Commentationes Mathematicae Universitatis Carolinae

Sergej M. Berger

On deductive varieties of locally convex spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 465--475

Persistent URL: `http://dml.cz/dmlcz/106662`

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1988

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://project.dml.cz`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,3 (1988)

ON DEDUCTIVE VARIETIES OF LOCALLY CONVEX SPACES

S.M. BERGER

Abstract: In this paper we study the deductive varieties of (real or complex) locally convex spaces (LCS´s), i.e. the varieties whose subprevarieties are the subvarieties, too. The structure of these varieties is characterized in two categories: the category LC of all LCS´s and the category HLC of all Hausdorff LCS´s. Our characterization is categorically-oriented. Also, these varieties prove to be generated by the spaces with the "strongest l.c. topology in some prevariety", so they are categorically similar to the varieties, generated by LCS´s with the strongest l.c. topology.

Key words and phrases: Deductive varieties, prevariety, reflector, reflection, strongest in prevariety l.c. topology.

Classification: 18A40, 46M10, 54B10

---

1. Introduction. The main result of this paper is the characterization in terms of reflectors of the deductive varieties of locally convex spaces (LCS´s). More precisely, we shall write out the collections of the deductive varieties dividing them owing to their generation, using only some special spaces and reflectors.

This topic is induced by appropriate general problems of algebra and algebraic logic such as the problem of hereditarily structurally complete algebraic systems in the finitary sense (in the Russian literature) and near questions (see, e.g. [1]).

However, our results show how far is the structure of the deductive varieties of LCS´s from the same in an algebraic case (cf. [1]). In particular, the deductive varieties of LCS´s admit a full description in terms of reflectors of varieties, which is impossible even in the category of modules.

Following [1], we give the definition of the basic object of our paper (all necessary notions are either defined below or are folklore in category theory or theory of LCS´s):

Definition 1.1. A variety of LCS´s is called deductive if every its subprevariety is a subvariety.

We shall construct some examples of the deductive and non-deductive varieties in Section "Examples". At the end of the paper we shall formulate two open questions concerning the structure of the deductive varieties in alternative categories.

The study of varieties of LCS´s was initiated in [2], where some general results concerning the structure of varieties had been obtained by J. Diestel, S.A. Morris and S.A. Saxon (see also [10]; for prevarieties - [11]).

2. Terminology and notation. Our terminology of category theory is standard (see, e.g. [3],[12]). The symbol R_K stands for the reflector from LC (HLC) into K, where K is a reflective subcategory of LC (HLC). We write also $K \triangleright K_1$ if K is a full subcategory of LC and K_1. Also, $r_K: X \longrightarrow R_K(X)$ denotes the corresponding reflection of an object X from LC (HLC) into K; while the symbol $R_K(X)$ denotes the object in K, corresponding to X by the action of the reflector R_K.

These denotations are essentially used throughout the paper. In what follows, all morphisms (maps) are assumed to be continuous linear (linear respectively), all categories are saturated (i.e. closed under the taking isomorphic images); all subcategories are full.

We write $E \hookrightarrow F$ if E is a subspace of F, where $E,F \in$ HLC (LC); we usually omit the symbol Obj, and so on.

Recall that a variety (a prevariety) of LCS´s is the subcategory of LC or HLC, closed under the operations S,C,Q (S,C only), where S, C, Q are the operations of the taking a subspace (not necessarily closed), the Tychonoff product and a separated quotient respectively.

A variety $\mathfrak{M}$ (prevariety P$\mathfrak{M}$) is said to be generated by a class of LCS´s $\mathfrak{K}$ (denotations: $\mathfrak{M} = \mathcal{V}(\mathfrak{K})$ and P$\mathfrak{M}$ =V($\mathfrak{K}$) respectively), whenever $\mathfrak{M}$ (P$\mathfrak{M}$) is an intersection of all varieties (prevarieties), containing $\mathfrak{K}$. It is well known that a (pre)variety is a reflective subcategory of LC or HLC. All (pre)varieties under consideration are assumed to be nonzero, i.e. do not consist of $\{0\}$ only.

