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THE FACTORIZATION THEOREM FOR PARACOMPACT Σ-SPACES

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Abstract: Factorization theorems and some corollaries are obtained for several classes of paracompact spaces.

Key words and phrases: Uniform, topological, metric, Lindelöf, Tychonoff spaces; p-spaces, σ-spaces, Σ-spaces, closed and perfect maps.

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1. Introduction. The factorization theorem for a class of spaces $\mathcal{C}$ is the following statement.

(FT). For every map $f:X \to Y$ into a member of $\mathcal{C}$, there exists Z in $\mathcal{C}$ and maps $g:X \to Z$ and $h:Z \to Y$ such that $f=h \circ g$, $wZ \leq wY$ and $\dim Z \leq \dim X$.

FT is known to hold for several classes of spaces such as compact spaces, metric spaces and paracompact p-spaces [10]. It is not known whether it holds for the class of all paracompact spaces. Bregman [1] asks whether FT holds for every map $f:X \to Y$ between paracompact σ-spaces, having proved it for a restrictive class of such maps called σ-discrete. We show in Section 3 that a stronger version of FT holds for paracompact Σ-spaces. In fact, it holds for a bigger class of spaces that includes Lindelöf spaces. The class of paracompact Σ-spaces is an important class of generalized metric spaces, and includes all paracompact p-spaces, all paracompact locally compact spaces and all paracompact σ-spaces (see [8] and the articles of Burke and Gruenhage in [6]). In Section 4, we prove FT for a class of maps between paracompact σ-spaces that includes perfect maps. In Section 5, we establish FT for a more general class of paracompact spaces that includes closed images of paracompact, locally compact spaces. Some corollaries of FT such as universal theorems are pointed out in Sections 3 and 5.

In this paper, all spaces are Tychonoff, N denotes the set of positive integers, I the unit interval $[0,1]$, βX and wX the Stone-Čech compactification and weight of a space X, respectively,and $|Y|$ the cardinality of a set

Y. For standard results in Dimension Theory the reader is referred to [5] and [11].

2. Preliminary results. Our factorization theorems follow from three results concerning the covering dimension, Dim X, of a uniform space X [2,3]. A uniformly open set of X is a set of the form $f^{-1}(G)$ where $f:X \to M$ is a uniformly continuous function into a metric space M (with its natural uniformity) and G is an open set of M. The set of all uniformly open sets of X is a base and it is closed under finite intersections and countable unions. DimX is defined in terms of uniformly open sets. Thus, Dim $X \leq n$ iff every finite uniformly open cover of X has a finite uniformly open refinement of order $\leq n$. If every cozero set of X is uniformly open, then Dim X=dim X. This happens, e.g., when X is Lindelöf or metric.

Theorem 1. Let $f:X \to Y$ be a uniformly continuous function and $\{X_\alpha : \alpha < \tau\}$ a collection of subspaces of X, where τ is a cardinal not less than W(Y), the uniform weight of Y. Then there exists a uniformly continuous $g:X \to Y \times I^\tau$ such that $f = \pi \circ g$, where $\pi : Y \times I^\tau \to Y$ is the canonical projection, and Dim $g(X_\alpha) \leq$ Dim X_α for each $\alpha < \tau$ [3, Theorem 5].

Theorem 2. Let $f:X \to Y$ be a closed uniformly continuous function with Lindelöf fibers into a (paracompact) space Y with the property that every open cover of Y has a σ-locally finite uniformly open refinement. Then X is paracompact and dim $X \leq$ Dim X [3, Theorem 10].

Theorem 3. If $Y \subset X$, then Dim $Y \leq$ Dim X [2, Proposition 3].

3. FT for paracompact Σ'-spaces. In this section, we prove a stronger version of FT for paracompact Σ'-spaces, a class of spaces that includes all Lindelöf spaces as well as all paracompact Σ-spaces. If $\mathcal{C}$ and $\mathcal{F}$ are covers of a space X, $\mathcal{F}$ is called a (mod $\mathcal{C}$)-net for X if whenever $C \subset U$ with C in $\mathcal{C}$ and U open in X, there is some F in $\mathcal{F}$ such that $C \subset F \subset U$. We call X a Σ'-space if it has a closed cover $\mathcal{C}$ consisting of Lindelöf subspaces and a σ-locally finite (mod $\mathcal{C}$)-net $\mathcal{F}$. Recall that if each C in $\mathcal{C}$ is countably compact (respectively, compact), then X is called a Σ-space (respectively, a strong Σ-space) [8]. Since a paracompact countably compact space is compact, every paracompact Σ-space is a Σ'-space.

