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AN APPLICATION OF THE THEORY OF ISOSCELES (ULTRAMETRIC) SPACES TO THE TRNKOVÁ - VINÁREK THEOREM

A.J. LEMIN

Abstract: For every commutative semigroup (S,+), there is constructed a family $\{X(s);s\in S\}$ of complete isosceles spaces of the diameter ≤ 1 satisfying the following conditions:

1. X(s+s´) is isometric to X(s).X(s´),
2. X(s) is homeomorphic to X(s´) iff s=s´.

Key words: Semigroup, representations, product and sum in a category, isosceles (ultrametric) space.

Classification: Primary 54B10, 54H10

Secondary 20M30

Introduction. In the present paper we give a stronger form of a profound and difficult theorem by Trnková and Vinárek on representations of commutative semigroups. This form is an easy consequence of the general theory of isosceles spaces.

We recall that a metric space (X,d) is called an isosceles (or ultrametric, or non-archimedean) if its metric satisfies the strengthened triangle axiom:

$$d(x,z)\leq \max\ (d(x,y),d(y,z)).$$

The inequality means precisely that every three points x, y, z form an isosceles triangle whose base is less or equal to its sides.

Studies of isosceles spaces have, in recent years, resulted in a considerable and interesting theory containing many deep and bright results. For the purpose of this paper, only some basic facts of the theory will be needed. For the reader´s convenience they are listed in § 1. Readers acquainted with the theory of isosceles spaces may omit this paragraph.

§ 2 relates to our application of the lemmas in § 1 to the representation theory as well as to the proof of the theorem.

In § 3 we publish, with kind permission of J. Vinárek, an example of a zero-dimensional space (X,d) whose countable degree is not zero-dimensional in the category METR.

§ 1. Basic properties of the category of isosceles spaces. Isosceles spaces were defined above (cf. Introduction). These spaces are of importance in many branches of mathematics: in number theory (rings of p-adic numbers), in abstract algebra (theory of non-archimedean rings), in general topology (the Baire space $B_{\aleph_0}$ and its generalization B_{τ}), in p-adic analysis, in complex analysis etc. Their general (axiomatic) description was given by M. Krasner [2]. Their basic geometrical and topological properties have been studied in [1],[2],[3],[7]; de Groot, in particular, has proved that any isosceles space is zero-dimensional and that any metric zero-dimensional space is homeomorphic to an isosceles space. For representation theory applications we shall need those of the properties of the category of isosceles spaces which are described in [5]. For clarity sake, we give first some propositions on arbitrary metric spaces and secondly we point out those ones which apply to isosceles spaces.

Let METR ($METR_c$) be the category of all metric spaces (of diameter $\leq c$) and of non-stretching mappings (cf. [4],[8]), $METR^{\bullet}$ the category of all the metric spaces with a marked point and non-stretching mappings transforming marked points to marked points.

Proposition 1. In $METR_c$ there is defined a sum $(X,d_{\Sigma})=m\sum_{\alpha\in A}(X_{\alpha},d_{\alpha})$ of an arbitrary number of spaces (X_{α},d_{α}) ("the metric sum"). In particular, $X=\sum_{\alpha\in A}X_{\alpha}$ is a sum of sets X in the category SET where $d_{\Sigma}(x_{\alpha},y_{\alpha})=$ $=d_{\alpha}(x_{\alpha},y_{\alpha})$, $d_{\Sigma}(x_{\alpha},y_{\beta})=c$ for $\alpha\neq\beta$.

Proof is obvious.

Property 1. Metric sum of complete spaces is complete.

Proposition 2. In $METR_c$ there is defined a product $(X,d_{\pi})=$ $=m\prod_{\alpha\in A}(X_{\alpha},d_{\alpha})$ of an arbitrary number of spaces (X_{α},d_{α}) ("the metric product"), viz. $X=\prod_{\alpha\in A}X_{\alpha}$ in SET, $d_{\pi}((x_{\alpha}),(y_{\alpha}))=\sup\{d_{\alpha}(x_{\alpha},y_{\alpha});\alpha\in A\}$.