By L(X) we denote the free LCS on the Tychonoff space X (see, in particular, [13]). We need essentially only the following property of the functor L:HW $\longrightarrow$ HLC, where HW is the category of Tychonoff spaces; namely: L is a reflector: HW $\longrightarrow$ HLC.

By the Greek letters we denote the ordinals (ORD stands for the class of all ordinals; ω is the first infinite ordinal. The symbol $|\alpha|$ stands for the cardinal, corresponding to $\alpha \in$ ORD; the cardinality of the set S is denoted by card(S). The symbol **N** stands for the set of natural numbers. Lastly $E_+(\alpha)$ denotes LCS with the strongest l.c. topology and the Hamel basis of cardinality $|\alpha|$, i.e. an indexed by α locally convex sum of the basic field F=**R** or **C**. The remaining denotations of concrete LCS's in Section 4 are folklore; other denotations are obvious. The sign := means "is equal by definition to ...".

3. Results. At the beginning we shall reformulate a result which is necessary for us. Let $X \in$ HLC be endowed with l.c. topology $\mathcal{T}$. The topology $\mathcal{T}$ on X is called <u>final</u> (final in $P\mathfrak{M}$) relatively to the map $S: \mathrm{HLC} \ni Y \longrightarrow X$ if $\mathcal{T}$ is the strongest l.c. topology on X, for which S is continuous (and $(X,\mathcal{T}) \in P\mathfrak{M}$ respectively).

Theorem 3.1 [4]. Let $P\mathfrak{M}$ be a prevariety in HLC. Then for every $X= =(X,T) \in$ HLC, the object $R_{P\mathfrak{M}}(X)$ is described in the following way: $R_{P\mathfrak{M}}(X)= =(X,\mathcal{T})$, where $\mathcal{T}$ is the final in $P\mathfrak{M}$ l.c. topology on X relatively to the identity map $\mathrm{Id}:(X,T) \longrightarrow X$.

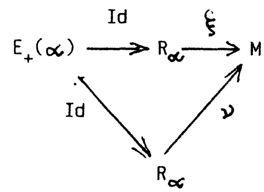
Every prevariety $P\mathfrak{M} \triangleright$ HLC contains the subcategory $P\mathfrak{M}_+$ of LCS's endowed with the strongest l.c. topology in $P\mathfrak{M}$ (i.e. if $E=(E,T) \in P\mathfrak{M}_+$ and $(E,T_1) \in P\mathfrak{M}$, then $T \geq T_1$. The theorem yields the following

Corollary 3.2. 1) Every space belonging to $P\mathfrak{M}_+$ has the form: $R_\alpha := R_{P\mathfrak{M}}(E_+(\alpha))$, $\alpha \in$ ORD.

2) Every subspace and quotient space of R_α have the form: R_β with $\beta \leq \alpha$.

3) Moreover, $V(P\mathfrak{M}_+)$ is a variety.

Proof: It is not hard to notice that every LCS $\in P\mathfrak{M}$ endowed with the strongest in $P\mathfrak{M}$ l.c. topology has the form: $R_\alpha = R_{P\mathfrak{M}}(E_+(\alpha))$, where $|\alpha|= =$cardinality of the Hamel basis in E; this is an immediate consequence of Theorem 3.1. This gives the first assertion of the corollary. Let us prove the second one. On the one hand, owing to Theorem 1 [5] and since all our categories are saturated, and a reflection of a prevariety is always a bijection, we may assume that the reflection $r_{P\mathfrak{M}}: E \longrightarrow R_{P\mathfrak{M}}(E)$ $(E \in \mathrm{HLC})$ is the(algebraically) identity map. On the other hand, every linear mapping $\xi: :R_\alpha \longrightarrow M \in P\mathfrak{M}$ is continuous. Indeed, let us consider the following diagram (it arises due to the reflectivity of $P\mathfrak{M}$):



Now we see that ξ is equal to ν - the continuous map. In particular, every subspace of R_α is closed. Thus every quotient space R_α/L (where L is a subspace of R_α) has the form: R_β ($\beta \leq \alpha$), because l.c. topology on R_β is the final one relatively to the natural map. Then every $L \hookrightarrow R$ has a complement in R_α (even if by Bourbaki´s Proposition 8.13 of [6]), henceforth, $L=R_\beta$ for some $\beta \leq \alpha$, too. This gives the second assertion.

