Commentationes Mathematicae Universitatis Carolinae

Jaroslav Ježek

Minimal bounded varieties

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 2, 261--265

Persistent URL: `http://dml.cz/dmlcz/106635`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,2 (1988)

MINIMAL BOUNDED VARIETIES

J. JEŽEK

Abstract: We describe all the minimal bounded varieties of algebras with one k-ary operation (for any $k \geq 2$). The collection of these varieties is found to be independent and to consist of the variety of constant algebras plus an infinite number of varieties that are equivalent with the varieties of point algebras studied by T. Evans and M. Saade.

Key words: Bounded variety, minimal variety, equational theory.

Classification: 08B15

A variety V of algebras of a finite similarity type ρ is said to be bounded if there exists a natural number m such that every term of the type ρ is equivalent, with respect to V, with some term of length not greater than m. An equational theory is said to be bounded if the corresponding variety is bounded. In [2] we have studied bounded equational theories. We have introduced a class of special equational theories which we called well-placed; we have proved that every well-placed equational theory is bounded and, conversely, in the case of a similarity type ρ consisting of a single operation symbol, that every bounded equational theory is contained in a well-placed equational theory; and we have proved, under the assumption on ρ, that an absorptive equational theory (an equational theory containing an equation $x \approx t$ where x is and t is not a variable) is bounded iff it is well-placed. This allowed us to find all the equationally complete bounded equational theories (for a type ρ with a single operation symbol). The aim of the present paper is to describe the corresponding varieties; they are, of course, just the minimal bounded varieties. We shall show that this collection coincides (up to the equivalence of varieties and after omitting the variety C of constant ρ-algebras) with the varieties of point algebras introduced and studied in the papers [1],[3],[4],[5] and that it is an infinite collection of independent varieties.

Throughout this paper let ρ be a fixed similarity type consisting of one operation symbol F which is of arity $k \geq 2$. By a place we mean a finite word over the alphabet $\{0,\dots,k-1\}$; infinite words over $\{0,\dots,k-1\}$ are called directions. The length of a place e is denoted by $\lambda(e)$. A nonempty place e is said to be irreducible if it is not a power of any place shorter than e. Given a place $e=c_0\dots c_{n-1}$ (where $c_i \in \{0,\dots,k-1\}$ for all i), we denote by $\mathrm{cyc}(e)$ the set of the places $c_i\dots c_{n-1}c_0\dots c_{i-1}$, with i ranging over $\{0,\dots,n-1\}$. Given a nonempty place e, we denote by e^{ω} the unique direction h such that e^i is an initial segment of h for any positive integer i; and we put $J(e)=\{f^{\omega} : f \in \mathrm{cyc}(e)\}$. For a ρ-term t and a place e we denote by $t[e]$ the subterm of t occurring at the place e; it can be defined recursively by $t[e]=t$ if t is empty, $t[e]=t_i[f]$ if $t=F(t_0,\dots,t_{n-1})$ and $e=if$, and $t[e]=\emptyset$ in all other cases. For a term t and a direction h denote by $\tau_t(h)$ the unique initial segment e of h such that $t[e]$ is a variable and put $t[h]=t[e]$.

We have proved in [2] that the following is the list of all minimal bounded varieties of type ρ: the variety C of constant ρ-algebras (satisfying the equation $F(x_0,\dots,x_{k-1}) \approx F(y_0,\dots,y_{k-1})$) and, for any irreducible place e, the variety V_e, the equational theory E_e of which consists of the equations $s \approx t$ such that

$$s[h]=t[h] \text{ and } \lambda(\tau_s(h)) \equiv \lambda(\tau_t(h)) \bmod \lambda(e)$$

for all $h \in J(e)$. Moreover, $V_e=V_f$ for two irreducible places e, f iff $\mathrm{cyc}(e)=\mathrm{cyc}(f)$. In order to be able to say more about the varieties V_e, we introduce the following notation.

Let e be an irreducible place of length n. Denote by $c(0),\dots,c(n-1)$ the sequence of the letters of e and put

$$e_0=c(0)\dots c(n-1)=e,$$
$$e_1=c(1)\dots c(n-1)c(0),$$
$$\dots$$
$$e_{n-1}=c(n-1)c(0)\dots c(n-2),$$

so that $\mathrm{cyc}(e)=\{e_0,\dots,e_{n-1}\}$. For $i \in \{0,\dots,n-1\}$ and $j \in \{1,\dots,n\}$ denote by $f_{i,j}$ the final segment of e_i of length j. Since e_i is irreducible, it is easy to see that none of the places $f_{i,1}e_i,\dots,f_{i,n}e_i$ is an initial segment of another one. From this it follows that there exists a term α_i such that $\mathrm{var}(\alpha_i)=\{x\}$ (where x is a fixed variable) and $f_{i,j}e_i$ is an occurrence of x in α_i for any $j \in \{1,\dots,n\}$. For every $i=0,\dots,n-1$ let us fix one such term α_i. Also, none of the places $e_0,\dots,e_{n-1}$ is an initial segment of another one and so there exists a term γ such that $\mathrm{var}(\gamma)=\{x_0,\dots,x_{n-1}\}$ and $\gamma[e_i]=x_i$ for $i=0,\dots,n-1$.