Proof is straightforward.

Property 2. Metric product of complete spaces is complete.

Corollary. The space of all the mappings of an arbitrary set A into a complete metric space (X,d) with the metric of a uniform convergence is complete.

Proof. For any $\alpha \in A$ put $(X_\alpha, d_\alpha)=(X,d)$ and use Property 2.

Proposition 3. In $METR_c$ there is defined a product
$m \prod^{\bullet}_{\alpha \in A} (X_\alpha, x^\bullet_\alpha, d_\alpha)=(X,x^\bullet,d^\bullet)$, namely $X=\prod_{\alpha \in A} X_\alpha$, $x^\bullet=(x^\bullet_\alpha)$, $d^\bullet_\pi = d_\pi$.

Proof is obvious.

Proposition 4. In METR there is defined a sum $\sum^{\bullet}_{\alpha \in A} (X_\alpha, x^\bullet_\alpha, d_\alpha)=$
$=(X,x^\bullet,d^\bullet_\Sigma)$ ("the pointed sum"), namely $(X,x^\bullet)=\sum_{\alpha \in A} (X_\alpha, x^\bullet_\alpha)$ (sum in $SET^\bullet$),
$d^\bullet_\Sigma(x_\alpha, x^\bullet)=d_\alpha(x_\alpha, x^\bullet_\alpha)$, $d^\bullet_\Sigma(x_\alpha, y_\alpha)=d_\alpha(x_\alpha, y_\alpha)$, $d^\bullet_\Sigma(x_\alpha, x_\beta)=d_\alpha(x_\alpha, x^\bullet_\alpha)+$
$+d_\beta(x_\beta, x^\bullet_\beta)$ for $\alpha \neq \beta$.

Proof is left to the reader.

The space $(X,x^\bullet,d^\bullet_\Sigma)$ could be thought of as "a fan" - i.e. as the union of spaces $(X_\alpha, x^\bullet_\alpha)$ whose $x^\bullet$ points are "glued" together.

Property 3. If $(X_\alpha, x^\bullet_\alpha, d_\alpha)$ are complete then $\sum^{\bullet}_{\alpha \in A} (X_\alpha, x^\bullet_\alpha, d_\alpha)$ is complete.

Now, let the set A of indices be a discrete metric space (A,D), i.e. for any $\alpha \in A$ there exists $r_\alpha \in R^+$ such that the ball $B(x,r_\alpha)$ contains only the point x. Let spaces (X_α, d_α) be such that $diam(X_\alpha, d_\alpha) \leq r_\alpha$. Then it is possible to define the sum of spaces (X_α, d_α) with respect to the space (A,D) as follows.

Definition. The sum of spaces (X_α, d_α) with respect to the space (A,D) is the space $(X, d_{A\Sigma})$ where $X=\sum_{\alpha \in A} X_\alpha$ in SET, $d_{A\Sigma}(x_\alpha, y_\alpha)=d_\alpha(x_\alpha, y_\alpha)$, $d_{A\Sigma}(x_\alpha, x_\beta)=D(\alpha,\beta)$ for $\alpha \neq \beta$.

It is easy to see that the definition of the metric sum commonly used (see Proposition 1 above) is a special case of the definition just given. It holds when $D(\alpha,\beta)=c$ for any pair α, β, $\alpha \neq \beta$.

Consider now the category of isosceles spaces.

Let ULTRAMETR ($ULTRAMETR_c$, $ULTRAMETR^\bullet$) be a full subcategory of METR ($METR_c$, $METR^\bullet$) consisting of isosceles spaces.

Lemma 1. $ULTRAMETR_c$ is closed in $METR_c$ with respect to sums, i.e. metric sums of isosceles spaces are isosceles.

Proof. Put $x_\alpha, y_\beta, z_\gamma \in m \sum_{\alpha \in A} (X_\alpha, d_\alpha)$. If $\alpha=\beta=\gamma$, then the triangle $x_\alpha, y_\alpha, z_\alpha$ is isosceles, because all spaces (X_α, d_α) are isosceles. If

$\alpha \neq \beta = \gamma$, then the triangle x_α, y_β, z_γ is equilateral.