By the main theorem of [2], $\mathcal{V}(\mathcal{K})=SCQP(\mathcal{K})$ (P stands for the operation of the taking the finite product). Hence we have the following chain of equalities:

$$V(P\,\mathfrak{M}_+)=SC(P\,\mathfrak{M}_+)=SCQP(P\,\mathfrak{M}_+)=\mathcal{V}(P\,\mathfrak{M}_+).$$

So $V(P\,\mathfrak{M}_+)$ is a variety and the corollary is proved.

Remark. Actually, it is possible to give a very short proof of Corollary 3.2 based on some properties of the functor L.But we prefer another one which uses $R_{P\mathfrak{M}}$ only. In fact, the proof of this statement could be easily extracted from [7].

Now we will prove our main result. As it is mentioned above, the result covers all deductive varieties.

Here (and further in Section 3) α is always a limit and β is a non-limit ordinal respectively. We define:

1) $\mathfrak{M} := \bigcup_{\alpha \in ORD} \mathfrak{M}_\alpha$ and $\mathfrak{M}_\alpha := \mathcal{V}(\prod_{\beta<\alpha} E_+(\beta))$

and $\mathfrak{M}_\beta := \mathcal{V}(E_+(\beta))$;

2) $\mathfrak{M}^* := \mathcal{V}(\mathfrak{M}, V)$, $\mathfrak{M}^*_\alpha := \mathcal{V}(\mathfrak{M}_\alpha, V)$ and $\mathfrak{M}^*_\beta := \mathcal{V}(\mathfrak{M}_\beta, V)$;

3) $R_{P\mathfrak{M}}\,\mathfrak{M}_\alpha := \mathcal{V}(\prod_{\beta<\alpha} R_{P\mathfrak{M}}(E_+(\beta)))$,

$R_{P\mathfrak{M}}\,\mathfrak{M}_\beta := \mathcal{V}(R_{P\mathfrak{M}}(E_+(\beta)))$, and

$R_{P\mathfrak{M}}\,\mathfrak{M} := \bigcup_{\alpha \in ORD} R_{P\mathfrak{M}}\,\mathfrak{M}_\alpha$,

and analogously $R_{P\mathfrak{M}}\,\mathfrak{M}^*$, $R_{P\mathfrak{M}}\,\mathfrak{M}^*_\alpha$, $R_{P\mathfrak{M}}\,\mathfrak{M}^*_\beta$.

Theorem 3.3. Let V be a prevariety of all LCS´s endowed with the antidiscrete l.c. topology. Then any deductive variety belongs to one of four collections of the deductive varieties in category LC:

1) $\mathfrak{M}$, $\mathfrak{M}_\alpha$, $\mathfrak{M}_\beta$;

2) $\mathfrak{M}^*$, $\mathfrak{M}^*_\alpha$, $\mathfrak{M}^*_\beta$;

3) $R_{P\mathfrak{M}}\mathfrak{M}$, $R_{P\mathfrak{M}}\mathfrak{M}_\alpha$, $R_{P\mathfrak{M}}\mathfrak{M}_\beta$;

4) $R_{P\mathfrak{M}}\mathfrak{M}^*$, $R_{P\mathfrak{M}}\mathfrak{M}^*_\alpha$, $R_{P\mathfrak{M}}\mathfrak{M}^*_\beta$.