Lemma 1. $f:X \to Y$ be a perfect surjection. Then X is a Σ'-space iff Y is a Σ'-space.

Proof. If $\mathscr{C}$ is a closed cover of X by Lindelöf subspaces and $\mathscr{F}$ is a σ-locally finite (mod $\mathscr{C}$)-net for X, it is routinely verified that $f(\mathscr{C})=$ $=\{f(C):C\in\mathscr{C}\}$ is a closed cover of Y by Lindelöf subspaces and $f(\mathscr{F})$ is a σ-locally finite (mod $f(\mathscr{C})$)net for Y. Conversely, if $\mathscr{C}$ is a closed cover of Y consisting of Lindelöf spaces and $\mathscr{F}$ a (mod $\mathscr{C}$)-net for Y then $f^{-1}(\mathscr{C})=$ $=\{f^{-1}(C):C\in\mathscr{C}\}$ is a closed cover of X consisting of Lindelöf spaces and $f^{-1}(\mathscr{F})$ is a σ-locally finite (mod $f^{-1}(\mathscr{C})$)-net for X.

Remark 1. For the converse, it is evidently sufficient to assume that f is closed and continuous with Lindelöf fibers.

Lemma 2. Let X be a paracompact Σ'-space. Then there is a continuous $\Phi:X\longrightarrow M$ onto a metric space M such that, if X is equipped with a uniformity that makes Φ uniformly continuous, then every open cover of X has a σ-locally finite uniformly open refinement.

Proof. Let $\mathscr{C}$ be a closed cover of X by Lindelöf spaces and $\mathscr{F}=$ $=\bigcup_{n=1}^{\infty}\mathscr{F}_n$ a σ-locally finite (mod $\mathscr{C}$)-net for X. Write $\mathscr{F}_n=\{F_\alpha:\alpha\in\Lambda_n\}$, and consider a locally finite cover $\mathscr{P}$ of the paracompact space X such that for each P in $\mathscr{P}$, $\overline{P}$ intersects only finitely many members of $\mathscr{F}_n$. If $H_\alpha=$ $=X-\bigcup(\overline{P}:P\in\mathscr{P}$ and $\overline{P}\cap F_\alpha=\emptyset)$, then $\{H_\alpha:\alpha\in\Lambda_n\}$ is a locally finite collection of open subsets of X with $F_\alpha\subset H_\alpha$. Let G_α be a cozero set of X with $F_\alpha\subset G_\alpha\subset H_\alpha$, $f_\alpha:X\longrightarrow I$ a continuous function with $G_\alpha=f^{-1}(0,1]$, and set

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\min\{1,\sum_{\alpha\in\Lambda_n}[f_\alpha(x)-f_\alpha(y)]\}.$$

Now d is a continuous pseudometric on X, and we let M be the metric space obtained by identifying x, y iff d(x,y)=0, and Φ the corresponding quotient map. Note that $G_\alpha=\Phi^{-1}(\Phi(G_\alpha))$ is open w.r.t. d and hence uniformly open, assuming X carries a uniformity that makes Φ uniformly continuous. Finally, given an open cover $\mathscr{U}$ of X, let $\mathscr{V}$ be a refinement of $\mathscr{U}$ by uniformly open sets, and consider $\mathscr{W}=\{\bigcup_{i=1}^{\infty}V_i:V_i\in\mathscr{V}\}$. For each C in $\mathscr{C}$, since C is Lindelöf, there is W in $\mathscr{W}$ such that $C\subset W$. Hence there is F in $\mathscr{F}$ with $C\subset F\subset$ $\subset W$. Let Λ'_n consist of all α in Λ_n for which we can fix C_α in $\mathscr{C}$ and W_α in $\mathscr{W}$ with $C_\alpha\subset F_\alpha\subset W_\alpha$. Clearly, $\{F_\alpha:\alpha\in\Lambda'_n, n\in N\}$ constitutes a cover of X. Also, if $W_\alpha=\bigcup_{i=1}^{\infty}V_{i\alpha}$ where $V_{i\alpha}\in\mathscr{V}$, then $\{G_\alpha\cap V_{i\alpha}:\alpha\in\Lambda'_n. i,n\in N\}$ is a σ-locally finite uniformly open refinement of $\mathscr{U}$.

