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# MORREY-CAMPANATO SPACES ON MANIFOLDS

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**Abstract:** The paper describes Morrey-Campanato spaces on compact manifolds.

**Keywords:** Function spaces, manifolds.

**Classification:** 46E35

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**1. Introduction and notations.** Usually, Morrey-Campanato spaces are defined on bounded domains of the Euclidean n-space $R_n$ (see [2, 3, 4]). Our aim is to extend two scales of spaces to more general underlying structures. Good candidates are closed Riemannian or - simpler - compact manifolds.

Let $\Omega \subset R_n$ be a bounded domain. For $0<\rho$ , $\sigma \le \infty$ , $0 \le \lambda < \infty$, $1 \le p < \infty$ , $x_0 \in R_n$ set

$$B_\rho(x_0)=\{x \in R_n \mid |x-x_0|<\rho\} , \quad \Omega_\rho(x_0)=\Omega \cap B_\rho(x_0),$$

$$f_{x_0,\rho} = \frac{1}{|\Omega_\rho(x_0)|} \cdot \int_{\Omega_\rho(x_0)} f(x)dx,$$

$$[f]_{p,\lambda,\Omega,\sigma} = \left[ \sup_{\substack{0<\rho\le\sigma \\ x_0 \in \Omega}} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f-f_{x_0,\rho}|^p \, dx \right]^{1/p} .$$

Hereby, for sets $A \subset R_n$, $|A|$ denotes the Lebesgue measure and, for $x \in R_n$, $|x|$ denotes the Euclidean norm.

For handling on a manifold N there are corresponding counterparts. Let d(P,Q) denote the geodesic distance of $P,Q \in N$. On complete Riemannian manifolds d(P,Q) coincides with the length of a minimizing geodesic, joining P and Q, according to the Hopf-Rinow-theorem. The ball $B_\rho^N(P)=\{Q \in N \mid d(P,Q)<\rho\}$ is open in the Hausdorff topology of N. Assume N to be orientable. Then we can integrate with respect to the standard n-form $\eta=\sqrt{|\det g_{ik}|}\,dx^1 \wedge \ldots \wedge dx^n$. Here $g_{ik}$ are the components of the metric g in local coordinates $x^1,\ldots,x^n$. Set

$|A| = \int_A \eta$ for measurable $A \subset N$.

$$f_{p,\varrho} = \frac{1}{|B_\varrho^N(P)|} \int_{B_\varrho^N(P)} f\eta ,$$

$$[f]_{p,\lambda,N,\delta} = \left[ \sup_{\substack{0<\varrho\leq\delta \\ P\in N}} \varrho^{-\lambda} \int_{B_\varrho^N(P)} |f-f_{p,\varrho}|^p \right]^{1/p} .$$

$\bar{A}$ stands for the closure of A (in a manifold N). We omit $\delta$ if $\delta = \infty$.

**2. Preliminaries: Morrey-Campanato spaces on domains.** For sake of redundance we recall first some usual basic material.

**Definition 0.** Let $L_p(\Omega)$ be the Lebesgue space on a bounded domain $\Omega \subset R_n$, $\varrho$, $\lambda$, p as above. The Morrey space $L^M_{p,\lambda}(\Omega)$ and the Campanato space $L^C_{p,\lambda}(\Omega)$ are defined as

$$L^M_{p,\lambda}(\Omega) = \{ f \in L_p(\Omega) \mid \| f | L^M_{p,\lambda}(\Omega) \| := \left[ \sup_{\substack{\varrho>0 \\ x_0\in\Omega}} \varrho^{-\lambda} \int_{\Omega_\varrho(x_0)} |f(x)|^p \, dx \right]^{1/p} < +\infty \}$$

and

$$L^C_{p,\lambda}(\Omega) = \{ f \in L_p(\Omega) \mid \| f | L^C_{p,\lambda}(\Omega) \| := \| f | L_p(\Omega) \| + [f]_{p,\lambda,\Omega} < \infty \} .$$

**Theorem o.** Let $\Omega, p, \lambda$ be as above. The following assertions hold:

(i) $L^M_{p,\lambda}(\Omega)$, $L^C_{p,\lambda}(\Omega)$ are Banach spaces. The following imbeddings are continuous:

$L^M_{p,\lambda}(\Omega) \subset L^C_{p,\lambda}(\Omega) \subset L_p(\Omega)$,

$L^C_{q,\nu}(\Omega) \subset L^C_{p,\lambda}(\Omega)$ for $1 \leq p \leq q < \infty$, $\frac{\lambda - n}{p} < \frac{\nu - n}{q}$ .

