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Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 457--463

Persistent URL: <http://dml.cz/dmlcz/106661>

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CHROMATIC NUMBER OF PRODUCTS OF GRAPHS

Vladimír PUŠ

Abstract: We give a description of all products $G * H$ of simple graphs (excepting the direct product) having the following property: the chromatic number $\chi(G * H)$ is a function of numbers $\chi(G)$ and $\chi(H)$. We also determine these functions.

Key words: Product of graphs, chromatic number.

Classification: 05C15

0. Introduction. L. Lovász's well-known problem is the following one: Is it true that the chromatic number of the direct product of simple graphs is given by the formula $\chi(G \times H) = \min(\chi(G), \chi(H))$? (In other words: Does the function f exist such that $\chi(G \times H) = f(\chi(G), \chi(H))$ for every pair G, H of simple graphs?)

In this paper we describe all products $G * H$ of simple graphs (excepting the direct product) for which there exists a function f such that the chromatic number of $G * H$ is given by the formula $\chi(G * H) = f(\chi(G), \chi(H))$. The explicit expressions of the functions f are also given.

1. Definitions. The graphs we consider are simple graphs, i.e. undirected graphs without loops and multiple edges. The set of vertices of a graph G is denoted by $V(G)$, $E(G)$ is the set of edges. We will consider only graphs with a non-empty set of vertices.

By $\chi(G)$ we denote the chromatic number of G .

K_n is the complete graph on n vertices, D_n is the discrete graph on n vertices and C_n is the circuit of the length n .

Let us recall the general definition of products of simple graphs (see [1]).

Let $p: \{1, -1, 0\} \times \{1, -1, 0\} \rightarrow \{1, -1, 0\}$ be a fixed mapping such that $p(i, j) = 0$ iff $i = j = 0$.

For a simple graph $G=(V,E)$ and a pair of vertices $x,y \in V$ define

$$s(x,y) = \begin{cases} 1 & \text{iff } \{x,y\} \in E \\ -1 & \text{iff } \{x,y\} \notin E \text{ and } x \neq y \\ 0 & \text{iff } x=y \end{cases}$$

(i.e. $s:V \times V \rightarrow \{1,-1,0\}$).

Given a pair G,H of simple graphs, define the product $G \times^p H$ as follows:

$$V(G \times^p H) = V(G) \times V(H)$$

and

$$E(G \times^p H) = \{ \{ (x,x'), (y,y') \}; p(s(x,y), s(x',y')) = 1 \}.$$

This definition covers all products of graphs (there exists $2^8=256$ different products).

For example, let $p(i,j)=1$ iff $i=j=1$. Then \times^p is the direct product; we denote \times instead of \times^p in this case.

Let $p(i,j)=1$ iff $i=1$ or $j=1$. The product we obtain is in [2], p. 52, called the cartesian sum and denoted by \oplus .

Let $p(i,j)=1$ iff $i=0$ and $j=1$, or $i=1$ and $j=0$. Then \times^p is the well-known cartesian product; this product will be denoted by \square .

Let $p(i,j)= -1$ iff $i= -1$ or $j= -1$. Then \times^p is the so-called strong product; we denote it by \boxtimes .

Let $p(i,j)=1$ iff either $i=1$, or $i=0$ and $j=1$. Then we obtain the so-called lexicographic product (or the substitution of the graph H into G). In this we denote $G \times^p H = G[H]$.

2. Auxiliary results. First we notice that

$$(G \times H) \cup (G \square H) = G \boxtimes H$$

and that

$$\chi(G \square H) = \max(\chi(G), \chi(H)).$$

In the following proposition we show that generally $\chi(G \oplus H) < \chi(G) \cdot \chi(H)$.

Proposition 1. $\chi(C_{2m+1} \oplus C_{2n+1}) \leq 8$ for $m, n \geq 2$.

Proof: Let G, H be graphs. For $v \in V(G)$ and $w \in V(H)$ denote $S_v = \{v\} \times V(H)$ and $R_w = V(G) \times \{w\}$. The mapping $\varphi: V(G) \times V(H) \rightarrow \{1, 2, \dots, k\}$ is a colouring of the graph $G \oplus H$ by k colours if and only if the following conditions hold:

$$\{v_1, v_2\} \in E(G) \implies S_{v_1} \cap S_{v_2} = \emptyset$$

and

$$\{w_1, w_2\} \in E(H) \implies R_{w_1} \cap R_{w_2} = \emptyset.$$

Hence, the following matrix (with $2n+1$ rows and $2m+1$ columns) represents a colouring of the graph $C_{2m+1} \oplus C_{2n+1}$ by 8 colours.