For any term t denote by $\alpha_i(t)$ the term obtained from α_i by substituting t for x. Denote by $\gamma(t_0,\dots,t_{n-1})$ the term obtained from γ by substituting $t_0,\dots,t_{n-1}$ for $x_0,\dots,x_{n-1}$.

Lemma 1. The following equations belong to the equational theory E_e:

(1) $\alpha_i \approx F(\alpha_i,\dots,\alpha_i)$ for $i=0,\dots,n-1$;

(2) $\gamma(\alpha_0,\dots,\alpha_{n-1}) \approx x$;

(3) $\alpha_i(\gamma) \approx \alpha_i(x_i)$ for $i=0,\dots,n-1$;

(4) $\alpha_i(F(x_0,\dots,x_{n-1})) \approx \alpha_m(x_{c(i)})$ for $i=0,\dots,n-1$ (where $m=i+1$ for $i<n-1$ and $m=0$ for $i=n-1$).

Proof. (1) Let $i \in \{0,\dots,n-1\}$; put $s=\alpha_i$ and $t=F(\alpha_1,\dots,\alpha_i)$. Let $h \in J(e)$. Then $h=f_{i,j}e_ie_i\dots$ for some $j \in \{1,\dots,n\}$. Evidently, $\tau_s(h)=f_{i,j}e_i$. If $j \neq 1$ then $f_{i,j-1}e_i$ is an occurrence of x in s, so that each of the places $0f_{i,j-1}e_i$, $1f_{i,j-1}e_i,\dots,(n-1)f_{i,j-1}e_i$ is an occurrence of x in t and especially $f_{i,j}e_i$ is an occurrence of x in t, which means that $\tau_t(h)=f_{i,j}e_i$. If $j=1$ then $f_{i,j}=c(n-1)$; since e_ie_i is an occurrence of x in s, $f_{i,j}e_ie_i$ is an occurrence of x in t and $\tau_t(h)=f_{i,j}e_ie_i$. In both cases we get $\lambda(\tau_s(h)) \equiv \lambda(\tau_t(h))$ mod n. Of course, $s[h]=t[h]=x$.

(2) Put $s=\gamma(\alpha_0,\dots,\alpha_{n-1})$ and $t=x$. Let $h \in J(e)$, so that $h=e_ie_i\dots$ for some i. We have $\tau_s(h)=e_ie_ie_i$, $\tau_t(h)=\emptyset$ and $s[h]=t[h]=x$.

(3) Let $i \in \{0,\dots,n-1\}$; put $s=\alpha_i(\gamma)$ and $t=\alpha_i(x_i)$. Let $h \in J(e)$. Then $h=f_{i,j}e_ie_i\dots$ for some $j \in \{1,\dots,n\}$. We have $\tau_s(h)=f_{i,j}e_ie_i$, $\tau_t(h)=f_{i,j}e_i$ and $s[h]=t[h]=x_i$.

(4) Let $i \in \{0,\dots,n-1\}$ and define m as above; put $s=\alpha_i(F(x_0,\dots,x_{n-1}))$ and $t=\alpha_m(x_{c(i)})$. Let $h \in J(e)$. We have $h=f_{i,j}e_ie_i\dots$ for some $j \in \{1,\dots,n\}$. We have $\tau_s(h)=f_{i,j}e_ic(i)$ and $s[h]=x_{c(i)}$. Of course, $t[h]=x_{c(i)}$. It remains to prove $\lambda(f_{i,j}e_ic(i)) \equiv \lambda(\tau_t(h))$ mod n. If $j<n$ then $h=f_{i,j}c(i)e_me_m\dots$, so that $\tau_t(h)=f_{i,j}c(i)e_m$. If $j=n$ then $h=e_ie_i\dots=c(i)e_me_m\dots$, so that $\tau_t(h)=c(i)e_m$; in this case we have $\lambda(f_{i,j})=n$.

For every nonempty set M and every irreducible place $e=c(0)\dots c(n-1)$ we define an algebra $\mathcal{A}_{M,e}$ of the type $\rho=\{F\}$ with the underlying set M^n by

$$F((a_{0,0},\dots,a_{0,n-1}),\dots,(a_{0,k-1},\dots,a_{n-1,k-1}))=$$
$$=(a_{1,c(0)},a_{2,c(1)},\dots,a_{n-1,c(n-2)},a_{0,c(n-1)}).$$

Theorem 2. Let e be an irreducible place. The following are equivalent for an algebra A of the type $\rho=\{F\}$:

(i) A belongs to the variety V_e;

(ii) A satisfies the $3n+1$ equations (1) - (4) from Lemma 1;

(iii) $A \cong \mathcal{A}_{M,e}$ for a nonempty set M.