(ii) For $0 \leq \lambda < n$ and $\Omega$ with the Lipschitz boundary $L^M_{p,\lambda}(\Omega)$ and $L^C_{p,\lambda}(\Omega)$ are isomorphic,

(iii) $L^M_{p,n}(\Omega)$ is isomorphic to $L_\infty(\Omega)$. There are $\Omega$ with $L^M_{p,n}(\Omega) \subsetneqq L^C_{p,n}(\Omega)$.

(iv) Let $n < \lambda \leq n+p$, $\alpha = \frac{\lambda - n}{p}$ . If $\Omega$ has a Lipschitz boundary then $L^C_{p,\lambda}(\Omega)$ is isomorphic to $C^\alpha(\bar{\Omega})$, the usual Hölder space with the norm

$$\| f | C^\alpha(\bar{\Omega}) \| = \sup_{x\in\bar{\Omega}} |f(x)| + \sup_{\substack{x,y\in\bar{\Omega} \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} .$$

For the proof see [3, 4].

The spaces $L^M_{p,\lambda}(\Omega)$ and $L^C_{p,\lambda}(\Omega)$ characterize local properties. In fact, we have the following

**Proposition 1.** Let $\Omega, \Omega_i' \subset R_n$, $i=1,\dots,m$, be bounded domains, $\bar{\Omega} = \bigcup_{i=1}^{m} \Omega_i'$, $\Omega_i = \Omega \cap \Omega_i'$, $\delta > 0$. Then

(i) $\|f|L_p(\Omega)\| + [f]_{p,\lambda,\Omega,\delta}$ is an equivalent norm in $L^C_{p,\lambda}(\Omega)$,

(ii) $\sum_{i=1}^{m} \|f|L^C_{p,\lambda}(\Omega_i)\|$ is an equivalent norm in $L^C_{p,\lambda}(\Omega)$.

**Proof.**

**Step 1.** Clearly $[f]_{p,\lambda,\Omega,\delta} \leq [f]_{p,\lambda,\Omega}$. On the other hand,

$$[f]_{p,\lambda,\Omega} \leq [f]_{p,\lambda,\Omega,\delta} + \left[\sup_{\substack{\rho>\delta \\ x_0 \in \Omega}} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f(x)-f_{x_0,\rho}|^p \, dx\right]^{1/p} \leq$$

$$\leq [f]_{p,\lambda,\Omega,\delta} + 2\,\delta^{-\lambda/p} \|f|L_p(\Omega)\|$$

by Hölder inequality. This proves (i).

**Step 2.** We have $\Omega_i \subset \Omega$ and therefore $[f]_{p,\lambda,\Omega_i} \leq [f]_{p,\lambda,\Omega}$. This yields $\sum_{i=1}^{m} \|f|L^C_{p,\lambda}(\Omega_i)\| \leq m \, \|f|L^C_{p,\lambda}(\Omega)\|$.

**Step 3.** We show that there exists a $\delta > 0$, such that for all $x \in \Omega$ and $\rho < \delta$ one can find a $\Omega_j$ with $\Omega_\rho(x) \subset \Omega_j$.

Assume the contrary. Then we have $x_k \in \Omega$ and $\rho_k \to 0$ such that $\Omega_{\rho_k}(x_k)$ is not contained in one of the $\Omega_j$'s. Since $\bar{\Omega}$ is compact there is an $x_0 \in \bar{\Omega}$ with $x_0 = \lim_{k\to\infty} x_k$ for some proper subsequence. Now $x_0 \in \Omega_i'$ for at least one i. Hence $B_\varepsilon(x_0) \subset \Omega_i'$ for some $\varepsilon > 0$. But for large k we get $\Omega_{\rho_k}(x_k) \subset B_\varepsilon(x_0) \cap \cap\, \Omega \subset \Omega_i' \cap \Omega = \Omega_i$, which yields a contradiction. (This is more or less the Lebesgue lemma.)