Remark. Notice that it is possible to consider V= $\mathfrak{M}^*_\beta$, $\beta=0\in$ ORD. This theorem is a consequence of the following

3.4. Theorem. The deductive varieties in the category HLC are $R_{P\mathfrak{M}}\mathfrak{M}$, $R_{P\mathfrak{M}}\mathfrak{M}_\alpha$ and $R_{P\mathfrak{M}}\mathfrak{M}_\beta$ only.

Proof of Theorem 3.3: Let E be LCS, belonging to a deductive variety $\mathfrak{V}$. E can be written as $E_+ \oplus E_-$, where $V \ni E_-$ is a closure of zero in E and $E_+ \in$ HLC is its complement in E. If $E_- \neq \{0\}$, then $V(\{E\}) \supseteq V$. Thus the collection 1 corresponds to the case $P\mathfrak{M}$ =HLC and $E_- = \{0\}$ for any LCS $E \in \mathfrak{V}$. The collection 2 corresponds to the case of the existence of $E \neq E_+$, $E \in \mathfrak{V}$. The collections 3, 4 appear from Theorem 3.4 analogously. The theorem is proved.

Before the proof of Theorem 3.4 we shall formulate the following almost obvious but useful

Lemma 3.5. Let a variety $\mathfrak{M}$ be deductive. Then for every class of LCS´s K, $K \subseteq \mathfrak{M}$ implies V(K)= $\mathfrak{V}$(K) or, equivalently, QP(K) $\subseteq$ SC(K).

Proof: Because of the equality $\mathfrak{V}$(K)=SCQP(K) and our assumption, we have V(K)=SC(K)=SCQP(K), so we have QP(K) $\subseteq$ SC(K). That is all.

In fact, Lemma 3.5 is a crucial observation.

Proof of Theorem 3.4: The scheme of the proof is analogous to [7, Therem 1].

For the convenience we shall divide our proof in 5 steps. In addition, for the sake of completeness all necessary auxiliary constructions and results are cited.

Step 1. Firstly, it is necessary to verify that all the varieties of the series of the theorem are deductive. To this end, the case of variety

$R_{P\mathfrak{M}}\, \mathfrak{M}_\beta$ (β is non-limit) is exactly the conclusion of Corollary 3.2, point 2, combined with the equality $\mathfrak{N}(K)$=SCQP(K), mentioned above, and the obvious fact that $R_{P\mathfrak{M}}(E_+(\alpha))$, $\alpha \in$ ORD, is isomorphic to its own square. Since $R_{P\mathfrak{M}}\, \mathfrak{M} = \bigcup_{\beta\in ORD} R_{P\mathfrak{M}}\, \mathfrak{M}_\beta$, it is a deductive variety, because a union of an increasing chain of deductive varieties is deductive. Let us prove that $R_{P\mathfrak{M}}\, \mathfrak{M}_\alpha$ is deductive; α is a limit ordinal. To this end, by the definition of $R_{P\mathfrak{M}}\, \mathfrak{M}_\alpha$ the following equality takes place: $R_{P\mathfrak{M}}\, \mathfrak{M}_\alpha$ = =SCQP($\prod_{\text{non-limit } \beta<\alpha} R_{P\mathfrak{M}}(E_+(\beta))$). So again by the virtue of Corollary 3.2 one has $R_{P\mathfrak{M}}\, \mathfrak{M}_\alpha$ =SC($\prod_{\beta<\alpha} R_{P\mathfrak{M}}(E_+(\beta))$). Thus, $R_{P\mathfrak{M}}\, \mathfrak{M}_\alpha = \bigcup_{\beta<\alpha} R_{P\mathfrak{M}}\, \mathfrak{M}_\beta$; the latter guarantees the deductivity again.