| | | | | | | | | | |
|---|---|---|---|-----|---|---|---|---|---|
| 1 | 2 | 1 | 2 | ... | 1 | 2 | 1 | 2 | 3 |
| 4 | 5 | 4 | 5 | ... | 4 | 5 | 4 | 5 | 6 |
| 1 | 2 | 1 | 2 | ... | 1 | 2 | 1 | 2 | 3 |
| 4 | 5 | 4 | 5 | ... | 4 | 5 | 4 | 5 | 6 |
| . | | | | | | | | . | . |
| . | | | | | | | | . | . |
| . | | | | | | | | . | . |
| 1 | 2 | 1 | 2 | ... | 1 | 2 | 1 | 2 | 3 |
| 4 | 5 | 4 | 5 | ... | 4 | 5 | 4 | 5 | 6 |
| 1 | 2 | 1 | 2 | ... | 1 | 2 | 1 | 2 | 8 |
| 4 | 5 | 4 | 5 | ... | 4 | 5 | 4 | 5 | 3 |
| 7 | 8 | 7 | 8 | ... | 7 | 8 | 6 | 7 | 8 |

Proposition 2. Suppose that there exists a function f such that

$$\chi(G \times H) \stackrel{p}{\leq} f(\chi(G), \chi(H)).$$

Then the following condition holds:

$$(T) \quad p(i, j) = 1 \implies i = 1 \text{ or } j = 1.$$

Conversely, if the condition (T) is fulfilled, then $\chi(G \times H) \stackrel{p}{\leq} \chi(G) \cdot \chi(H)$.

Proof: Suppose that there exists a function f such that $\chi(G \times H) \stackrel{p}{\leq} f(\chi(G), \chi(H))$ and that $p(i, j) = 1$. Assume that for contradiction $i \neq 1$ and $j \neq 1$.

If $(i, j) = (-1, -1)$ then $K_n \subseteq D_n \times D_n$, hence $n \leq \chi(D_n \times D_n) \leq f(1, 1)$ for every n , a contradiction.

If $(i, j) = (-1, 0)$ then $D_n \times D_1 \subseteq K_n$, hence $n = \chi(D_n \times D_1) \leq f(1, 1)$, a contradiction. Similarly, the case $(i, j) = (0, -1)$ leads to a contradiction.

Conversely, let the condition (T) be fulfilled. Then the product $A \times B$

of discrete sets $A \subseteq V(G)$ and $B \subseteq V(H)$ is a discrete set in $G \times^p H$, which implies that $\chi(G \times^p H) \leq \chi(G) \cdot \chi(H)$.

3. The main result

Theorem

(I) Suppose that p fulfils the following conditions:

(1) $p(i,j)=1 \Rightarrow i=1$ and

(2) $p(1,0)=1$.

Then $\chi(G \times^p H) = \chi(G)$.

(II) Suppose that

(3) $p(i,j)=1 \Rightarrow j=1$ and

(4) $p(0,1)=1$.

Then $\chi(G \times^p H) = \chi(H)$.

(III) If \times^p is the cartesian product, then $\chi(G \times^p H) = \max(\chi(G), \chi(H))$.

(IV) If p is identically equal to -1 , then $\chi(G \times^p H)$ is identically equal to 1 .

(V) Assume that there exists a function f such that $\chi(G \times^p H) = f(\chi(G), \chi(H))$ for every pair G, H of (finite) graphs. Then either \times^p is the direct product or some of the cases (I)-(IV) occurs.

(VI) Assume that there exists a function f such that $\chi(G \times^p H) = f(\chi(G), \chi(H))$. Then $\chi(G \times^p H) = \min(\chi(G), \chi(H))$.

Proof: Suppose that there exists a function f such that $\chi(G \times^p H) = f(\chi(G), \chi(H))$. Then the condition (T) from Proposition 2 is satisfied. Now we distinguish four cases (α), (β), (γ) and (σ).

(α) Let $p(1,0)=1$ and $p(0,1)=-1$.

Then $f(n,m) = \chi(K_n \times^p K_m) = n$. For this, let $V(K_n) = \{1, 2, \dots, n\}$ and $V(K_m) = \{1, 2, \dots, m\}$. Then the function φ defined by $\varphi(i,j)=i$ is a colouring of $K_n \times^p K_m$ by n colours and moreover $K_n \subseteq K_n \times^p K_m$.

