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NONISOLATED SINGULARITIES OF SOLUTIONS TO A
QUASILINEAR ELLIPTIC SYSTEM

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Abstract: There is presented an example of a quasilinear elliptic system which has solutions with nonisolated discontinuities.

Key words: Elliptic systems of partial differential equations, weak solutions, regularity.

Classification: 35J60, 35D10

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be an open set. We consider quasilinear elliptic systems

$$(1) \quad D_{\alpha} A_{ij}^{\alpha\beta}(u) D_{\beta} u^j = 0, \quad i=1, \dots, m.$$

(The summation convention concerning repeated indices is used throughout the paper; $i, j=1, \dots, m$, $\alpha, \beta=1, \dots, n$.) Referring to the system (1) we always assume the coefficients to be bounded uniformly continuous functions on \mathbb{R}^m satisfying the ellipticity condition

$$(2) \quad A_{ij}^{\alpha\beta} \xi_{\alpha}^i \xi_{\beta}^j \geq |\xi|^2 \quad \text{for each } \xi \in \mathbb{R}^{mn}.$$

By a (weak) solution of (1) we understand a (vector valued) function $u \in W_{loc}^{1,2}(\Omega, \mathbb{R}^m)$ such that

$$(3) \quad D_{\alpha} b_i^{\alpha} = 0, \quad i=1, \dots, m$$

holds in the sense of distributions for

$$(4) \quad b_i^{\alpha} = A_{ij}^{\alpha\beta}(u) D_{\beta} u^j.$$

The counterexample by E. Giusti and M. Miranda [3] shows that for $n \geq 3$ the discontinuous function

$$u: x \mapsto \frac{x}{|x|}$$

solves a system of type (1). Thus one cannot expect full regularity results in this general setting. Typical results estimate the Hausdorff dimension of singularities.

Theorem 1. (E. Giusti [2], see also [1].) Let u be a weak solution of (1). Then there is an open set $\Omega_0 \subset \Omega$ such that u is locally Hölder continuous on Ω_0 and the Hausdorff dimension of $\Omega \setminus \Omega_0$ is less than $n-2$.

An easy modification of the above mentioned counterexample shows that for every $n \geq 4$ there is a system of type (1) which has a solution discontinuous at every point of

$$\{x \in \mathbb{R}^n : x_1 = x_2 = x_3 = 0\}.$$

Thus, it has been known for a long time that the singular set can be nonisolated if $n \geq 4$. We shall prove the following theorem concerning $n=3$.

Theorem 2. There are a weak solution $s: \mathbb{R}^3 \rightarrow \mathbb{R}^6$ of a system of type (1) and a sequence $\{z_k\}$ of discontinuity points for s such that $z_k \neq 0$, $z_k \rightarrow 0$.

2. Reduction to two singularities. Let $z_0 \in \mathbb{R}^3$, $z_0 \neq 0$. Let u_0 be a bounded weak solution of (1). For each $k=0,1,\dots$ denote

$$\begin{aligned} z_k &= z_0 / 2^k, \\ u_k(x) &= u_0(2^k x), \\ B_k &= B(z_k, |z_k|/4) \end{aligned}$$

$B(z,r)$ denotes the open ball with center at z and radius r . By a simple homothety argument we see that u_k also solve (1) (the coefficients are fixed!). Now let us assume that

$$u_1 = u_0 \text{ outside } B_0 \cup B_1.$$

Put

$$s_1 = \begin{cases} u_0 & \text{outside } B_1, \\ u_1 & \text{on } B_1. \end{cases}$$

Of course, $s_1 = u_1$ outside B_0 . As the concept of weak solution is local, we see that s_1 solves (1). We define recurrently ($k=2,3,\dots$)

$$s_k = \begin{cases} s_{k-1} & \text{outside } B_k, \\ u_k & \text{on } B_k. \end{cases}$$

We see by induction that s_k solve (1) and

$$\|s_k - s_{k-1}\|_{W^{1,2}(\Omega)}^2 \leq C(\Omega) 2^{-k}$$

for every bounded domain $\Omega \subset \mathbb{R}^3$. Hence the sequence $\{s_k\}$ has a limit s in the sense of (strong) $W_{loc}^{1,2}$ convergence. By routine arguments we see that s solves (1), too. If u_0 is discontinuous at z_0 , then s is discontinuous at

all points z_k (and at the origin).