Step 2. Now we turn our attention to some constructions and results of [8]. Let X be a Tychonoff space, $|\alpha|$ is a cardinal. For $t\in X$ we put $X_t := \bigsqcup_{x\in X} A_x$, where $\bigsqcup$ denotes the disjoint union and if $x \neq t$, then A_x is a set of cardinality $> |\alpha|$; otherwise, $A_t = \{t\}$. The topology on X_t is the following: all points except t are isolated and every neighbourhood of the point t has the form: $p_t^{-1}(\mathcal{V}) \setminus \bigcup_{x\neq t} B_x$, where p_t is a natural mapping from X_t into X by the rule: $A_x \to x \in X$; $\mathcal{V}$ is a neighbourhood of t in X; $B_x \subset A_x$ and $\mathrm{card}(B_x) \leq |\alpha|$. Let $T(X,\alpha)$ be $\bigsqcup_{t\in X} X_t$ endowed with the topology of the disjoint union and $p:T(X,\alpha)\to X$ is $\bigsqcup_{t\in X} p_t$, i.e. corresponding to the disjoint mapping with $p|_{X_t}=p_t$.

Lemma 3.6 [8, Lemma 1]. The map $p:T(X,\alpha)\to X$ is a factorable map.

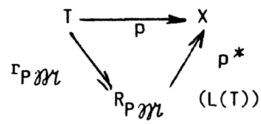
Recall that the map p is factorable if for every set V the set $p^{-1}(V)$ is open if and only if V is open. Recall also that L(X) denotes the free LCS on the Tychonoff space X.

Lemma 3.7 [8, Lemma 2]. For every Tychonoff space X and each LCS $E \hookrightarrow L(T(X,\alpha))$ with the Hamel basis S and $\mathrm{card}(E) \leq |\alpha|$ the following equality takes place:

$L(T(X,\alpha))=E \oplus H$ and $E=\bigoplus_{s\in S} F_s$, where $F_s=F$ for each $s\in S$.

Step 3. We need also the following

Lemma 3.8. Let $P\mathfrak{M}$ be a prevariety (or simply a reflective subcategory) in HLC. Then for every factorable map $p:T\to X$, $X\in P\mathfrak{M}$ and Tychonoff space T there is the following commutative diagram with a quotient map $p^{*}:R_{P\mathfrak{M}}(L(T)) \longrightarrow X$:



Proof: Let $\mathcal{U} \neq \{0\}$ be an absolutely convex neighbourhood of zero in $R_{P\mathfrak{M}}(L(T))$. Choose $\lambda \in F$ with $\lambda\mathcal{U} \cap r(T) \neq \emptyset$. Without loss of generality, we may assume that $\mathcal{U}$ has this property. It follows that the set $r^{-1}(\mathcal{U} \cap r(T))$ is open and nonempty. Henceforth, for $V=p(r^{-1}(\mathcal{U} \cap r(T)))$ there is the chain of equalities and an inclusion:

$$p^{-1}(V)=p^{-1}p(r^{-1}(\mathcal{U} \cap r(T)))=r^{-1}(p^{*\,-1})p^{*}(\mathcal{U} \cap r(T)) \supset r^{-1}(\mathcal{U}).$$

The last set is open. Since p is a factorable map, the set V is also open. Evidently, $V \subseteq p^{*}(\mathcal{U})$, so $p(\mathcal{U})$ is open, q.e.d.

Step 4. Let $\mathfrak{M}$ be a deductive variety. Fix $X \in \mathfrak{M}$, $\operatorname{card}(X) \geq \aleph_0$. Take $\alpha \in ORD$, $|\alpha| > \operatorname{card}(X)$. By Lemma 3.6, there are the factorable map $p:T=T(X,\alpha) \longrightarrow X$ and, as follows by Lemma 3.8, the quotient map $p^{*}:R_{P\mathfrak{M}}(L(T)) \longrightarrow X$. According to Lemma 3.5, LCS X is a subspace of some power ξ ($\xi \in ORD$) of the space $R_{\mathfrak{M}}(L(T))$ (let us denote the last space by R). Indeed, it is obvious that R^n is isomorphic to R for every $n \in \mathbf{N}$, so $QP(R)=Q(R) \subseteq SC(R)$.

The image of the restriction to X of the natural projection $\pi:R^{\xi} \longrightarrow R$ is the subspace of $R_{\mathfrak{M}}(L(T))$ with cardinality $< |\alpha|$.