We now record for future reference a result whose proof is contained in the proof of Lemma 2.

Lemma 3. Let $\mathscr{C}$ be a closed cover of a space X by Lindelöf subspaces and $\mathscr{F}=\{F_\alpha:\alpha\in\Lambda\}$ a σ-locally finite (mod $\mathscr{C}$)-net for X. If there is a σ-locally finite open cover $\{G_\alpha:\alpha\in\Lambda\}$ of X with $F_\alpha\subset G_\alpha$, then X is paracompact and, if it is endowed with a uniformity that makes every G_α uniformly open, then every open cover of X has a σ-locally finite uniformly open refinement.

The FT for paracompact Σ'-spaces generalizes Theorem 4 of [10], and we recall some definitions from this paper. The compact weight of X, bwX, is the smallest cardinal τ for which there is a space Z of weight τ, a metrizable space M and an embedding of X into MxZ. The metrizable weight of X, μwX, is the supremum of all cardinals τ for which there exists a map onto a metrizable space of weight τ. It is readily checked that wX=max$\{$bwX, μwX$\}$ and, if X is metrizable, μwX=wX and bwX=1, unless X=$\emptyset$, when bwX=0. Also, bwX $\leq\aleph_0$ implies X is metrizable, $Y\subset X$ implies bwY$\leq$bwX, X Lindelöf and infinite implies μwX= $\aleph_0$, and X Lindelöf and non-metrizable implies bwX=wX.

Lemma 4. Let X be a paracompact Σ'-space, $\mathscr{C}$ a closed cover of X by Lindelöf subspaces and $\mathscr{F}$ an infinite σ-locally finite (mod $\mathscr{C}$)-net for X. Then μwX=$|\mathscr{F}|$.

Proof. Write $\mathscr{F}=\{F_\alpha:\alpha\in\Lambda\}$ with $|F|=|\Lambda|$, let $\{G_\alpha:\alpha\in\Lambda\}$ be a σ-locally finite cozero cover of X with $F_\alpha\subset G_\alpha$, and $\Phi:X\to M$ the quotient map constructed in Lemma 2. Then $\{\Phi(G_\alpha):\alpha\in\Lambda\}$ is a point-countable open cover of M. Let D be a dense subset of the metric space M with $|D|$=wM and for each $x\in D$, let $\Lambda(x)=\{\alpha\in\Lambda:x\in\Phi(G_\alpha)\}$. We can assume that $F_\alpha=\emptyset$ for at most one α in Λ and hence that $G_\alpha\neq\emptyset$ for all α in Λ. Then $\Lambda=\bigcup(\Lambda(x):x\in D)$ with each $\Lambda(x)$ countable. Hence, if D is infinite, $|\mathscr{F}|=|\Lambda|\leq|D|$ = =wM $\leq\mu$wX; and if D is countable, then $\mathscr{F}$ is countably infinite, which implies that X is Lindelöf and infinite so that $|\mathscr{F}|=\aleph_0=\mu$wX. Thus, in any case, $|\mathscr{F}|\leq\mu$wX.

To prove μwX$\leq|\mathscr{F}|$, consider a continuous surjection $f:X\to S$ onto a metric space S. Let $\{U_\beta:\beta<wS\}$ be a discrete collection of non-empty open sets of S, for each $\beta<$wS, pick x_β in U_β, let $U'=S-\{x_\beta:\beta<wS\}$, $\mathscr{U}=\{U_\alpha:\alpha<wS\}\cup\{U'\}$, and note that a refinement of $\mathscr{U}$ must have cardinality at least wS. Now consider the open cover

