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SEMIPRIME IDEALS IN ORTHOMODULAR LATTICES

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Abstract: We prove that, in an orthomodular lattice, an ideal is semiprime if and only if it contains the commutator ideal. This result allows us to characterize the distributivity of an orthomodular lattice.We also give some properties of kernels of conguence relations in orthomodular lattices.

Key words: Commutator ideal, distributivity criterion, orthomodular ideal, orthomodular lattice, semiprime ideal.

Classification: 06C15.

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1. Introduction. In [2], L. Beran has characterized the distributivity of a finitely generated orthomodular lattice by using the concept of semiprime ideal. In Section 3 of this paper we prove that, in an OML (an abbreviation for orthomodular lattice), an ideal is semiprime if and only if it contains the commutator ideal. This fact allows us to generalize the main result of [2].In Section 2, two binary relations on an OML are introduced, the position p' and the relation π. These relations permit us to obtain some useful charaterizations of kernels of congruence relations in OML.In the final Section, some other characterizations of semiprime ideals and kernels of congruence relations are given.

Notations and definitions are borrowed from [1]. There is an exception: kernels of congruence relations in OMLs are called *orthomodular ideals*.

2. Some characterizations of orthomodular ideals.

In [5], it is show that a congruence relation in an OML L is completely determined by its kernel and that an ideal I of L is the kernel of a congruence relation if and only if it fulfils:

$$a \in I \text{ and } b \in L \text{ imply } s_b(a) = b \wedge (b^\perp \vee a) \in I.$$

There are a lot of other conditions which can be used to define orthomodular ideals. More precisely, we say that a binary relation R on an OML L *characterizes the orthomodular ideals* of L if, for an ideal I of L, the following statements are equivalent:

1) I is an orthomodular ideal,

2) $a \in I$, $b \in L$ and $R(a,b)$ imply $b \in I$.

It is well-known that the relations of perspectivity and strong perspectivity characterize the orthomodular ideals.Two other binary relations are very useful:

Elements a and b of an OML L are said to be in *position p'* if $a^\perp \wedge b = a \wedge b^\perp = 0$.They satisfy the *relation π* if $a \wedge b = a^\perp \wedge b = a \wedge b^\perp = 0$

Obviously, if $\pi(a,b)$ then a and b are in position p' and it is proved in [4] that a and b are in position p' if and only if they are strongly perspective in the subalgebra generated by {a,b}. In particular, position p' implies stong perspectivity.

Recall that, for elements a and b of an OML L, we have ([4]):

* $s_a(b)$ and $s_b(a)$ are in position p' and $a \wedge \overline{com}(a,b)$ and $b \wedge \overline{com}(a,b)$ satisfy the relation π.
* a and b are in position p' if and only if $a = s_a(b)$ and $b = s_b(a)$.
* $\pi(a,b)$ is equivalent to $a = a \wedge \overline{com}(a,b)$ and $b = b \wedge \overline{com}(a,b)$.

Proposition 1 ([4]). Let L be an OML. Any binary relation contained between position p' and perspectivity characterizes the orthomodular ideals of L.

□ Let I be an ideal of L such that $x \in I$, $y \in L$ and $\pi(x,y)$ imply $y \in I$. For elements a and b of L, define $c = a \wedge \overline{com}(a,b)$ and $d = b \wedge \overline{com}(a,b)$. We have $\pi(c,d)$ and from $c \le a$ it follows that $c \in I$. Therefore, $d \in I$ and hence $d \vee a \in I$. Now:

$d \vee a = (b \wedge \overline{com}(a,b)) \vee a = (a \vee b) \wedge (a \vee b^{\perp}) \geq b \wedge (a \vee b^{\perp}) = s_b(a)$ and hence $s_b(a) \in I$. Then, the ideal I is orthomodular. Since perspectivity characterizes orthomodular ideals, the proof is complete. □

The next proposition, which is another characterization of orthomodular ideals, will be useful for the proof of Proposition 4.

Proposition 2. Let I be an ideal of an OML L. The following statements are equivalent:

1) I is an orthomodular ideal,
2) The relation $s_x(y) \in I$ is symmetric.