We shall prove now that $\pi(X)=R_{\mathfrak{M}}(E_{+}(\beta^{*}))$, $\beta^{*} \in ORD$ and $|\beta^{*}| < |\alpha|$. Really, the space $\pi(X)$ lies on a subspace $E \hookrightarrow R$ with the Hamel basis S, $\operatorname{card}(E) < |\alpha|$. By Lemma 3.7, we have $L(T)=(\bigoplus_{s\in S} F_s) \oplus H$ and $E=\bigoplus_{s\in S} F_s$. Since

$$R_{\mathfrak{M}}(L(T))=R_{\mathfrak{M}}(\bigoplus_{s\in S} F_s) \oplus R_{\mathfrak{M}}(H)=R_{\mathfrak{M}}(E_{+}(\alpha_1)) \oplus R_{\mathfrak{M}}(H),$$

where $\alpha_1=\operatorname{card}(S) < |\alpha|$ and $H \cap S=\emptyset$, it follows that $\pi(X)$ is a subspace of $R_{\mathfrak{M}}(E_{+}(\alpha_1))$.

Now $\pi(X)$ has the form: $R_{\mathfrak{M}}(E_{+}(\gamma))$ ($|\gamma| < |\alpha_1|$) by Corollary 3.2. Consequently, X is a subspace of the product of spaces of the form $R_{\mathfrak{M}}(E_{+}(\gamma))$, and so of the form $R_{P\mathfrak{M}}(E_{+}(\gamma))$ simultaneously. Lastly, we have for our deductive variety $\mathfrak{M}$

$$(*) \quad \mathfrak{M} \subseteq SC\Big(\prod_{\gamma\in ORD} R_{P\mathfrak{M}}(E_{+}(\gamma))\Big)=\mathcal{V}\Big(\prod_{\gamma\in ORD} R_{P\mathfrak{M}}(E_{+}(\gamma))\Big).$$

Step 5. Now we get a conclusion of the theorem by means of successive examinations of all possible cases. Namely: let $\varepsilon := \inf\{\alpha \in \mathrm{ORD},$ $\mathcal{T}(R_{P\mathfrak{M}}(E_+(\alpha))) \supsetneq \mathfrak{M}\}$. If ε exists, then $\mathfrak{M}$ has the form $R_{\mathfrak{M}}\, \mathfrak{M}_\alpha$ or $R_{\mathfrak{M}}\, \mathfrak{M}_\beta$ respectively to the cases ε is a limit or non-limit ordinal. Actually, the ordinal γ in (∗) is the indexing ordinal $\gamma < \varepsilon$. So the product $\prod_\gamma$ will be $\prod_{\gamma<\varepsilon}$ or $\prod_{\gamma\leq\varepsilon}$. The latter leads to $R_{P\mathfrak{M}}\, \mathfrak{M}_\varepsilon$ by using spaces with the form $R_{P\mathfrak{M}}(E_+(\alpha))$. If ε does not exist, then $\mathfrak{M} = R_{\mathfrak{M}}\, \mathfrak{M}$; it is even more easier than the case of the existence of ε. Thus we have got all the collections of the theorem.

Corollary 3.9. The deductive varieties in HLC are exactly varieties, generated by the class of LCS´s with the strongest in some prevariety l.c. topology or (just the same) by the class of projective objects in this prevariety.

Proof: The first assertion is implied by combining Theorem 3.4 and Corollary 3.2. The second one is a straightforward consequence of Theorem 1 of [7] which contains the suitable characterization of projective objects in the prevariety $P\mathfrak{M}$: they are exactly the spaces with the strongest in the prevariety $P\mathfrak{M}$ l.c. topology. The corollary is proved.