So we have $[f]_{p,\lambda,\Omega,\delta} \leq \max_{i=1,\dots,m} [f]_{p,\lambda,\Omega_i,\delta}$ which completes the proof.

**Remark 1.** With obvious modifications the above proposition is also true for $L^M_{p,\lambda}(\Omega)$.

Next we show that Morrey-Campanato spaces are invariant under diffeomorphism.

**Proposition 2.** Let $\Omega \subset R_n$ be a bounded domain and $\varphi : \Omega \to \Omega' = \varphi(\Omega)$ be a diffeomorphism, such that

$\left| \frac{\partial \varphi^j(x)}{\partial x^k} \right|$ and $\left| \frac{\partial (\varphi^{-1})^j}{\partial y^k} \right|$ are bounded on $\Omega$, and $\Omega'$, respectively, for all $1 \leq j,\, k \leq n$. Then $\varphi^* : f \to f \circ \varphi$ is an isomorphism of $L^M_{p,\lambda}(\Omega')$ onto

$L^C_{p,\lambda}(\Omega)$.

**Proof.** We prove the proposition for $L^C_{p,\lambda}$. Let $f\in L^C_{p,\lambda}(\Omega')$. Then

$$\|\varphi^* f|L_p(\Omega)\|^p=\int_\Omega |f(\varphi(x))|^p\,dx=\int_{\Omega'}|f(y)|^p\,\mathcal{J}_{\varphi^{-1}}(y)\,dy\le c\,\|f|L_p(\Omega')\|^p$$

by virtue of the boundedness condition. Here $\mathcal{J}_{\varphi^{-1}}$ denotes the Jacobian of $\varphi^{-1}$. From the well-known inequality (see [3])

$$[g]_{p,\lambda,\Omega}\le 2\left[\sup_{\substack{\rho>0\\ x_0\in\Omega}}\rho^{-\lambda}\left(\inf_C\int_{\Omega_\rho(x_0)}|g(x)-c|^p\,dx\right)\right]^{1/p}$$

we conclude

$$[\varphi^* f]_{p,\lambda,\Omega}\le 2\left[\sup_{\substack{\rho>0\\ x_0\in\Omega}}\rho^{-\lambda}\int_{\Omega_\rho(x_0)}|f\circ\varphi(x)-s|^p\,dx\right]^{1/p},$$

for an s we fix later.

Our boundedness condition provides a constant c which is independent of $x_0$ and $\rho$ such that $\varphi(\Omega_\rho(x_0))\subset\Omega'_{c\rho}(\varphi(x_0))$. Consequently

$$[\varphi^* f]_{p,\lambda,\Omega}\le c'\left[\sup_{\substack{\rho>0\\ y_0\in\Omega'}}\rho^{-\lambda}\int_{\Omega'_\rho(y_0)}|f(y)-s|^p\mathcal{J}_{\varphi^{-1}}(y)\,dy\right]^{1/p}.$$

Putting $s=f_{\rho,y_0}$ it follows

$$[\varphi^* f]_{p,\lambda,\Omega}\le c''[f]_{p,\lambda,\Omega'}.$$

Now one can replace $\varphi$ by $\varphi^{-1}$. The proof is complete.

The last propositions enable us to define $L^M_{p,\lambda}$ and $L^C_{p,\lambda}$ on compact manifolds via a local procedure.

### 3. Morrey-Campanato spaces on compact manifolds

**Definition 2'.** Let N be a compact manifold, and $(U_i,\varphi_i)$, $i=1,\dots,m$ a collection of charts which cover N (i.e.

