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THE CLASS OF $\aleph$-SPACES IS INVARIANT OF CLOSED MAPPINGS WITH LINDELÖF FIBRES

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Abstract: In this paper, we give a new characterization of $\aleph$-spaces and show the result listed in the title.

Key words: $\aleph$-spaces, closed mapping, s-mapping.

Classification: 54C10, 54E18

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1. Introduction. $\aleph$-spaces as an interesting generalization of metric spaces were introduced by O'Meara in [O_1, O_2], and were studied by several authors ([F_1], [F_2],[T]). In [F_1], L. Foged gave some characterizations of $\aleph$-spaces and showed that the classes of $\aleph$-spaces and of cs-σ-spaces, introduced by Guthrie in [G], coincide. But some open questions remain, such as the question, whether each closed s-image of metric spaces is an $\aleph$-space, posed by Tanaka in [T].

Here, we shall give a new characterization of $\aleph$-spaces and show the result listed in the title; in particular, we answer the above question of Tanaka. Throughout this paper, we assume that all spaces are regular and all maps are continuous surjections. N denotes the set of all positive integers.

Definition. A collection $\mathcal{P}$ of subsets of a topological space X is a k-network for X if, given any compact subset C of X and any neighbourhood U of C, there is a finite subcollection $\mathcal{P}^*$ of $\mathcal{P}$ so that $C \subseteq \bigcup \mathcal{P}^* \subseteq U$. A collection $\mathcal{P}$ is a cs-network for X if, given any sequence S converging to $x \in X$ and any neighbourhood U of x, there is a $P \in \mathcal{P}$ so that $P \subseteq U$ and S is eventually in P. A regular space is an $\aleph$-space if it has a σ-locally finite k-network, because of regularity, this collection can be chosen to consist of closed sets. Recall that a map $f:X \longrightarrow Y$ is an s-map if each $f^{-1}(y)$ is Lindelöf.

2. Results. The following theorem gives a new characterization of $\aleph$-spaces.

Theorem 1. The following conditions are equivalent:

(a) X is an $\aleph$-space.

(b) X has a σ-discrete cs-network.

(c) X has a σ-closure preserving, point-countable closed k-network.

Proof. The fact that (a) and (b) are equivalent is well known (see [F_1]). We only show (c) $\Rightarrow$ (b). First note that X is a σ-space [SN]. For each $n\in N$, let $\mathcal{P}_n$ be a closure preserving, point-countable collection of closed subsets of X so that $\bigcup\mathcal{P}_n$ is a k-network for X. Without loss of generality, let $\mathcal{P}_n$ be closed under finite intersections and $\mathcal{P}_n\subseteq\mathcal{P}_{n+1}$ for each n. Clearly, $\mathcal{P}_n$ is locally countable for each n. Thus there is an open cover, each element of which meets only countably many members of $\mathcal{P}_n$ only. Furthermore, there is a σ-discrete closed cover $\mathcal{V}_n=\bigcup_{m=1}^{\infty}\mathcal{V}_{n,m}$, each element of which meets only countably many members of $\mathcal{P}_n$, where each $\mathcal{V}_{n,m}$ is discrete. Thus for each $V\in\mathcal{V}_{n,m}$, we can denote $\{P_{n,k}(V):k\in N\}$ the family of all the unions of finite subcollections of $\{P\in\mathcal{P}_n:P\cap V\neq\emptyset\}$. For each $h\in N$, let $F_{n,m,h}(V)=\bigcup\{P\in\mathcal{P}_h:P\subseteq X\setminus(\bigcup\mathcal{V}_{n,m}\setminus\{V\})\}$. Then each $F_{n,m,h}(V)$ is clearly closed. For each $l\in N$, let $F_{n,m,h,l}=\bigcup\{P\in\mathcal{P}_l:P\subseteq(X\setminus\bigcup\mathcal{P}^*_{n,m,h})\}$, where $\mathcal{P}^*_{n,m,h}=\{P\in\mathcal{P}_h:P\cap(\bigcup\mathcal{V}_{n,m})=\emptyset\}$. Now let $P_{n,k,h,l}(V)=P_{n,k}(V)\cap F_{n,m,h}(V)\cap F_{n,m,h,l}$ for each $V\in\mathcal{V}_{n,m}$ and let $\mathcal{W}_{n,m,k,h,l}=\{P_{n,k,h,l}(V):V\in\mathcal{V}_{n.m}\}$. It is clear that $\mathcal{W}_{n,m,k,h,l}$ is pairwise disjoint and closure-preserving, i.e., it is discrete since each element is closed.

