient to show that $\varphi(a_1a_1 \ldots a_k) = 0$ for all finite pro-
ducts of $a_1$ and $a_2$, i.e. for all $k \in \{1, 2, \ldots \}$, $i_1, \ldots, i_k \in \{1, 2\}$. This is clear
if $a_1 = a_2$ as the third row is then equal to zero. From the same reason
$\varphi(a_1 \ldots a_k) = 0$ if $a_1 = a_1$, $a_2 = a_2$. The rest follows from the relations
$\varphi(a_1^2) = \varphi(a_1 a_2) = 0$, $a_1^3 = a_2 a_2 = 0$ and $a_1^2 a_2 a_1 = a_1^2$ which can be checked directly.
Thus $(0, 0) = (\varphi(a_1), \varphi(a_2)) \in \mathfrak{C}^B(a_1, a_2)$ and $p(0, 0) = p(\varphi(a_1), \varphi(a_2)) =
\varphi(p(a_1, a_2)) \in \mathfrak{C}^B(p(a_1, a_2))$ for every polynomial $p \in F_2$.

Further $\mathfrak{C}^B(p(a_1, a_2)) \subset \mathfrak{C}^A(p(a_1, a_2))$ as $\dim B < \infty$.

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Hence $(0,0) \not\in \sigma^A(a_1,a_2)$ and $\sigma^A(a_1,a_2)$ is not the rationally convex hull of $\sigma^A(a_1,a_2) = \emptyset$.

**Example 2.** Let $K = \{ (z_1,z_2) \in \mathbb{C}^2, |z_2| \leq |z_1| \}$. Then $K$ is compact but not rationally convex. Its rationally convex hull $\tilde{K}$ is equal to

$$\tilde{K} = \{ (z_1,z_2) \in \mathbb{C}^2, |z_1| \leq 1, |z_2| \leq 1 \}$$

(see [1], p. 76).

Let $A = C(K)$ be the algebra of all continuous complex-valued functions on $K$. Then the bicommutant joint spectrum $\sigma^\ast$ (cf. [4]) coincides with the Hart's spectrum on this algebra. Put $\sigma_1^K(z_1,z_2) = z_1$ and $\sigma_2^K(z_1,z_2) = z_2$. Then

$$\sigma^\ast(\sigma_1^K, \sigma_2^K) = \sigma(\sigma_1^K, \sigma_2^K) = \tilde{K} \neq \sigma^\ast(\sigma_1^K, \sigma_2^K) = \sigma(\sigma_1^K, \sigma_2^K).$$

Thus we see that the rationally convex joint spectrum is larger than the bicommutant spectrum.

**References**


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