$N=\bigcup_{i=1}^m U_i$, $U_i$ open, $\varphi_i:U_i\to\varphi_i(U_i)\subset R_n$ are homeomorphic maps, $\varphi_i\circ\varphi_j^{-1}$ are diffeomorphisms, n=dim N). For a function $f:N\to R$ we put $\varphi_i^* f:=f|_{U_i}\circ\varphi_i^{-1}$, $i=1,\dots,m$, and define

$$L^C_{p,\lambda}(N):=\{f:N\to R|\ \varphi_i^* f\in L^C_{p,\lambda}(\varphi_i(U_i)),\ i=1,\dots,m,$$

$$\|f|L^C_{p,\lambda}(N)\|:=\sum_{i=1}^m\|\varphi_i^* f|L^C_{p,\lambda}(\varphi_i(U_i))\|\}.$$

Analogously for $L^M_{p,\lambda}(N)$.

We have to verify that the above definition is independent of the charts $(U_i, \varphi_i)$, especially, that different charts yield equivalent norms.

Indeed, let $(U_i, \varphi_i)$, $i=1,\dots,m$, $(V_j, \psi_j)$, $j=1,\dots,k$ be two collections of charts which cover N. Put $W_{ij}=U_i \cap V_j$, $\Phi_{ij}:= \varphi_i \circ \psi_j^{-1}$. As a consequence of Proposition 1 and Proposition 2 we obtain

$$\sum_{i=1}^{m} \| \varphi_i^* f | L^C_{p,\lambda}(\varphi_i(U_i))\| \sim \sum_{\substack{i=1,\dots,m\\ j=1,\dots,k}} \| \varphi_i^* f | L^C_{p,\lambda}(\varphi_i(W_{ij}))\| \sim$$

$$\sim \sum_{\substack{i=1,\dots,m\\ j=1,\dots,k}} \| \Phi_{ij}^*(\varphi_i^* f) | \Phi_{ij}^{-1} \varphi_i(W_{ij})\| =$$

$$= \sum_{\substack{i=1,\dots,m\\ j=1,\dots,k}} \| \psi_j^* f | L^C_{p,\lambda}(\psi_j(W_{ij}))\| \sim \sum_{j=1}^{k} \| \psi_j^* f | L^C_{p,\lambda}(\psi_j(V_j))\| .$$

Clearly one can replace "C" by "M". As an immediate consequence of Theorem 0 we obtain

**Proposition 3.** Let N be a compact manifold, $1 \leq p < \infty$, $0 < \lambda$, n=dim N. Then the following assertions hold:

(i) $L^M_{p,\lambda}(N)$ and $L^C_{p,\lambda}(N)$ are Banach spaces. The following imbeddings are continuous:

$L^M_{p,\lambda}(N) \subset L^C_{p,\lambda} \subset L_p(N)$,

$L^C_{q,\nu}(N) \subset L^C_{p,\lambda}(N)$ for $1 \leq p \leq q < \infty$, $\frac{\lambda-n}{p} < \frac{\nu-n}{q}$.

(ii) For $0 \leq \lambda < n$ $L^M_{p,\lambda}(N)$ and $L^C_{p,\lambda}(N)$ are isomorphic,

(iii) $L^M_{p,n}(N)$ is isomorphic to $L_\infty(N)$.

A more interesting question is the relation of $L^C_{p,\lambda}(N)$ to the space of Hölder continuous functions.

We recall the fact that the geodesic distance d(P,Q)= "inf of length of all piecewise smooth curves joining P and Q" makes a Riemannian manifold to a metric space such that the metric topology is equivalent to the original Hausdorff topology. Hölder continuity can be defined as follows.

**Definition 3.** Let $0 < \alpha \leq 1$ and N be a connected Riemannian manifold with geodesic distance.d. The Hölder space $C^\alpha(N)$ is defined as

$$C^\alpha(N) = \{ f: N \to R \mid f \text{ continuous}, \| f | C^\alpha(N) \| :=$$

$$:= \sup_{P \in N} |f(P)| + \sup_{\substack{P,Q \in N\\ P \neq Q}} \frac{|f(P)-f(Q)|}{d(P,Q)^\alpha} < \infty \} .$$

Clearly $C^{\alpha}(N)$ is a Banach space.