Lemma 2. The sum $m A\,\Sigma(X_\alpha, d_\alpha)$ of isosceles spaces (X_α, d_α) with respect to an isosceles space (A,D) is isosceles.

Proof is similar to the proof of Lemma 1.

Lemma 3. $ULTRAMETR_C$ ($ULTRAMETR_C^{\bullet}$) is closed in $METR_C$ ($METR_C^{\bullet}$) with respect to products.

Proof. Suppose $(x_\alpha), (y_\alpha), (z_\alpha) \in m \prod_{\alpha \in A} (X_\alpha, d_\alpha) = (X, d_\pi)$. Then $d_\pi((x_\alpha),(z_\alpha)) = \sup\{d_\alpha(x_\alpha, z_\alpha); \alpha \in A\} \leq \sup\{\max\{d_\alpha(x_\alpha, y_\alpha), d_\alpha(y_\alpha, z_\alpha)\}; \alpha \in A\} = \max\{\sup\{d_\alpha(x_\alpha, z_\alpha); \alpha \in A\}, \sup\{d_\alpha(y_\alpha, z_\alpha); \alpha \in A\}\} = \max\{d_\pi((x_\alpha),(y_\alpha)), d_\pi((y_\alpha),(z_\alpha))\}$.

Subcategory $ULTRAMETR^{\bullet}$ ($ULTRAMETR_C^{\bullet}$) is not closed in $METR^{\bullet}$ ($METR_C^{\bullet}$) with respect to products. However, if we consider $ULTRAMETR^{\bullet}$, then there is defined an isosceles sum of spaces $(X_\alpha, x_\alpha^{\bullet}, d_\alpha)$.

Lemma 4. In the category $ULTRAMETR^{\bullet}$ there is defined the sum $(X, x^{\bullet}, \Delta) = m\Sigma(X_\alpha, x_\alpha^{\bullet}, d_\alpha)$ of an arbitrary number of isosceles spaces. Namely, $(X, x^{\bullet}) = \Sigma(X_\alpha, x_\alpha^{\bullet})$ in $SET^{\bullet}$, $\Delta(x_\alpha, x^{\bullet}) = d_\alpha(x_\alpha, x_\alpha^{\bullet})$, $\Delta(x_\alpha, y_\alpha) = d_\alpha(x_\alpha, y_\alpha)$; $\Delta(x_\alpha, x_\beta) = \max\{d_\alpha(x_\alpha, x_\alpha^{\bullet}), d_\beta(x_\beta, x_\beta^{\bullet})\}$ for $\alpha \neq \beta$.

Proof. The triangles $(x_\alpha, y_\alpha, z_\alpha)$ and $(x_\alpha, y_\alpha, x_\alpha^{\bullet})$ are isosceles. It remains to prove that also the triangles $(x_\alpha, y_\beta, x^{\bullet})$, $(x_\alpha, y_\alpha, y_\beta)$, $(x_\alpha, x_\beta, x_\gamma)$ are isosceles for $\alpha \neq \beta \neq \gamma \neq \alpha$. We leave proving of the first two examples to the reader, and are going to prove the third case. Choose α, β, γ in such a way that $d_\alpha(x_\alpha, x_\alpha^{\bullet}) \geq d_\beta(x_\beta, x_\beta^{\bullet}) \geq d_\gamma(x_\gamma, x_\gamma^{\bullet})$. Then $\Delta(x_\alpha, x_\beta) = \Delta(x_\alpha, x_\gamma) = d_\alpha(x_\alpha, x_\alpha^{\bullet}) \geq \max\{d_\beta(x_\beta, x_\beta^{\bullet}), d_\gamma(x_\gamma, x_\gamma^{\bullet})\} = \Delta(x_\beta, x_\gamma)$.