$$\mathscr{W}=\{f^{-1}(U_1\cup U_2\cup\dots):U_i\in\mathscr{U}\}$$

of X. For each C in $\mathcal{C}$, there is some F in $\mathcal{F}$ and W in $\mathcal{W}$ with $C\subset F\subset W$. Let Λ' consist of all α's in Λ for which we can fix C_α in $\mathcal{C}$ and W_α in $\mathcal{W}$ with $C_\alpha\subset F_\alpha\subset W_\alpha$. If $W_\alpha=f^{-1}(U_{1\alpha}\cup U_{2\alpha}\cup\ldots)$, where $U_{i\alpha}\in\mathcal{U}$, then clearly $\{f(F_\alpha)\cap U_{i\alpha}:\alpha\in\Lambda', i\in N\}$ refines $\mathcal{U}$. Hence $wS\le\max\{\aleph_0,|\Lambda'|\}\le$ $\le|\Lambda|=|\mathcal{F}|$. This implies $\mu wX=|\mathcal{F}|$, which completes the proof.

Lemma 5. Let $f:X\longrightarrow Y$ be a closed, continuous surjection with Lindelöf fibers between infinite paracompact Σ'-spaces. Then $\mu wX=\mu wY$.

Proof. Let $\mathcal{C}$ be a closed cover of Y consisting of Lindelöf spaces,and $\mathcal{F}$ a σ-locally finite (mod $\mathcal{C}$)-net for Y. If necessary,we add to $\mathcal{F}$ a countably infinite collection of singletons so that it becomes infinite and, by Lemma 4, $\mu wY=|\mathcal{F}|$. Clearly $f^{-1}(\mathcal{C})$ is a closed cover of X consisting of non-empty Lindelöf spaces and $f^{-1}(\mathcal{F})$ is an infinite σ-locally finite (mod $f^{-1}(\mathcal{C})$)-net for X, and Lemma 4 implies $\mu wX=|f^{-1}(\mathcal{F})|=|\mathcal{F}|=\mu wY$.

Lemma 6. Let E be an F_σ -set of a paracompact Σ'-space X. Then E is a paracompact Σ'-space with $\mu wE\le\mu wX$.

Proof. E is paracompact and we may assume that it is also infinite. Let $\mathcal{C}$ be a closed cover of X by Lindelöf spaces and $\mathcal{F}$ a σ-locally finite (mod $\mathcal{C}$)-net for X which contains countably infinitely many singletons from E. Then $\mathcal{C}\cap E=\{C\cap E:C\in\mathcal{C}\}$ is a closed cover of E by Lindelöf spaces and $\mathcal{F}\cap E$ is an infinite σ-locally finite (mod $\mathcal{C}\cap E$)-net for E. By Lemma 4, $\mu wE=|\mathcal{F}\cap E|\le|F|=\mu wX$.

Proposition 1. Let $f:X\longrightarrow Y$ be a continuous function into a paracompact Σ'-space. Then there is a paracompact Σ'-space Z and continuous $g:X\longrightarrow Z$ and $h:Z\longrightarrow Y$ such that h is perfect, $f=h\circ g$, $\dim Z\le\dim X$, $\mu wZ\le\mu wY$ and $bwZ\le bwY$.

Proof. We can clearly assume that Y is infinite. Note that if βf is the extension of f to Stone-Čech compactifications, $\dim\beta f^{-1}(Y)=\dim\beta X=$ $=\dim X$ and $\beta f:\beta f^{-1}(Y)\longrightarrow Y$ is perfect. Thus, we can also assume that $f:X\longrightarrow$ $\longrightarrow Y$ is perfect and, in view of Lemma 6, surjective.