For a compact manifold N a function $f:N \to R$ belongs to $C^{\alpha}(N)$ if and only if f belongs to $C^{\alpha}$ in any chart. More precisely we have

**Proposition 4.** Let $(V_i, \varphi_i)$, $i=1,\dots,m$ be a finite system of charts which cover the compact manifold N such that

(i) $\varphi_i(V_i) \subset R_n$ is convex for $i=1,\dots,m$,

(ii) for every $\overline{V}_i$ there is a coordinate neighbourhood $U_i \supset \overline{V}_i$, i.e. $(U_i, \varphi_i)$ is a chart over N.

Then $f \in C^{\alpha}(N)$ if and only if $\varphi_i^* f \in C^{\alpha}(\varphi_i(V_i))$ for all $i=1,\dots,m$.

**Proof.** The proof is standard. We sketch the idea.

**Step 1.** N admits a Riemannian structure and, by the Hopf-Rinow theorem, two points $P,Q \in N$ can be joined by a minimizing geodesic of length $d(P,Q)$. According to a lemma of Lebesgue, there is $\delta > 0$, such that all sets with diameter less than $\delta$ are contained in one of the $V_i$'s.

Furthermore, the eigenvalues of the matrix of the metric tensor in every $\varphi_i(\overline{V}_i)$ can be estimated from below and above by positive constants $0 < C_0 \leq C_1 < \infty$ :

$C_0 = \inf\{\mu \mid \mu = \text{eigenvalue of } g_{kl}(x),\ x \in \varphi_i(\overline{V}_i),\ i=1,\dots,m\}$,

$C_1 = \sup\{\mu \mid \mu = \text{eigenvalue of } g_{kl}(x),\ x \in \varphi_i(\overline{V}_i),\ i=1,\dots,m\}$.

**Step 2.** In Definition 3 we can assume $d(P,Q) < \delta$ . Let $f \in C^{\alpha}(N)$ and let the geodesic joining P and Q of length $d(P,Q)$ be contained in $V_i$. The image of this geodesic is a smooth curve in $\varphi_i(V_i)$. Hence

$$d(P,Q) = \int_0^{d(P,Q)} \left(g_{kl} \frac{dx^k}{ds} \cdot \frac{dx^l}{ds}\right)^{1/2} ds \leq c_1 |\varphi_i(P) - \varphi_i(Q)|,$$

because the image of the straight line joining $\varphi_i(P)$ and $\varphi_i(Q)$ under $\varphi_i^{-1}$ is a curve on N. This yields

$$\max_{i=1,\dots,m} \| \varphi_i^* f | C^{\alpha}(\varphi_i(V_i) \| \leq \max(1, C_1^{\alpha}) \| f | C^{\alpha}(N) \| .$$

Conversely, let $\varphi_i^* f \in C^{\alpha}(\varphi_i(V_i))$. We have

$c_0 |\varphi_i(P) - \varphi_i(Q)| \leq d(P,Q)$ because a geodesic of the length $d(P,Q)$ is contained in $V_i$. Hence

$$\min(1, C_0^{\alpha}) \| f | C^{\alpha}(N) \| \leq \max_{i=1,\dots,m} \| \varphi_i^* f | C^{\alpha}(\varphi_i(V_i)) \| .$$

**Remark.** Condition (i) may be removed by:

(i)* $\varphi_i(V_i)$ has a Lipschitz boundary for $i=1,\ldots,m$.

This ensures that two points of $\varphi_i(V_i)$ can be joined by a curve contained in $\varphi_i(V_i)$ and with a length that can be estimated by Euclidean distance from below and above.

As a consequence of the $R_n$-results respectively (Theorem 0,( iv)) we have

**Proposition 5.** Let N be a compact manifold, $n=\dim N$ and $n<\lambda\leq n+p$, $\alpha = \frac{\lambda-n}{p}$. Then $L^C_{p,\lambda}(N)$ is isomorphic to $C^{\alpha}(N)$.

**Proof.** It is sufficient to construct a finite system of charts covering N which fulfils the assumptions of Proposition 4. This can be done as follows.

Let $P\in W$, $(W,\varphi)$ be a chart. Then $\varphi(W)$ contains an open ball with the centre $\varphi(P)$, the image of which under $\varphi^{-1}$, we denote by $V_P$. Now, the collection cf charts $(V_P,\varphi)$ admits a finite subcovering of N which obviously has the desired properties.