Conclusion. Theorem 2 is proved if we construct a system (1) (i.e. coefficients $A_{ij}^{\alpha, \beta}$) and its bounded weak solution u such that

- (5) u is discontinuous at some point $z \in \mathbb{R}^3$, $z \neq 0$,
 (6) $u(2x) = u(x)$ for all $x \in B(z, |z|/4) \cup B(z/2, |z|/8)$.

3. Construction. Fix a decreasing function $\varphi \in C^2([0, 1])$ with

$$\varphi'(1_-) = 0, \quad \varphi(0) = 1, \quad \varphi(1) = 0$$

and denote by ψ its inverse. Now, prolong the functions φ and ψ to $[0, \infty)$ putting

$$\varphi(r) = 0, \quad \psi(t) = 0 \quad \text{for } r, t \in (1, \infty).$$

Fix a point $z \in \mathbb{R}^3$, $|z| = 4$. Denote

$$y = x - z \quad \text{for every } x \in \mathbb{R}^3.$$

Let c be a fixed constant,

$$(7) \quad c \geq 2 \sup_{r \in (0, 1)} (4+r) |\varphi'(r)|.$$

Put

$$(8) \quad u^i(x) = \begin{cases} c \frac{x_i - 3}{|x|} & \text{if } i=4, 5, 6, \\ \varphi(|y|) \frac{y_i}{|y|} & \text{if } i=1, 2, 3. \end{cases}$$

We have

$$(9) \quad D_{\alpha} u^i = \begin{cases} \frac{c}{|x|} \left(\sigma_{i-3}^{\alpha} - \frac{x_i - 3}{|x|^2} \right) & \text{if } i=4, 5, 6, \\ \frac{\varphi(|y|)}{|y|} \left(\sigma_i^{\alpha} - \frac{y_i y_{\alpha}}{|y|^2} \right) + \varphi'(|y|) \frac{y_i y_{\alpha}}{|y|^2} & \text{if } i=1, 2, 3. \end{cases}$$

Denote

$$(10) \quad b_i^{\alpha} = \begin{cases} \frac{c}{|x|} \left(\sigma_{i-3}^{\alpha} + \frac{x_i - 3}{|x|^2} \right) & \text{if } i=4, 5, 6, \\ \frac{\varphi(|y|)}{|y|} \left(\sigma_i^{\alpha} + \frac{y_i y_{\alpha}}{|y|^2} \right) + \varphi'(|y|) \left(\sigma_i^{\alpha} - \frac{y_i y_{\alpha}}{|y|^2} \right) & \text{if } i=1, 2, 3. \end{cases}$$

By a routine calculation we obtain the validity of (3). Obviously, the function u satisfies (5) and (6). It remains only to find coefficients $A_{ij}^{\alpha, \beta}$ such that (4) holds.

4. Coefficients. In this section we construct coefficients $A_{ij}^{\alpha\beta}(v,w)$ ($v,w \in R^3$) such that the functions u^i, b_i^{α} given by (8), (10) satisfy

$$(11) \quad b_i^{\alpha} = A_{ij}^{\alpha\beta}((u^1, u^2, u^3), (u^4, u^5, u^6)) D_{\beta} u^j.$$