By Lemma 2, there is a continuous function $\Phi:Y\longrightarrow M$ into a metric space M such that, if Y is endowed with a uniformity that makes Φ uniformly continuous, every open cover of Y has a σ-locally finite uniformly open refinement. Let $\Psi:Y\rightarrow L\times I^\tau$ be an embedding, where L is metrizable and $\tau=bwY$. We endow M, L, I^τ and $M\times L\times I^\tau$ with their natural uniformities, X with its finest uniformity and Y with the uniformity induced by the embedding $\Phi \times \Psi : Y \to M \times L \times I^{\tau}$. Evidently, every cozero set of X is uniformly continuous so that dim X=Dim X, $f: X \to Y$ and $\Phi : Y \to M$ are uniformly continuous and hence every open cover of Y has a σ-locally finite uniformly open refinement. Now, by Theorem 1, there are uniformly continuous $g: X \to Z$ and $h: Z \to Y$ such that $Z=g(X) \subset M \times L \times I^{\tau} \times I^{\tau}$, $f=h \circ g$ and Dim $Z \leq$ Dim X=dim X. Since f is perfect and f and g are onto, then h is a perfect surjection and hence Z is paracompact and, by Lemma 1, a Σ'-space. Now applying Theorem 2 and Lemma 5, we obtain, respectively, that dim $Z \leq$ Dim $Z \leq$ dim X and $\mu wZ \leq \mu wY$. Finally, the inequality $bwZ \leq bwY = \tau$ follows from that fact that Z is a subspace of $M \times L \times I^{\tau} \times I^{\tau}$.

Our next two results are corollaries of Proposition 1. The first of these results follows from Proposition 1 by a straightforward application of a method due to Pasynkov [9].

Proposition 2. The class $\mathcal{C}$ of all paracompact Σ'-spaces X with $bwX \leq \alpha$, $\mu wX \leq \beta$ and dim $X \leq n$ has a universal element which is a paracompact p-space.

Proof. We may clearly assume that α and β are infinite. If M is a universal metrizable space of weight β, it is readily seen that every member of $\mathcal{C}$ is embeddable in $M \times I^{\alpha}$, $bw(M \times I^{\alpha}) \leq \alpha$ and, by Lemma 5 applied to the projection of $M \times I^{\alpha}$ onto M, $\mu w(M \times I^{\alpha}) = \beta$. Let $\{X_{\lambda} : \lambda \in \Lambda\}$ be the collection of all subspaces of $M \times I^{\alpha}$ in $\mathcal{C}$, X their topological sum and $f: X \to M \times I^{\alpha}$ the map whose restriction to each X_{λ} is its inclusion into $M \times I^{\alpha}$. By Proposition 1, there are continuous $g: X \to Z$ and $h: Z \to Y$ such that h is perfect, $f=h \circ g$, dim $Z \leq$ dim $X \leq n$, $bwZ \leq \alpha$ and $\mu wZ \leq \beta$. Evidently, Z is a universal element of $\mathcal{C}$.

Proposition 3. For every paracompact Σ'-space Y, there is a paracompact Σ'-space Z with dim $Z \leq 0$, $bwZ \leq bwY$, $\mu wZ \leq \mu wY$, and a perfect surjection $h: Z \to Y$.

Proof. Consider a cardinal α such that I^{α} contains a copy of βY, and hence of Y. Let $f: C^{\alpha} \to I^{\alpha}$ be a surjection, where C is Cantor's discontinuum, and $X=f^{-1}(Y)$. Let X, Y be endowed with the subspace uniformities inherited from C^{α}, I^{α}, respectively. Note that every cozero set of Y is uniformly open. Furthermore, $f: X \to Y$ is uniformly continuous and perfect, and by Theorem 2, dim $X \leq$ Dim X. But, by Theorem 3, Dim $X \leq$ Dim C^{α}=dim $C^{\alpha} \leq 0$. Hence dim $X \leq 0$. Now, by Proposition 1, there is a paracompact Σ'-space Z and continuous $g:X \longrightarrow Z$ and $h:Z \longrightarrow Y$ such that $f=h \circ g$, $\dim Z \leq 0$, $bwZ \leq bwY$ and $\mu wZ \leq \mu wY$. Note that because $f:X \longrightarrow Y$ is a perfect surjection, the same is true of g and h.

4. FT for paracompact σ-spaces. In this section, we prove FT for the class of paracompact σ-spaces and σ-locally finite maps, which strengthens [1, Theorem 3]. A continuous $f:X \longrightarrow Y$ onto a paracompact σ-space will be called σ-discrete (resp. σ-locally finite) if there is a σ-discrete (resp. σ-locally finite) network $\mathcal{F}$ for X such that $f(\mathcal{F})$ is a σ-discrete (resp. σ-locally finite) network for Y. Here, it is understood that $f(\mathcal{F})$ should be σ-discrete or σ-locally finite as a collection indexed by the same set as $\mathcal{F}$. Thus, as the example of the projection of an uncountable discrete space onto a singleton shows, it is false that every closed surjection between paracompact σ-spaces is σ-discrete or even σ-locally finite. This casts doubt on the validity of FT for such maps [1, Corollary 1]. However, a perfect map between paracompact σ-spaces is σ-locally finite, which leads to a factorization theorem for these maps.