It remains to prove that $\mathcal{W}=\{\mathcal{W}_{n,m,k,h,l}:n,m,k,h,l\in N\}$ is a cs-network. Suppose that S is a sequence converging to $x\in X$ and U is an open set containing x. Then there is a finite subcollection $\mathcal{P}^*$ of $\mathcal{P}_{n_0}$ for some $n_0\in N$ so that $\bigcup\mathcal{P}^*\subseteq U$ and S is eventually in $\bigcup\mathcal{P}^*$. We also can assume $x\in\bigcap\mathcal{P}^*$. But $\mathcal{V}_{n_0}=\bigcup_{m=1}^{\infty}\mathcal{V}_{n_0,m}$ is a cover, so there is a $V_0\in\mathcal{V}_{n_0,m_0}$ with $v\in V_0$ for some $m_0\in N$, in particular, $(\bigcap\mathcal{P}^*)\cap V_0\neq\emptyset$, thus $\bigcup\mathcal{P}^*=P_{n_0,k_0}(V_0)$ for some $k_0\in N$. Note that x is in $X\setminus(\bigcup\mathcal{V}_{n_0,m_0}\setminus\{V_0\})$ which is open, so there is a finite subcollection $\mathcal{P}'$ of $\mathcal{P}_{h_0}$ for some $h_0\in N$ so that S is eventually in

$\bigcup \mathcal{P}'$ and $\bigcup \mathcal{P}' \subseteq (X \setminus (\bigcup \mathcal{V}_{n_0,m_0} \setminus \{V_0\})) \cap U$, in particular,

$\bigcup \mathcal{P}' \subseteq F_{n_0,m_0,h_0}(V_0)$. Likewise, there is an $l_0 \in N$ so that S is eventually in F_{n_0,m_0,h_0,l_0}; i.e., S is eventually in $P_{n_0,k_0}(V_0) \cap F_{n_0,m_0,h_0}(V_0) \cap$ $\cap F_{n_0,m_0,h_0,l_0} = P_{n_0,m_0,k_0,h_0,l_0}(V_0)$ and $P_{n_0,m_0,k_0,h_0,l_0}(V_0) \subseteq U$. The proof is complete.

Remark. If we say that a collection $\mathcal{P}$ is a weak cs-network for X if, given any sequence S converging to x and any neighborhood U of x, there is a finite subcollection $\mathcal{P}^*$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}^* \subseteq U$ and S is eventually in $\bigcup \mathcal{P}^*$, then a k-network is, of course, a weak cs-network and the above proof shows that the following conditions are equivalent:

(a) X is an $\aleph$-space.

(b) X has a σ-closure preserving, point-countable closed weak cs-network.

(c) X has a σ-discrete cs-network.

(d) X has a σ-discrete k-network.

Recall that a continuous map $f:X \to Y$ is called compact-covering if for every compact subset $B \subseteq Y$ there exists a compact $A \subseteq X$ such that $f(A)=B$.

Lemma 1. Every closed s-map is compact covering.

Proof. First we note that the preimage of a compact subset (or Lindelöf subset) under a closed s-map f is a Lindelöf subset. Then by virtue of a theorem of E. Michael stating that every closed mapping of a paracompact space X onto an arbitrary space Y is compact-covering, the lemma is proved.

Lemma 2. Every closed s-image of an $\aleph$-space has a σ-closure preserving, point-countable closed k-network.

Proof. For each $n \in N$, let $\mathcal{P}_n$ be a discrete collection of closed subsets of X so that $\bigcup \mathcal{P}_n$ is a k-network for X and $f:X \to Y$ is a closed s-map. Then one easily verifies that $\{f(B): B \in \mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n\}$ is σ-closure-preserving and, since f is compact-covering by Lemma 1, that $\{f(B): B \in \mathcal{P}\}$ is a closed k-network for X. It remains to show that $\mathcal{P}$ is point-countable. Let $y \in Y$, for each $x \in f^{-1}(y)$, $n \in N$, there is an open neighborhood of x which meets only one element of $\mathcal{P}_n$; furthermore, there is an open subset U_y of X with $U_y \supseteq f^{-1}(y)$ so that U_y meets only at most countably many elements of $\mathcal{P}_n$. Thus the point y belongs to countably many elements of $\{f(B):B \in \mathcal{P}\}$ at most.

Theorem 2. Every closed s-image of an $\aleph$-space is an $\aleph$-space.

Corollary. Every closed s-image of a metric space is an $\aleph$-space.

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