We follow essentially the method due to J. Souček [6]. Denote

$$Y = \psi(|v|),$$

$$X = \begin{cases} |z+Y| \frac{v}{|v|^2}, & v \neq 0, \\ 4, & v=0, \end{cases}$$

$$h = \frac{cY}{cY + X\varphi(Y)},$$

$$f = \frac{X\varphi(Y)}{cY + X\varphi(Y)},$$

$$g = \frac{XY\varphi'(Y)}{cY + X\varphi(Y)}.$$

Using the conventions

$$\frac{v_i v_{\alpha}}{|v|^2} = 0 \text{ if } v=0, \quad \frac{w_i - 3w_{\alpha}}{|w|^2} = 0 \text{ if } w=0,$$

$$v_i = w_i = 0 \text{ if } i \notin \{1, 2, 3\},$$

we define

$$B_i^{\alpha} = h \left(\delta_{i-3}^{\alpha} + \frac{w_i - 3w_{\alpha}}{|w|^2} \right) + f \left(\delta_i^{\alpha} + \frac{v_i v_{\alpha}}{|v|^2} \right) + g \left(\delta_i^{\alpha} - \frac{v_i v_{\alpha}}{|v|^2} \right),$$

$$T_i^{\alpha} = h \left(\delta_{i-3}^{\alpha} - \frac{w_i - 3w_{\alpha}}{|w|^2} \right) + f \left(\delta_i^{\alpha} - \frac{v_i v_{\alpha}}{|v|^2} \right) + g \frac{v_i v_{\alpha}}{|v|^2}$$

($i=1, \dots, 6; \alpha=1, 2, 3$). Finally we put

$$Q = \min(1, |w|) (3B_i T_i - T_i T_i)^{-1},$$

$$A_{ij} = \delta_i^j \delta_{\alpha}^{\beta} + Q(3B_i^{\alpha} - T_i^{\alpha})(3B_j^{\beta} - T_j^{\beta})$$

($i, j=1, \dots, 6; \alpha, \beta=1, 2, 3$). By (7) we verify that Q is a nonnegative bounded function on $R^3 \times R^3$. Indeed, we have

$$3B_i T_i - T_i T_i = \frac{\frac{3}{4}c^2 Y^2 + X^2 \varphi^2(Y) + 3(X\varphi(Y) + 2XY\varphi'(Y))^2 + \frac{13}{4}Y^2(c^2 - 4X^2(\varphi'(Y))^2)}{(cY + X\varphi(Y))^2}.$$

The coefficients are continuous: discontinuities $\frac{v_i v_{\alpha}}{|v|^2}, \frac{w_i - 3w_{\alpha}}{|w|^2}, X$ are always

neutralized being multiplied by vanishing continuous functions. We observe that the supremum of $|A_{ij}^{\alpha\beta}|$ as well as the modulus of continuity of $A_{ij}^{\alpha\beta}$ are estimated by the same quantities on $\{v: |v| \leq 1\} \times \{w: |w| \leq 1\}$. Hence the coefficients are bounded and uniformly continuous. By a direct calculation we see that (11) is satisfied.

5. **Some remarks.** A) If we admit discontinuous coefficients (Borel measurable only) then Theorem 1 does not hold.

Let $a_{ij}^{\alpha\beta}$ be bounded Borel measurable functions on R^3 satisfying

$$a_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \geq |\xi|^2 \text{ for each } \xi \in R^9.$$

Let v be a weak solution to the linear system

$$(12) \quad D_\alpha a_{ij}^{\alpha\beta}(x) D_\beta v^j = 0.$$

Then the function u defined by

$$u^i = v^i, \quad i=1,2,3, \quad u^i = x_{i-3}, \quad i=4,5,6$$

solves the quasilinear system

$$D_\alpha A_{ij}^{\alpha\beta}(u) D_\beta u^j = 0$$

where

$$A_{ij}^{\alpha\beta}(u) = \begin{cases} a_{ij}^{\alpha\beta}(u^4, u^5, u^6) & \text{if } i, j \in \{1, 2, 3\}, \\ \sigma_i^j \sigma_\alpha^\beta & \text{otherwise.} \end{cases}$$

However, solutions of (12) can be everywhere discontinuous (see [4]).

B) Our example does not fill the gap between the estimate of Hausdorff dimension of singular sets given in Theorem 1 and $n-3$ -dimensionality of singular sets in the known counterexamples. It is not even clear whether the singular set for $n=3$ can be uncountable.

C) It would be nice to have counterexamples (or further positive regularity results) in case when the quasilinear system (1) is obtained as a system in variation. The only known counterexamples (see e.g. J. Nečas [5]) have one singular point.

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