§ 2. A stronger form of the Trnková - Vinárek theorem. Let $(S,+)$ be a commutative semigroup. It is called to be represented in the category $\mathcal{K}$ if there exists a family $\{X(s) \in \mathrm{Ob}\,\mathcal{K}; s \in S\}$ such that $X(s+s')$ is isomorphic to $X(s)$. $X(s')$ and $X(s)$ is isomorphic to $X(s')$ iff $s=s'$.

In 1977 V. Trnková ([9]) proved that for every commutative semigroup $(S,+)$ there exists a family of complete metric spaces of the diameter ≤ 1 such that $X(s+s')$ is isometric to $X(s)$. $X(s')$ ($X(s+s') \cong X(s).X(s')$) and $X(s)$ is homeomorphic to $X(s')$ ($X(s) \sim X(s')$) iff $s=s'$. In 1982 J. Vinárek ([11]) proved a stronger form of the theorem by showing that only zero-dimensional spaces can be taken into account in the theorem. The long and difficult

Vinárek's proof is far from being trivial, since infinite products in METR are used in constructing spaces X(s) and metric products of infinite number of zero-dimensional spaces need not be necessarily zero-dimensional. The latter is illustrated by a very interesting example constructed by J. Vinárek. The example is quoted in § 3.

The following stronger form of the theorem mentioned could be easily obtained from the theory of isosceles spaces.

Theorem. For any commutative semigroup (S,+) there exists a family $\{X(s); s\in S\}$ of complete isosceles spaces of a diameter ≤ 1 such that

$$X(s+s')\cong X(s).X(s'),$$
$$X(s)\sim X(s') \text{ iff } s=s'.$$

Proof. Using Vinárek's notations and reasoning (see [11]), we show which alterations of his argument are needed to obtain the theorem formulated. According to V. Trnková theórem ([10]), every abelian semigroup is isomorphic to the semigroup exp $N^{\aleph_0.\text{card } S}$. Thus, as noted by Vinárek, it is sufficient to construct for every subset A of $N^{\aleph_0.\text{card } S}$ a complete zero-dimensional (isosceles, resp.) space X(A) satisfying the conditions:

(i) $X(A+A')\cong X(A).X(A')$

(ii) $X(A)\sim X(A')$ iff $A=A'$.

Hence, it suffices to construct for any function $f\in N^{\aleph_0.\text{card } S}$ an (isosceles) complete zero-dimensional space X(f) with a diameter ≤ 1 such that for every $f,g\in N^{\aleph_0.\text{card } S}$ and $a,A'\subseteq N^{\aleph_0.\text{card } S}$ the following conditions hold:

(1) $X(f+g)\cong X(f).X(g)$,

(2) $\coprod_{2^{\aleph_0.\text{card } S}} (\coprod_{h\in A} X(h))$ is (isosceles) 0-dimensional,

(3) $\coprod_{2^{\aleph_0.\text{card } S}} (\coprod_{h\in A} X(h)) \sim \coprod_{2^{\aleph_0.\text{card } S}} (\coprod_{k\in A'} X(k))$

iff $A=A'$.

Trnková's general method for constructing such spaces is the following: find a collection $\{X_a; a\in\aleph_0.\text{card } S\}$ of objects of a given category such that for every $A,A'\subseteq N^{\aleph_0.\text{card } S}$ the following condition holds:

$$\coprod_{2^{\aleph_0.\text{card } S}} (\coprod_{h\in A} \prod_{a\in\aleph_0.\text{card } S} X_a^{h(a)}) \sim \coprod_{2^{\aleph_0.\text{card } S}} (\coprod_{k\in A'} \prod_{a\in\aleph_0.\text{card } S} X_a^{k(a)})$$

iff $A=A'$.