J. Vilímovský [14] proved that there exists the greatest (by inclusion) reflective subcategory $K \rhd$ HLC with $K \cap \mathrm{NORM} = \{\mathbf{F}^n \mid n \in \mathbf{N}\}$, where NORM is the category of all normed spaces. It is not hard to see that K is a prevariety. Indeed, for $K \subseteq V(K)$ and infinite-dimensional normed space N we have: if $N \hookrightarrow \prod F_i$, $F_i \in \mathrm{LC}$, then $N \hookrightarrow \prod_{i=1}^{n} F_i$ for some $n \in \mathbf{N}$; and there exists $j \in I$ (I is an indexed set, $i \in I$, too) so that F_j has an infinite-dimensional normed space as a subspace. Thus, V(K) also (hereditarily) does not contain infinite-dimensional normed spaces.

The prevariety $P\mathfrak{M}$, generated by the class of LCS with the form $R_K(E_+(\alpha))$ ($\alpha \in$ ORD), is the greatest deductive subprevariety in K. So we can formulate this observation as follows:

Corollary 3.10. There exists the greatest deductive variety in HLC, not containing infinite-dimensional normed spaces.

4. Examples

Example 1. (Here **F**=**R**.) As has been proved above (Corollary 3.2), the prevariety $V(\{\varphi\})$, where $\varphi = E_+(\omega)$, is deductive. It implies, in particular, the result in [2]: $V(\{\varphi\})$ is the second smallest variety in HLC, besides the variety $V(\{\mathbf{R}\})$ of LCS with the weak l.c. topology, in the following sense: any variety, containing $V(\{\mathbf{R}\}) = \mathcal{V}(\{\mathbf{R}\})$ and which is not equal to $V(\{\mathbf{R}\})$, contains $V(\{\varphi\})$, too.

It is obvious that $V(\{\mathbf{R}\})$ is deductive.

Example 2. The variety $\mathfrak{N}$ of all nuclear LCS´s and the variety S of all Schwartz LCS´s are not deductive. It is enough to prove this claim for $\mathfrak{N}$ because $\mathfrak{N} \subset S$. In order to prove this suppose on the contrary that $\mathfrak{N}$ is deductive. It is well known (this is a result of T. and Y. Kömura) that $\mathfrak{N}$ has a universal generator - the Fréchet space s of rapidly decreasing sequences (see, e.g. [2]). (Recall also that LCS E is said to be a universal generator in the variety $\mathfrak{M}$ if $\mathfrak{M} = SC(E)$.) This is a contradiction with the following

Proposition 4.1. Let $\mathfrak{M}$ be a deductive variety in HLC. Then $\mathfrak{M}$ has not the metrizable LCS as a universal generator if $\mathfrak{M} \neq V(\{F\})$.

Proof (by contradiction): By Theorem 3.4 there is an equality
$\mathfrak{M} = V\left(\prod_{\beta < \alpha} R_{\mathfrak{M}}(E_+(\beta))\right),\ \alpha \geq \omega$.

Let P be a universal generator of $\mathfrak{M}$. If it were metrizable, then it would be a product of metrizable spaces in $\mathfrak{M}$ (in general). Thus, by our assumption, P is a product of metrizable subspaces in $\prod_{\beta < \alpha} R_{\mathfrak{M}}(E_+(\beta))$. Since $\mathfrak{M} \neq V(\{F\})$, it follows that $V(\{\varphi\}) \subseteq \mathfrak{M}$ (as cited above), which means that $R_{\mathfrak{M}}(E_+(\omega)) = \varphi$. Because φ is not metrizable LCS, one easily sees that any metrizable subspace E of $\prod_{\beta < \alpha} R_{\mathfrak{M}}(E_+(\beta))$ is either some power of the space F or a subspace of this power, i.e. E is LCS, having the weak topology. This is a contradiction with our assumption $\mathfrak{M} \neq V(\{F\})$, so the proposition is proved.

The proof allows to establish the following

Corollary 4.2. If $\mathfrak{M} = \mathcal{V}(K)$, where K is a class of metrizable spaces, then $\mathfrak{M}$ is not deductive.

Proof: The argument is the same as in the proof of Proposition 4.1, because it suffices to observe that $\mathfrak{M}$ contains $\mathfrak{M}_1 = \mathcal{V}(\{P\})$, where $P \in K$ and so $\mathfrak{M}_1$ must be a deductive variety.