Lemma 7. Let a uniform function $f:X \longrightarrow Y$ be σ-locally finite, where Y is endowed with its finest uniformity. Then X is paracompact and $\dim X \leq$ $\leq \operatorname{Dim} X$.

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a σ-locally finite network for X with $f(\mathcal{F})$ a σ-locally finite network for the paracompact space Y. As in Lemma 2, there is a σ-locally finite cozero cover $\{G_\alpha : \alpha \in \Lambda\}$ of Y with $f(F_\alpha) \subset G_\alpha$ for each α in Λ. Now, since each cozero set of Y is evidently uniformly open, $\{f^{-1}(G_\alpha) : \alpha \in \Lambda\}$ is a σ-locally finite uniformly open cover of X with $F_\alpha \subset f^{-1}(G_\alpha)$. By Lemma 3, X is paracompact and every open cover of X has a σ-locally finite uniformly open refinement. Finally, by Theorem 2 applied to the identity $X \longrightarrow X$, $\dim X \leq \operatorname{Dim} X$.

Proposition 4. Let $f:X \longrightarrow Y$ be a σ-locally finite map. Then there are σ-locally finite maps $g:X \longrightarrow Z$ and $h:Z \longrightarrow Y$ such that $f=h \circ g$, $\dim Z \leq \dim X$, $bwX \leq bwY$ and $\mu wZ \leq \mu wY$.

Proof. Proposition 1 provides a paracompact Σ'-space W and continuous $g:X \longrightarrow W$ and $h:W \longrightarrow Y$ such that $f=h \circ g$, $\dim W \leq \dim X$, $bwW \leq bwY$ and $\mu wW \leq$ $\leq \mu wY$. Let X, Y, W be endowed with their finest uniformities and $Z=g(X)$ with the subspace uniformity inherited from W. Let $\mathcal{F}$ be a σ-locally finite network for X with $f(\mathcal{F})$ σ-locally. Then $g(\mathcal{F})$ is a σ-locally finite network for Z with $h(g(\mathcal{F}))=f(\mathcal{F})$ σ-locally finite. Hence $h:Z \longrightarrow Y$ is σ-locally finite and, by Lemma 7, Z is a paracompact space so that $g:X \to Z$ is σ-locally finite. Also, Theorem 3 implies $\operatorname{Dim} Z \leq \operatorname{Dim} W = \dim W \leq \dim X$ and, by Lemma 7, $\dim Z \leq \operatorname{Dim} Z \leq \dim X$. Finally, $bwZ \leq bwW \leq bwY$ and, by Lemma 4, since we may clearly assume that Y and hence $f(\mathcal{F})$ and $g(\mathcal{F})$ are infinite, $\mu wZ = |g(\mathcal{F})| \leq$ $\leq |h(g(\mathcal{F}))| = \mu wY$.

The following result follows immediately from Proposition 4, or, more directly, from Proposition 1.

Proposition 5. Let $f:X \to Y$ be a perfect surjection between paracompact σ-spaces. Then there is a paracompact σ-space Z and perfect surjections $g:X \to Z$ and $h:Z \to Y$ such that $f = h \circ g$, $\dim Z \leq \dim X$, $bwZ \leq bwY$ and $\mu wZ \leq \mu wY$.

5. FT for more general paracompact spaces. In this section, we prove FT for the class $\mathcal{C}$ consisting of all paracompact spaces X containing a closed subset E with a base of neighbourhoods of cardinality $\leq wX$ such that E and every closed set of X disjoint from E is a Σ'-space. If X is the topological sum of ω_1 copies of the space of ordinals $\leq \omega_1$, the first uncountable ordinal, and Y is obtained from X by identifying ω_1 in each copy to a single point, then X is a paracompact Σ-space while its closed image Y is, of course paracompact, but not a Σ'-space [6, p. 452, Example 4.18]. However, Y is in $\mathcal{C}$. Note that $\mathcal{C}$ is closed w.r.t. perfect preimages.