The "local" Definition 2′ yields the well-known properties of Morrey-Campanato spaces on compact manifolds with the help of $R_n$-results, respectively. However, it seems convenient to give a more intrinsic description of these spaces.

First we recall some technical prerequisites. For a $P\in N$ and a tangent vector X at P let $\gamma:R\to N$ be the unique geodesic with $\gamma(0)=P$ and tangent X at P. Put $\exp_P X := \gamma(1)$, which is well defined at least for small X. The map $\exp_P$ is diffeomorphic near the origin of the tangential space at P and depends smoothly on P. $r_N := \inf_{P\in N} \sup\{r \mid \exp_P X \text{ is injective for } g(X,X)<r^2\}$ is called injectivity radius of N. Hereby g stands for the Riemannian metric. For compact N we have $r_N>0$. Clearly $B^N_\varrho(P)=\{\exp_P X \mid g(X,X)<\varrho^2\}$ for $\varrho\leq r_N$. (A good reference is [1].)

**Proposition 6.** Let N be a compact orientable manifold and $0<\delta\leq\infty$, p, $\lambda$ as above. Then it holds

$$L^M_{p,\lambda}(N)=\{f\in L_p(N) \mid \|f|L^M_{p,\lambda}\|_1 := \Big[\sup_{\substack{\varrho>0\\ P\in N}} \varrho^{-\lambda}\int_{B^N_\varrho(P)} |f|^p \eta\Big]^{1/p}<\infty\},$$

$$L^C_{p,\lambda}(N)=\{f\in L_p(N) \mid \|f|L^C_{p,\lambda}(N)\|_1 := \|f|L_p(N)\| + [f]_{p,\lambda,N\delta}<\infty\}$$

and the norms $\|\cdot\|$ are equivalent to the norms $\|\cdot\|_1$.

**Proof**

**Step 1.** By the same argument as in the proof of Proposition 1, Step 1, one can assume $\delta$ small, i.e. $0<\delta<r_N$.

Let $N=\bigcup_{i=1}^{m} B^N_\rho(P_i)$ for a $0<\rho<\frac{1}{2}r_N$. For every $P_i\in N$ choose a basis $(Y_1,\dots,Y_n)$, the tangential space at $P_i$ such that $g(Y_j,Y_k)=\delta_{jk}$ and put $\exp^{-1}_{P_i}(\exp_{P_i}(h_1 Y_1+\dots+h_n Y_n))=(h_1,\dots,h_n)$. Clearly $(B^N_\rho(P_i),\exp^{-1}_{P_i})$ is a chart. We prove

$f^i:=(\exp^{-1}_{P_i})^* f\in L^C_{p,\lambda}(B_\rho(0))$ if $\|f|L^C_{p,\lambda}(N)\|_1<\infty$. Indeed,

$$\|f^i|L_p(B_\rho(0))\|^p=\int_{B_\rho(0)}|f^i(x)|^p\,dx\le$$

$$\le(\sup_{x\in\overline{B}_\rho(0)}|\det g_{jk}(x)|^{-1/2}\cdot\int_{B_\rho(0)}|f^i(x)|^p|\det g_{jk}(x)|^{1/2}\,dx\le C\,\|f|L_p(N)\|^p$$

since $\overline{B}_\rho(0)$ is compact and $|\det g_{jk}(x)|$ continuous and strictly positive.

For any a we have

$$\frac{1}{2}[f^i]_{p,\lambda,B_\rho(0),\delta}\le\left[\sup_{\substack{0<s<\delta\\ x\in B_\rho(0)}} s^{-\lambda}\int_{B_\rho(0)\cap B_s(x)}|f^i(y)-a|^p\,dy\right]^{1/p}\le$$

$$\le\left[\sup_{\substack{0<s<\delta\\ x\in B_\rho(0)}} C\cdot s^{-\lambda}\int_{B_\rho(0)\cap B_s(x)}|f^i(y)-a|^p|\det g_{jk}(y)|^{1/2}\,dy\right]^{1/p}.$$

Let $x=\exp^{-1}_{P_i}P$ and $y=\exp^{-1}_{P_i}Q$. Then it follows (cf. Prop. 4) that $\beta|x-y|\le$ $\le d(P,Q)\le\alpha|x-y|$ where

$\alpha=\sup\{\mu\mid\mu=\text{eigenvalue of } g_{jk}(x),\ x\in B_\rho(0)\}$,

$\beta=\inf\{\mu\mid\mu=\text{eigenvalue of } g_{jk}(x),\ x\in B_\rho(0)\}$.