Now we have also an immediate

Corollary 4.3. The class of non-deductive varieties in HLC(LC) is a proper class. The class of the deductive varieties in HLC(LC) is a proper class, too.

Proof: The first claim is a consequence of Corollary 4.2. To prove the second one it suffices to look at the varieties $\mathfrak{M}_\alpha = \mathcal{V}(E_+(\alpha))$, $\alpha \in$ ORD.

Example 3. There exists just-non-singly generated deductive variety $\mathfrak{M}$, for which every its proper subvariety is a singly generated variety (i.e. it is equal to $\mathcal{V}(\{E\})$, $E \in$ HLC) and deductive. Namely $\mathfrak{M} = \mathcal{V}(E_+)$, where E_+ is a class of all spaces with the strongest l.c. topology [9].

Notice again that this result is a consequence of Theorem 3.4 and v.v. Theorem 3.4 may be considered as a natural generalization of the structure of the variety $\mathfrak{M}$.

This note with Corollary 3.9 is an explanation of our last phrase in Abstract.

5. Problems

1. What is a characterization of the structure of the deductive varieties of topological vector spaces?

It seems that this problem is more topological, because there is no functor like L in the category of topological vector spaces. In addition, it seems worthwhile to mention that there exist some characterizations in terms of reflectors.

2. Is there a suitable characterization of the structure of the deductive varieties of topological groups?

It is possible that this characterization is based on the correspondence basis of laws (topological laws) (cf., e.g. [9]).

The author expresses his gratitude to V.G. Pestov and especially to A.G. Leiderman for encouragement; he also would like to thank E.R. Leonov for his assistance.

References

[1] L. HOGBEN, C. BERGMAN: Deductive varieties of modules and universal algebras, Trans. Amer. Math. Soc. 289(1985), 303-320.

[2] J. DIESTIEL, S.A. MORRIS, S.A. SAXON: Varieties of linear topological spaces, ibid. 172(1972), 207-230.

[3] S. MACLANE: Categories for the working mathematician, Springer-Verlag, N.Y., 1971.

[4] H.J. JUNEK: Über lokalkonvexe Räume und Operatorenideale, Math. Nachr. 71(1976), 7-22.

[5] V.A. GEĬLER: O proektivnykh ob"ektakh v kategoriyakh lokal´no vypuklykh prostranstv, Funkc. analiz i prilozh. 6(1972), No 2, 79-80.

[6] N. BURBAKI : Topologicheskie prostranstva, Moskva, "IL", 1959.

[7] S.M. BERGER: O proektivnykh ob"ektakh v kategoriyakh LVP, Matematicheskie zametki 39(1986), No 1, 113-120.

[8] V.A. GEĬLER, V.B. GISIN: Obobshchënnaya dvoïstvennost´ dlya LVP, Funkcional´nyï analiz 11(1978), Ul´yanovsk, 41-50.

[9] S.A. MORRIS: Just-non-singly generated varieties of locally convex spaces, Colloq. Math. 29(1974), 151-153.

[10] : Varieties of topological groups. A survey, ibid. 46(1982), 147-165.

[11] S. DIEROLF: Über assoziierte lineare und lokalkonvexe Topologien, Manuscr. Math. 16(1975), 27-46.

[12] H. HERRLICH, G. STRECKER: Category theory, Boston, Allyn and Bacon, 1973.

[13] J. FLOOD: Free locally convex spaces, Dissert. Math. CCXXI, 1984, Warzawa, PWN.

[14] J. VILÍMOVSKÝ: Locally convex spaces not containing normed-like subspaces, "Categorical Topology. Proc. Int. Conf., Toledo, Ohio, Aug. 1-5, 1983", Berlin, 1984, 588-598.

Department of Higher Mathematics, Novosiberian branch of Moscow Technological Institute of light industry, Novosibirsk, Potaninsky Lane 5, 630 099, USSR

(Oblatum 17.11. 1986, revisum 7.10. 1987)