Proposition 6. Let $f:X \to Y$ be a continuous function into a member of $\mathcal{C}$. Then there is a member Z of $\mathcal{C}$ and continuous $g:X \to Z$ and $h:Z \to Y$ with h perfect, $f = h \circ g$, $\dim Z \leq \dim X$ and $wZ \leq wY$.

Proof. As in Proposition 1, we can assume that $\tau = wY$ is infinite and f is a perfect surjection. Then there is a closed cover $\{E_\alpha : \alpha < \tau\}$ of X by paracompact Σ'-spaces such that each closed subset of X disjoint from E_0 is contained in some E_α.

Let $\mathcal{C}_\alpha$ be a cover of E_α by Lindelöf sets and $\mathcal{F}_\alpha = \{F_{\alpha\beta} : \beta < \tau\}$ a σ-locally finite (mod $\mathcal{C}_\alpha$)-net for E_α. As in Lemma 2, let $\{G_{\alpha\beta} : \beta < \tau\}$ be a σ-locally finite cozero cover of E_α with $F_{\alpha\beta} \subset G_{\alpha\beta}$. It can be seen that Y can be embedded in I^τ in such a manner that $G_{\alpha\beta} = E_\alpha \cap H_{\alpha\beta}$ for some cozero set $H_{\alpha\beta}$ of I^τ. Letting each subset of Y carry the subspace uniformity induced by I^τ, we see that each $G_{\alpha\beta}$ is uniformly open in E_α so that, in view of Lemma 3, every open cover of E_α has a σ-locally finite uniformly open refinement. Also, $W(Y) \leq \tau$ and if X is endowed with its finest uniformity, then $f:X \to Y$ is uniformly continuous and Theorem 1 provides a subspace

Z of I^{τ} and uniformly continuous surjections $g:X \longrightarrow Z$ and $h:Z \longrightarrow Y$ such that $f=h \circ g$ and $\mathrm{Dim}\, gf^{-1}(E_{\alpha}) \leq \mathrm{Dim}\, f^{-1}(E_{\alpha})$ for $\alpha < \tau$. Note that, by Theorem 3, $\mathrm{Dim}\, f^{-1}(E_{\alpha}) \leq \mathrm{Dim}\, X = \dim X$ and hence $\mathrm{Dim}\, gf^{-1}(E_{\alpha}) \leq \dim X$. Also, since f is a perfect surjection, the same is true of g and h and hence of $h : h^{-1}(E_{\alpha}) \longrightarrow E_{\alpha}$ for each $\alpha < \tau$. Now Theorem 2 applies and gives $\dim h^{-1}(E_{\alpha}) \leq \mathrm{Dim}\, h^{-1}(E_{\alpha}) = \mathrm{Dim}\, gf^{-1}(E_{\alpha}) \leq \dim X$. Thus, $\dim h^{-1}(E_{\alpha}) \leq \dim X$ and if F is a closed subspace of Z disjoint from $h^{-1}(E_0)$, then $F \subset f^{-1}(E_{\alpha})$ for some α so that, as Z is paracompact and hence normal, $\dim F \leq \dim X$. Hence $\dim Z \leq \dim X$ [4].

Proposition 6 like Proposition 1 has corollaries analogous to Propositions 2 and 3.

Finally, by a subset theorem for dim [3, Proposition 2], if X is the union of a σ-locally finite collection of cozero Lindelöf subspaces, then $\dim X \leq \mathrm{Dim}\, X$. It follows that Proposition 6 holds if $\mathscr{C}$ is the class of all paracompact spaces X containing a closed set E such that E and every closed set of X disjoint from E can be expressed as the union of a σ-locally finite collection of cozero Lindelöf subspaces. If $f:X \longrightarrow Y$ is a closed map from a paracompact and locally compact space X onto a space Y, then Y contains a closed discrete subset E such that $f^{-1}(y)$ is compact for each y in Y-E [7]. Hence, for any closed subset F of Y disjoint from E, $f:f^{-1}(F) \longrightarrow F$ is perfect, which readily implies that F is paracompact and locally compact and Y is in $\mathscr{C}$.

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