□ For elements a and b of L, $s_a(b)$ and $s_b(a)$ are in position p'. Hence, by Proposition 1, the relation $s_x(y) \in I$ is symmetric if I is an orthomodular ideal. Conversely, assume that $s_x(y) \in I$ is a symmetric relation and let $a \in I$ and $b \in L$ in position p'. We have $s_a(b) = a$ and $s_b(a) = b$. Thus, $b \in L$ and, by Proposition 1, I is an orthomodular ideal. □

3. Semiprime ideals in orthomodular lattices.

In [8], Y. Rav introduced the concept of a *semiprime ideal* of a lattice which is an ideal satisfying:

$$a \wedge b \in I \text{ and } a \wedge c \in I \text{ imply } a \wedge (b \vee c) \in I.$$

Recall that, in an OML, any ideal which contains the ideal I_c generated by the upper commutators is orthomodular and, for an orthomodular ideal I of L, the quotient L/I is boolean if and only if $I_c \subset I$. The ideal I_c is called the *commutator ideal* and, for the most part, these results are proved in [6].

Theorem Let I be an ideal of an OML L. The following statements are equivalent:

1) I is semiprime,

2) I satisties the condition:

$a \wedge b \in I$ and $a \wedge b^{\perp} \in I$ imply $a \in I$,

3) I contains the commutators ideal.

□ Obviously, 1) implies 2). Let I be an ideal satisfying 2) and let $a, b \in L$, $c = a \wedge \overline{\text{com}}(a,b)$ and $d = b \wedge \overline{\text{com}}(a,b)$. We have $\pi(c,d)$, and as $0 \in I$, c and d are elements of I. Since $c \vee d = \overline{\text{com}}(a,b)$, I contains the commutator ideal.

Consider now an ideal I which contains the commutator ideal. The quotient L/I is distributive and, therefore, {0} is semiprime in L/I. This fact is equivalent to the semiprimality of I. □

Corollary 1 Let L be an orthomodular lattice. The following statements are equivalent:

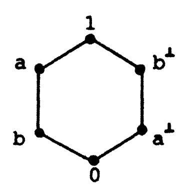
1) L is distributive (i.e. L is a boolean lattice),

2) For elements a, b, c of L, $a \wedge b = a \wedge c = 0$ implies $a \wedge (b \vee c) = 0$,

3) For elements a, b of L, $a \wedge b = a \wedge b^{\perp} = 0$ implies $a = 0$.

□ An OML L is distributive if and only if its commutator ideal is {0}. Hence, by the preceding theorem, L is distributive is equivalent to the semiprimality of {0}. □

Remark. Let L be an orthocomplemented lattice. If L is not orthomodular then L contains a subalgebra L' isomorphic to the following orthocomplemented lattice:



It is easy to see that {0} is a semiprime ideal of L'. Hence, Corollary 1 cannot be extended to orthocomplemented lattices. More precisely, if L is an orthocomplemented lattice then the following statements are equivalent:

1) L is orthomodular,

2) A subalgebra L' of L is distributive if and only if {0} is a semiprime ideal of L'.

Corollary 2. If L is an OML then the following statements are equivalent:

1) L is distributive,

2) The ideal of lower bounds of the set of lower commutators is semiprime.

□ Obviously, 1) implies 2). Conversely, let I be the ideal of lower bounds of the set of lower commutators. If I is semiprime then I contains the commutator ideal and therefore, for elements a and b of L, $\overline{com}(a,b) \le \underline{com}(a,b)$. As $\underline{com}(a,b) = \overline{com}(a,b)^{\perp}$, we have $\overline{com}(a,b) = 0$ and L is distributive. □

In [7], S. Pulmannovà has generalized the concept of a commutator of a finite subset of an OML L. For every subset M of L, $\overline{com}(M)$ and $\underline{com}(M)$ are defined by:

$$\overline{com}(M) = \bigvee\{\overline{com}(F) \mid F \text{ finite} \subset M\}$$

$$\underline{com}(M) = \bigwedge\{\underline{com}(F) \mid F \text{ finite} \subset M\} = \overline{com}(M)^{\perp}$$

Evidently, such elements need not exist if L is not complete.