Hence $y\in B_\rho(0)\cap B_s(x)$ yields $Q\in B^N_{\alpha s}(P)$. Consequently

$$[f^i]_{p,\lambda,B_\rho(0),\delta}\le c'\left[\sup_{0<s<\delta/\alpha} s^{-\lambda}\int_{B^N_s(P)}|f-a|^p\eta\right]^{1/p}\le c'[f]_{p,\lambda,N}.$$

Here we replaced s by $s/\alpha$ and a by $f_{p,s}$.

**Step 2.** Let $f\in L^C_{p,\lambda}(N)$. Assume that there are $P_k\in N$, $\rho_k\to 0$ such that

$$\left[\rho_k^{-\lambda}\int_{B^N_{\rho_k}(P_k)}|f-f_{P_k,\rho_k}|^p\eta\right]^{1/p}>k.$$

Since N is compact, there is a subsequence such that $P_k\to P\in N$ (convergence in the metric topology). For large k we obtain $B^N_{\rho_k}(P_k) \subset B^N_{\sigma}(P)$. This implies

$$\left[\rho_k^{-\lambda} \int_{\exp_p^{-1} B^N_{\rho_k}(P_k)} |(\exp_p^{-1})^* f - a|^p |\det g_{ij}|^{1/2}\, dx\right]^{1/p} \geq k/2$$

for arbitrary a (cf. the proof of Proposition 2). The same arguments as above yield

$$\left[\rho_k^{-\lambda} \int_{B_{\rho_{k/\beta}}(\exp_p^{-1}(P_k))} |(\exp_p^{-1})^* f - a|^p\, dx\right]^{1/p} \geq k/2 \cdot c$$

which contradicts $f \in L^C_{p,\lambda}(N)$ for $a = (\exp_p^{-1})^* f_{\rho_{k/\beta}, B_{\rho_{k/\beta}}(\exp_p^{-1}(P_k))}$.

**Step 3.** The assertions with respect to $L^M_{p,\lambda}(N)$ can be proved analogously but simpler.

**Remark 2.** Proposition 6 characterizes Morrey-Campanato spaces on compact manifolds via a very natural translation procedure: all ingredients of Definition 0 are replaced by their counterparts on the manifold.

**Remark 3.** It is not hard to see that a compact manifold has in some sense the "type-A" property. (A domain $\Omega \subset R_n$ is of type A, $A > 0$, if $|\Omega_\rho(x)| \geq A\rho^n$ for all x provided $\rho \leq$ const.) Indeed, let $(x_1, \ldots, x_n)$ be local geodesic coordinates , $P \sim (0, \ldots, 0)$.
Then $\exp_p^{-1}(B^N_\rho(P)) = \{x \in R_n \mid |x| < \rho\}$, $g_{ij}(x) = \delta_{ij} + \Delta_{ij}(x)$, $|\Delta_{ij}(x)| \leq c|x|^2$ for $|x| < \rho_0$. Consequently

$$|B^N_\rho(P)| = \int_{|x|<\rho} \det(\delta_{ij} + \Delta_{ij})^{1/2} dx = \int_{|x|<\rho} (1 + \Delta(x))dx$$

with $|\Delta(x)| < c'|x|^2$. From this it follows that

$$c''(1 - a(\rho))\rho^n \leq |B^N_\rho(P)| \leq c''(1 + a(\rho))\rho^n \text{ with } a(\rho) = \rho^{-n} \int_{|x|<\rho} |\Delta(x)|dx \leq$$

$\leq c'''\rho$ . This proves $C_1\rho^n \leq |B^N_\rho(P)| \leq C_2\rho^n$ for some $C_1, C_2 > 0$, $\rho < \rho_0$.

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