It follows that $p(-1,1) = -1$. Indeed, $p(-1,1)=1$ implies $K_n \subseteq D_n \times^p K_n$ for every n , and so $n = \chi(D_n \times^p K_n) = f(1,n) = 1$, a contradiction. Hence, according to

(T), the following condition holds:

$$(1) \quad p(i,j)=1 \implies i=1.$$

Since, moreover, by the assumption, $p(1,0)=1$, the conditions (1) and (2) in Part (I) of Theorem are fulfilled. Conversely we show that under these conditions $\chi(G \times^p H) = \chi(G)$.

Indeed, (1) follows from the fact that $A \times V(H)$ is a discrete set for every discrete set $A \subseteq V(G)$. Hence, $\chi(G \times^p H) \leq \chi(H)$. Further, (2) follows from $G \subseteq G \times^p H$ and so $\chi(G \times^p H) \geq \chi(G)$.

$$(\beta) \quad \text{Let } p(1,0) = -1 \text{ and } p(0,1) = 1.$$

Then, similarly as in the case (α), the conditions (3) and (4) in Part II of Theorem follow. Conversely, these conditions imply that $\chi(G \times^p H) = \chi(H)$.

Now we suppose that

$$(P) \quad p(1,0) = p(0,1).$$

We divide this case into two partial cases (γ) and (δ).

$$(\gamma) \quad \text{In addition, let } p(1,1) = -1.$$

By (P), either $p(1,0) = p(0,1) = 1$ or $p(1,0) = p(0,1) = -1$.

(γ_1) In the first case we have $K_n \times^p K_m \cong K_n \square K_m$, hence $\max(n,m) = \chi(K_n \times^p K_m) = f(\chi(K_n), \chi(K_m)) = f(n,m)$. It follows that $p(1,-1) = p(-1,1) = -1$. Indeed, if for example $p(1,-1) = 1$ then $K_2 \times^p (K_2 + K_2)$ contains K_4 (see the figure) and so $4 \leq \chi(K_2 \times^p (K_2 + K_2)) = f(2,2) = 2$, a contradiction. Thus, \times^p is the cartesian product; hence, the case (III) in Theorem has occurred.



Figure

(γ_2) In the second case we have $K_n \times^p K_m \cong D_{n \cdot m}$, so $f(n,m) = 1$. But this means that p is identically equal to -1 and $\chi(G \times^p H)$ is identically equal to

-1 and $\chi(G \times^p H)$ is identically equal to 1, which is the situation described in Theorem, Part (IV).

(\mathcal{O}) Let $p(1,1)=1$.

By (P) we again consider two cases.

(\mathcal{O}_1) Let $p(1,0)=p(0,1)=1$. Then $K_n \times^p K_m \cong K_{n \cdot m}$. Hence $f(n,m) = \chi(K_n \times^p K_m) = n \cdot m$. Further, by (P), $G \times^p H \cong G \oplus H$ (more exactly, this case includes the strong product, the lexicographic product and the cartesian sum). Therefore, by Proposition 1, we have

$$9 = 3 \cdot 3 = f(3,3) = \chi(C_{2m+1} \times^p C_{2n+1}) \neq \chi(C_{2m+1} \oplus C_{2n+1}) = 8,$$

a contradiction.

(\mathcal{O}_2) Let $p(1,0)=p(0,1) = -1$. Then $\chi(K_n \times^p K_m) \leq \min(n,m)$. For this, if $V(K_n) = \{1,2,\dots,n\}$ and $V(K_m) = \{1,2,\dots,m\}$, then the function φ defined by $\varphi((i,j))=i$ (or $\varphi((i,j))=j$) is a colouring of the graph $K_n \times^p K_m$ by n (or m) colours. Conversely, since $p(1,1)=1$, we have $K_{\min(n,m)} \subseteq K_n \times^p K_m$. Therefore, $f(n,m) = \chi(K_n \times^p K_m) = \min(n,m)$.

The last formula implies that $p(-1,1)=p(1,-1) = -1$. To see this, let us suppose without loss of generality that $p(-1,1)=1$. Then $n = \chi(D_n \times^p K_n) = f(1,n) = \min(1,n) = 1$ for every n , a contradiction. Thus, \times^p is the direct product (the case (VI) in Theorem).

It is clear that the discussion just given includes proofs of all propositions (I)-(VI) in Theorem.

The previous theorem gives also the answer to the question of C. Thomassen whether there exists a product \times^p such that $\chi(G \times^p H) = \chi(G) \cdot \chi(H)$.

Corollary. There is no product \times^p of simple graphs with the following property: $\chi(G \times^p H) = \chi(G) \cdot \chi(H)$ for every pair G,H of (finite) graphs.

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(Oblatum 25.5. 1988)