Recall that Corollary VII.1.11 of [1] asserts that the lower commutator of a finite subset is a meet of a finite set of lower commutators of two elements. Therefore, if $\underline{com}(L)$ exists (for example, if L is complete or finitely generated) then $\underline{com}(L)$ is the meet of $\{\underline{com}(a,b) \mid a, b \in L\}$ and $[0,\underline{com}(L)]$ is the ideal of lower bounds of the set of lower commutators. Thus, Corollary 2 is a generalization of the main result of [2].

Corollary 3. Let L be an OML.

1) A proper ideal I of L is a prime ideal if and only if I is a maximal semiprime ideal.

2) A proper semiprime ideal is the intersection of the prime ideals that contain it.

□ Recall that an ideal I of an OML L is prime if and only if I is orthomodular and $L/I = \{0,1\}$ ([3]).Hence, a prime ideal is a maximal ideal which contains the commutator ideal and 1) is proved. The congruence lattice of an OML L is isomorphic to the lattice of orthomodular ideals of L. Then, by using the third isomorphism theorem, the lattice of ideals which contain the commutator ideal I_c is isomorphic to the lattice of ideals of the boolean lattice L/I_c. Since, in a boolean lattice, a proper ideal is the intersection of the maximal ideals that contain it the proof is complete. □

4) Other characterizations of orthomodular and semiprime ideals.

In a boolean lattice, $x \wedge y = 0$ is equivalent to $x \leq y^{\perp}$ and, in an OML, we have $x \leq y^{\perp}$ if and only if $s_x(y) = 0$ or $s_y(x) = 0$. Thus, it is natural to consider, for an ideal I of an OML, the following conditions:

(C_0) : $x \wedge y \in I$ and $x \wedge z \in I$ imply $s_{y \vee z}(x) \in I$,

(C_1) : $x \wedge y \in I$ and $x \wedge z \in I$ imply $s_x(y \vee z) \in I$,

(C_2) : $s_x(y) \in I$ and $s_x(z) \in I$ imply $s_{y \vee z}(x) \in I$,

(C_3) : $s_y(x) \in I$ and $s_z(x) \in I$ imply $s_{y \vee z}(x) \in I$,

(C_4) : $s_y(x) \in I$ and $s_z(x) \in I$ imply $s_x(y \vee z) \in I$,

(C_5) : $s_y(x) \in I$ and $s_z(x) \in I$ imply $x \wedge (y \vee z) \in I$.

Proposition 4. Let I be an ideal of an OML L.

a) The ideal I is semiprime if and only if it satisfies one of the conditions (C_i), $i = 0, 1$.

b) The ideal I is orthomodular if and only if it satisfies one of the conditions (C_i), $i = 2, 3, 4, 5$.

□ a) For elements a and b of an OML, $s_a(b) \geq a \wedge b$ holds. Therefore, an ideal which satisfies (C_0) or (C_1) is semiprime.

Conversely, if I is a semiprime ideal then L/I is a boolean lattice Since, for elements a and b of such a lattice, $s_a(b) = a \wedge b$ it is easy to see that I fulfils (C_0) and (C_1).

b) In order to show that an orthomodular ideal satifies (C_i), $2 \le i \le 5$, it suffices to use Proposition 2 and the properties $a \wedge (b \vee c) \le s_a(b \vee c) = s_a(b) \vee s_a(c)$.

By choosing y = z in (C_2) or (C_4), we infer that these conditions imply the symmetry of the relation $s_x(y) \in I$. Hence, by Proposition 2, an ideal satisfying (C_2) or (C_4) is orthomodular. Now, let I be an ideal satisfying (C_5) and let $y \in I$, $x \in L$. As $s_y(x) \le y$ and $s_{x^\perp}(x) = 0$, we have $s_y(x) \in I$ and $s_{x^\perp}(x) \in I$. Therefore, $s_x(y) = x \wedge (y \vee x^\perp) \in I$ and I is an orthomodular ideal. As (C_3) implies (C_5), every ideal satisfying (C_3) is orthomodular.

□

(There exist two other conditions similar to the conditions (C_i). Every ideal satisfies these conditions.)

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