Vinárek's construction is made in such a way. Let S be an arbitrary commutative semigroup and γ the first ordinal with card $\gamma = \aleph_0$.card S. For every $a \in \gamma$ choose a set $B_a = \{\beta_{a,n}; n \in N^+\}$ of cardinal numbers such that the following condition holds:

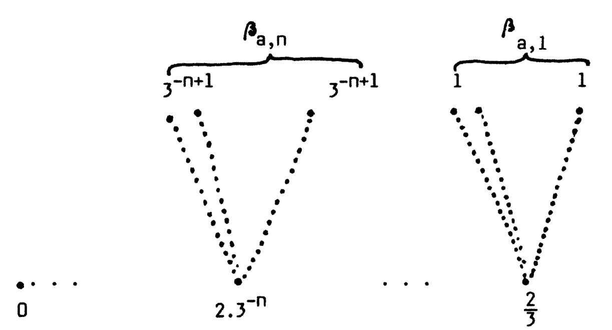
$$2^{\gamma} < \beta_{0,1};\ \beta_{a,n} < \beta_{a,n+1};\ \beta_{a,1} > (\sup\{\beta_b; b < a\})^{\gamma} \text{ where}$$

$\beta_b = \sup\{\beta_{b,n}; n \in N^+\}$.

Spaces are constructed for the Cantor set

$$C = [0,1] \setminus \bigcup_{n=1}^{+\infty} \bigcup_{i=1}^{\frac{3^n-1}{2}} \left]\frac{2i-1}{3^n}, \frac{2i}{3^n}\right[\text{ as follows:}$$

Let $C_n = [2.3^{-n}, 3^{-n+1}] \cap C$ be the n-th "Cantor interval" and $D = \{2.3^{-n}; n \in N^+\} \cup \{0\}$ the set of "left" ends of these intervals. For any $a \in \gamma$, $n \in N^+$ take a pointed sum of $\beta_{a,n}$ copies of C_n with the common points which are left ends of these intervals. The space X_a is defined as a sum of fans obtained with respect to D as shown at the picture



In this construction, sets C, C_n and D are taken with the usual real-line metric. Pointed sums of "Cantor intervals" are taken in METR$^\bullet$ as well as sums of "fans" with respect to D. (J. Vinárek has not introduced the concept of pointed metric sum with respect to a set. Hence, he has to define the metric explicitly.)

Now, we are going to show how to change the metric on Vinárek's spaces in order to obtain isosceles zero-dimensional spaces. It is necessary to take C with the metric induced from the Baire space of its intervals and the set of countable sequences of nonnegative integers with an isosceles metric. Pointed sums of pointed Cantor intervals ("left" ends as chosen points) need to be taken in ULTRAMETR$^{\bullet}$. By Lemma 4, they are isosceles. By Lemma 2, the metric sum of "fans" with respect to an isosceles space is isosceles as well. Metric product $\prod_{a \in \aleph_0 . \mathrm{card}\, S} X_a^{f(a)}$ is isosceles according to Lemma 3. Finally, Lemma 1 implies that any X(A) is isosceles. Q.E.D.

§ 3. Non-preservation of zero-dimensionality by infinite products in METR_c. In this paragraph, we quote with kind permission of J. Vinárek an example of a zero-dimensional metric space V whose countable power is not zero-dimensional.

Suppose $V=Q\cap[0,1]$, i.e. V is the set of rational points of interval $[0,1]$ with the usual real-line metric. We are going to prove that $V^{\aleph_0}$ is not zero-dimensional in METR_1. Denote by B(x,r) a spherical neighbourhood of a point x with a diameter r. Let U be an arbitrary open neighbourhood of $\bar{0}=(0,\dots,0,\dots)$. We are going to prove that U is not closed. (This proof is analogous to the famous Erdös´ proof of 1-dimensionality of rational points of the Hilbert space.)

By the method of mathematic induction, let us construct an increasing sequence $\{r_n\}$ of rationals such that $x_n=(r_1,\dots,r_n,r_n,\dots)\in U$ for any $n\in N$ and $\mathrm{dist}(x_n,V\setminus U)< 2^{-n}$. Then $x=(r_1,\dots,r_n,r_{n+1},\dots)\in \overline{U}\cap(\overline{V\setminus U})$ and U is not closed. Hence, V is not zero-dimensional.

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