Proof. (i) implies (ii) by Lemma 1. in order to prove that (ii) implies (iii), let A be an algebra satisfying the equations (1) - (4) and denote by M the set of idempotents of A, i.e. elements a such that $F(a,\dots,a)=a$. For $a \in A$ put $\varphi(a)=(\alpha_0(a),\dots,\alpha_{n-1}(a))$. By (1), φ is a mapping of A into M^n. By (4), φ is a homomorphism of A into $\mathcal{A}_{M,e}$. By (2), φ is injective. By (3), φ maps A onto M^n.

It remains to prove that (iii) implies (i). Since V_e is a nontrivial variety, it contains arbitrarily large algebras; but (i) implies (iii) and so V_e contains an algebra $\mathcal{A}_{K,e}$ for a set K such that M is a subset of K. Clearly, $\mathcal{A}_{M,e}$ is a subalgebra of $\mathcal{A}_{K,e}$ and hence $\mathcal{A}_{M,e}$ belongs to V_e, too.

Theorem 3. Let e be an irreducible place. The following are true:

(i) A mapping φ of $\mathcal{A}_{M,e}$ into $\mathcal{A}_{k,e}$ is a homomorphism iff there is a mapping f of M into K such that $\varphi(a_0,\dots,a_{n-1})=(f(a_0),\dots,f(a_{n-1}))$ for all $(a_0,\dots,a_{n-1}) \in M^n$.

(ii) An algebra A is a subalgebra of $\mathcal{A}_{M,e}$ iff $A=\mathcal{A}_{K,e}$ for a nonempty subset K of M.

(iii) The congruence lattice of $\mathcal{A}_{M,e}$ is isomorphic to the lattice of equivalences of the set M.

(iv) The product of a family $\mathcal{A}_{M(i),e}$ $(i \in I)$ is isomorphic with the algebra $\mathcal{A}_{M,e}$ where M is the product of the sets M_i $(i \in I)$.

(v) Every infinite algebra in V_e is V_e-free; an algebra $\mathcal{A}_{M,e}$ with M finite is V_e-free iff the cardinality of M is a multiple of n.

Proof. It follows from the equations (1) and (2) that a homomorphism between two algebras A, B from V_e is uniquely determined by its restriction to the set of idempotents of A. One can easily verify for any mapping f that the mapping φ from (i) is a homomorphism. From this we get (i). The other assertions are easy consequences. The assertion (v) is proved in [5].

Theorem 4. Let e, f be two irreducible places of the same length n. Then the varieties V_e and V_f are equivalent.

Proof. It follows easily from Theorem 3. Let us remark that the varieties V_e and V_f are equivalent even if they are of different similarity types $\rho = \{F\}$ and $\sigma = \{G\}$. The varieties V_e are thus all equivalent to varieties of groupoids.

A finite collection $V_0,\dots,V_{m-1}$ of varieties of a type ρ is said to be independent if there exists a ρ-term t such that $\mathrm{var}(t)=\{x_0,\dots,x_{m-1}\}$ and the equation $t\approx x_i$ is satisfied in V_i for any $i=0,\dots,m-1$. An infinite collection $\mathcal{C}$ of varieties is said to be independent if any finite subcollection of $\mathcal{C}$ is independent.

For a survey of various properties of independent collections of varieties see [6].

Theorem 5. The collection of all the minimal bounded varieties of the type ρ that are different from C is independent.

Proof. Let $V_0,\dots,V_{m-1}$ be a finite subcollection of this collection. For any $i=0,\dots,m-1$ we can write $V_i=V_{e(i)}$ where e(i) is an irreducible place. The sets $J(e(0)),\dots,J(e(m-1))$ are pairwise disjoint. There exists a positive integer p such that p is a common multiple of the lengths of e(i) $(i=0,\dots,m-1)$ and whenever h, k are two different directions from $J(e(0))\cup\dots\cup J(e(m-1))$ then the initial segments of h and k of length p are different. There exists a term t such that $\mathrm{var}(t)=\{x_0,\dots,x_{m-1}\}$ and whenever e is an initial segment of a direction from J(e(i)) of length p then $t[e]=x_i$. Clearly, the equation $t\approx x_i$ is satisfied in V_i.

References

[1] T. EVANS: Products of points - some simple algebras and their identities, Amer. Math. Monthly 74(1967), 362-372.

[2] J. JEŽEK, G.F. McNULTY: Bounded and well-placed theories in the lattice of equational theories (to appear).

[3] M. SAADE: A comment on a paper by Evans, Zeitschr. f. math. Logik u. Grundl.d. Math. 15(1969), 97-100.

[4] M. SAADE: Generating operations of point algebras, Journal Comb. Theory 11(1971), 93-100.

[5] M. SAADE: A note on some varieties of point algebras, Czechoslovak Math. J. 29(1979), 21-26.

[6] W. TAYLOR: The fine spectrum of a variety, Algebra Universalis 5(1975), 263-303.

Matematicko-fyzikální fakulta, Univerzita Karlova, Sokolovská 83, 18600
Praha 8, Czechoslovakia

(Oblatum 1.2. 1988)